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Math 20580 L.A. and D.E. Tutorial
Worksheet 7

1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -12 & 15 \\ -10 & 13 \end{bmatrix}$ and $B = \begin{bmatrix} -12 & 5 \\ -30 & 13 \end{bmatrix}$, and then check if they are similar

Solution: We first find the eigenvalues and eigenvectors for A:

$$\det(A - \lambda I) = \det \begin{bmatrix} -12-\lambda & 15 \\ -10 & 13-\lambda \end{bmatrix} = (\lambda - 3)(\lambda + 2)$$

Setting the characteristic polynomial to 0, we get 3 and -2 are two eigenvalues for A.

When $\lambda = 3$:

$$(A - 3I)x = 0 \Leftrightarrow \begin{bmatrix} -15 & 15 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is an eigenvector corresponds to $\lambda = 3$.

We do the same for $\lambda = 2$ and get the corresponding eigenvector to be $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$.

So according to the theory of diagonalization,

$$A = PDP^{-1}$$

for $P = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

We do a similar computation and find $\lambda = 3, -2$ are two eigenvectors of B and their corresponding eigenvectors are $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$.

So the diagonalization of $B = P'DP'^{-1}$ is for $P' = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

Then solving D and get $D = P'^{-1}BP'$. We then plug D into $A = PDP^{-1}$ to get $A = PDP^{-1} = PP'^{-1}BP'P^{-1} = PP'^{-1}B(PP'^{-1})^{-1}$.

We say A and B are similar if we can find a matrix P'' such that A=$P''BP''^{-1}$.

Note that $P'' = PP'^{-1}$ is a candidate. So A and B are similar.

2. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

   (a) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

   (b) What is A^{2025}?

Solution:

(a) • First, we find all eigenvalues of A:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3),$$

which implies that the eigenvalues are $\lambda = 2$ and $\lambda = 3$. Since the eigenvalues of A are **distinct**, A is **diagonalizable**.

• Next, we find eigenvectors corresponding to each eigenvalue:

- For $\lambda = 2$:

$$(A - 2I_2)\mathbf{x} = 0 \iff \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

and hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

- For $\lambda = 3$: Using the same method, the eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$

• According to the theory of diagonalization, the following matrices

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

satisfies $A = PDP^{-1}$.

(b) Using that $A = PDP^{-1}$, we can simplify:

$$\begin{aligned} A^{2024} &= (PDP^{-1})^{2025} = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \qquad \text{(2025 times)} \\ &= PD(P^{-1}P)D(P^{-1}P)D\dots(P^{-1}P)DP^{-1} \\ &= PD^{2025}P^{-1} \end{aligned}$$

Let's compute P^{-1} and D^{2025}:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Longrightarrow P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix};$$
$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Longrightarrow D^{2025} = \begin{bmatrix} 2^{2025} & 0 \\ 0 & 3^{2025} \end{bmatrix}.$$

Finally,

$$\begin{aligned} A^{2025} &= PD^{2025}P^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{2025} & 0 \\ 0 & 3^{2025} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

3. Let $A = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$.

(a) Determine all the eigenvalues of A.

(b) For each eigenvalue λ of A, find the eigenspace E_λ.

(c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

(d) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution:

(a) We have

$$\det(A - \lambda I_3) = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$.

(b) Recall that a λ-eigenvector is an element of the kernel of $A - \lambda I$.

We have that

$$A-\lambda_1 I = \begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix} \quad A-\lambda_2 I = \begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \quad A-\lambda_3 I = \begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

The eigenspaces E_λ of A will be the null spaces $A - \lambda I$. We find these using row reduction on the homogeneous systems $[A - \lambda I | \vec{0}]$.

For $\lambda_1 = 1$, we have $E_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

For $\lambda_2 = 2$, we have $E_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_3 = 3$, we have $E_3 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(c) A basis for $\mathbb{R}^3$ consisting of eigenvectors of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(d) The matrices P and D we need to find are

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

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4. Eigenvalues and eigenvectors only make sense for square matrices. Fortunately, even if A is a nonsquare matrix, AA^T and A^TA will be square. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$.

   (a) Find the characteristic polynomial and eigenvalues of AA^T.

   (b) Find the characteristic polynomial and eigenvalues of A^TA.

   (c) What do you notice about the eigenvalues of the two square matrices?

Solution:

(a) $AA^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & -4 & 8 \end{bmatrix}$. Using Laplace expansion along the top row, we find that the characteristic polynomial of AA^T is $\det(AA^T - \lambda I) = (2-\lambda)((2-\lambda)(8-\lambda) - 16) = (2-\lambda)(\lambda^2 - 10\lambda) = (2-\lambda)(\lambda)(\lambda - 10)$. The eigenvalues of AA^T are the roots of the characteristic polynomial, namely $\lambda = 0, 2, 10$.

(b) $A^TA = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}$. The characteristic polynomial of A^TA is $\det(A^TA - \lambda I) = (6-\lambda)(6-\lambda) - 16 = \lambda^2 - 12\lambda + 20 = (\lambda - 2)(\lambda - 10)$. The eigenvalues of A^TA are the roots of the characteristic polynomial, namely $\lambda = 2, 10$.

(c) Each eigenvalue of A^TA is also an eigenvalue of AA^T, but not all eigenvalues of AA^T is an eigenvalue of A^TA.