Name: SOLUTIONS Date: 03/27/2025

M20580 L.A. and D.E. Tutorial
Worksheet 8

1. Determine if the given vectors form an orthogonal set

$$\begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\4\\-2\end{bmatrix}.$$

Solution: Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the given vectors respectively. Then we have

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= -4 + 1 + 3 = 0\\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= 2 + 4 - 6 = 0\\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= -2 + 4 - 2 = 0.\end{aligned}$$

Therefore they form an orthogonal set.

2. Determine if the given vectors form an orthogonal basis for $\mathbb{R}^2$

$$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}-4\\6\end{bmatrix}.$$

Solution:

$$\begin{bmatrix}3\\2\end{bmatrix} \cdot \begin{bmatrix}-4\\6\end{bmatrix} = -12 + 12 = 0$$

thus they are orthogonal. Next we need to check they form a basis. We provide two different proofs of this.

- Proof 1 using direct computation: Since we have two vectors and $\mathbb{R}^2$ is 2-dimensional, we only need to check that they are linearly independent.

$$\begin{aligned}\begin{bmatrix}3 & -4\\2 & 6\end{bmatrix} &\sim \begin{bmatrix}3 & -4\\6 & 18\end{bmatrix}\\ &\sim \begin{bmatrix}3 & -4\\0 & 26\end{bmatrix}\end{aligned}$$

  and thus they are linearly independent and we've shown *they are an orthonormal basis for* $\mathbb{R}^2$.

- Proof 2 using Theorem 5.1 in Poole: *If* $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ *is an orthogonal set of nonzero vector in* $\mathbb{R}^n$*, then these vectors are linearly independent.* Since our vectors are nonzero and form an orthogonal set, they are also linearly independent. Thus they form an orthogonal basis for $\mathbb{R}^2$.

3. Find the orthogonal complement $W^\perp$ of W in $\mathbb{R}^3$ where

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 3y = z \,, x + 2z = 0 \right\}.$$

Hint: $(\text{null}A)^\perp = \text{col}A^T$ and $(\text{col}A)^\perp = \text{null}A^T$.

Solution: Note that $W = \text{null}\left\{ \begin{bmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \right\}$. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$. Then, by the hint, $W^\perp = \text{col}A^T$. We have that

$$A^T = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus $W^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

4. Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

   (a) Find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

   (b) Find $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution:

(a) $\mathbf{u} \cdot \mathbf{v} = 0 - 3 - 4 = -7 = \mathbf{v} \cdot \mathbf{u}$.

(b)

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{-7}{0+1+4}\mathbf{u} = \frac{-7}{5}\mathbf{u}$$
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{-7}{1+9+4}\mathbf{v} = \frac{-7}{14}\mathbf{v} = \frac{-1}{2}\mathbf{v}.$$

5. Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$ onto the subspace $W = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$ in $\mathbb{R}^3$, where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$. Find the distance from $\mathbf{v}$ to W.

Solution: Note that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthogonal basis of W. Thus,

$$\begin{aligned}
\text{proj}_W(\mathbf{v}) &= \frac{(1,-2,1)\cdot(1,-2,7)}{\|(1,-2,1)\|^2}(1,-2,1) + \frac{(3,0,-3)\cdot(1,-2,7)}{\|(3,0,-3)\|^2}(3,0,-3) \\
&= \frac{12}{6}(1,-2,1) + \frac{-18}{18}(3,0,-3) \\
&= (-1,-4,5) \\
\text{perp}_W(\mathbf{v}) &= \mathbf{v} - \text{proj}_W(\mathbf{v}) = (2,2,2).
\end{aligned}$$

Then distance from $\mathbf{v}$ to W is $\|\text{perp}_W(\mathbf{v})\| = 2\sqrt{3}$.

6. Apply the Gram-Schmidt process to the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix},$$

to obtain an orthonormal basis.

Solution: Let $\mathbf{v}_1 = (1, 0, -1)$. Then

$$\mathbf{v}_2 = (1, 1, 3) - \frac{(1, 1, 3) \cdot (1, 0, -1)}{\|(1, 0, -1)\|^2}(1, 0, -1) = (2, 1, 2),$$

and

$$\begin{aligned} \mathbf{v}_3 &= (-4, 2, -6) - \left(\frac{(-4, 2, -6) \cdot (1, 0, -1)}{\|(1, 0, -1)\|^2}(1, 0, -1) + \frac{(-4, 2, -6) \cdot (2, 1, 2)}{\|(2, 1, 2)\|^2}(2, 1, 2) \right) \\ &= (-1, 4, -1) \end{aligned}$$

To get orthonormal basis, we simply normalize the above vector, i.e.

$$\begin{aligned} \mathbf{u}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}(1, 0, -1) \\ \mathbf{u}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{3}(2, 1, 2) \\ \mathbf{u}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{3\sqrt{2}}(-1, 4, -1). \end{aligned}$$