Name: Date: 04/03/2025

Math 20580 (L.A. and D.E.) Tutorial
Worksheet 9

1. Determine the QR-factorization of the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution: Apply Gram–Schmidt to the columns of A:

$$u_1 = [1, -1, 0]^T.$$

Hence,

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}[1, -1, 0]^T.$$

Now for e_2:

$$u_2 = [1, 0, 1]^T - \text{proj}_{u_1}[1, 0, 1]^T = [1, 0, 1]^T - ([1, 0, 1]^T \cdot e_1)e_1;$$

$$u_2 = [1, 0, 1]^T - \frac{1}{2}[1, -1, 0]^T;$$

$$u_2 = [1/2, 1/2, 1]^T.$$

Hence,

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}}[1, 1, 2]^T.$$

Thus, we get that

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

To get the matrix R, we use that fact that $Q^TQ = I_{2\times 2}$. Hence,

$$R = Q^TA = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}.$$

2. Find a least squares solution to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Furthermore, is the solution unique or not?

Solution: We compute

$$A^T A = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix} \quad \text{and} \quad A^T \mathbf{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

So the solution to the equation $A^T A\overline{\mathbf{x}} = A^T\mathbf{b}$ is

$$\overline{\mathbf{x}} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{15} \end{bmatrix}.$$

Furthermore, the solution is unique since $A^T A$ has full rank.

3. Find the quadratic function that gives the best least squares approximation to the points $(1,1)$, $(2,-2)$, $(3,3)$, $(4,4)$.

Solution: Let the equation of the quadratic function be $y = a+bx+cx^2$. Substituting the given points into this quadratic, we obtain the linear system $A\mathbf{x} = \mathbf{b}$ that we want the least squares approximation of, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}.$$

We compute

$$A^TA = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \quad \text{and} \quad A^T\mathbf{b} = \begin{bmatrix} 6 \\ 22 \\ 84 \end{bmatrix}.$$

So the solution to the equation $A^TA\overline{\mathbf{x}} = A^T\mathbf{b}$ is

$$\overline{\mathbf{x}} = \begin{bmatrix} 3 \\ -\frac{18}{5} \\ 1 \end{bmatrix}.$$

Thus, the least squares approximating parabola has the equation

$$y = 3 - \frac{18}{5}x + x^2.$$

4. The function $y_1(x) = \frac{1}{16}x^4$, $-\infty < x < \infty$ and

$$y_2(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

have the same domain but are clearly different. Show that both functions are solutions of the initial-value problem (IVP)

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(2) = 1$$

on the interval $(-\infty, \infty)$. Explain why this IVP fails to have unique solutions.

Solution: Note that

$$\frac{d}{dx}\left(\frac{1}{16}x^4\right) = \frac{1}{4}x^3 = x \cdot \left(\frac{1}{4}x^2\right),$$

y_1 is a solution. Note that

$$\frac{d}{dx}(0) = 0 = x \cdot 0,$$

and that y_2 is differentiable on $(-\infty, \infty)$, y_2 is also a solution.

Let

$$F(x, y) = x\sqrt{y}.$$

This IVP fails to have unique solutions as

$$\frac{\partial F}{\partial y} = \frac{x}{2\sqrt{y}}$$

does not exist on $y = 0$.