MATH 40720: TOPICS IN ALGEBRA
ORDER AND LATTICES

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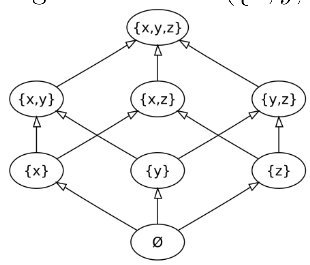
This course will provide an introduction to partially ordered sets ("posets") and lattices, which are mathematical notions based on the idea of an "ordering" of a collection of objects. Posets and lattices have broad relevance in mathematics and many applications to other fields including computer science, social science, sports and games, philosophy and logic.

A partial order on a set P specifies that certain elements of P precede others, in a compatible way. We write $p \leq q$ if p precedes (or equals) q. It is assumed that $\leq$ has the properties (1) reflexivity: $p \leq p$ for all p in P; (2) transitivity: if $p \leq q$ and $q \leq r$, then $p \leq r$; and (3) anti-symmetry: if $p \leq q$ and $q \leq p$, then $p = q$. The pair $(P, \leq)$ is then called a partially ordered set or poset. We say $\leq$ is a total order if any two elements are comparable: that is, for p, q in P, either $p \leq q$ or $q \leq p$. The standard orders of familiar systems of numbers such as the integers, rational numbers and real numbers are total orders. For any set A, one may partially order the set $P = \mathcal{P}(A)$ of all subsets of A by letting $X \leq Y$ mean $X \subseteq Y$ (that is, X is a subset of Y), for subsets X and Y of A.

A finite poset $(P, \leq)$ may be visualized by a Hasse diagram, with one vertex for each element of P, and in which $p \leq q$ if there is a path upward from p to q along edges of the diagram. For example, a Hasse diagram of $P = \mathcal{P}(\{x, y, z\})$ is given by

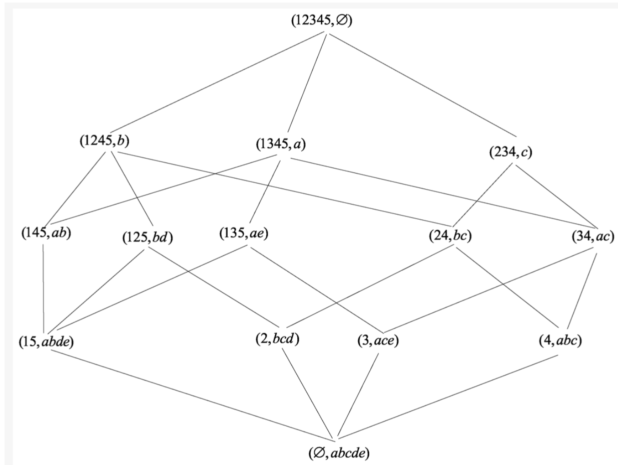


where $\emptyset = \{\}$ is the empty set.

A lattice is a poset $(P, \leq)$ in which for any elements p, q of P, there is a smallest element r (called the join, sup or lub of p and q, denoted $p \vee q$) of P such that $r \geq p$ and $r \geq q$, and a largest element s (called the meet, inf or glb of p and q, denoted $p \wedge q$) of P such that $s \leq p$ and $s \leq q$. A lattice can be regarded as an algebraic structure $(P, \vee, \wedge)$ of a set P with two operations $\vee$ and $\wedge$. For any set A, the "Boolean poset" $\mathcal{P}(A)$ is a complete lattice, where complete means that any subset of P has a join and a meet, defined similarly.

The first part of the course will discuss basic properties, examples and special types of posets and lattices, their special elements and subsets, their visualization by Hasse diagrams, constructions involving them, and special types of functions between them. The second part will treat a variety of special topics from amongst formal concept analysis, Galois connections, representations of lattices, congruences and quotient lattices, and fixed point theory, possibly depending partly on background and interests of attending students.

We illustrate some of the later topics with an example. Write 15, *abde* as shorthand notations for the sets $\{1,5\}$ and $\{a,b,d,e\}$, etc. Consider the poset with the following Hasse diagram:



It is not a lattice since $(15, abde) \vee (4, abc)$ does not exist (check!). However, the poset $(P, \leq)$ with the Hasse diagram obtained by adding an extra edge from $(4, abc)$ to $(145, ab)$ above is a lattice, which arises in formal concept analysis as follows.

Suppose given a set $\{1,2,3,4,5\}$ of objects and a set $\{a,b,c,d,e\}$ of attributes (properties) where objects $1,3,4,5$ (but not 2) have attribute a; $1,2,4,5$ have b; $2,3,4$ have c; $1,2,5$ have d and $1,3,5$ have e. A concept is defined to be a pair (A, B) where A is a subset of the set of objects, B is the set of all attributes shared by all objects of A, and A is the set of all objects having all the attributes in B. Concepts are partially ordered by $(A, B) \leq (A', B')$ if and only if $A \subseteq A'$ or equivalently, $B' \subseteq B$. In the above example, $(P, \leq)$ is the lattice of concepts. It is a basic fact of formal concept analysis, generalized in the theory of Galois connections, that any set of objects, each satisfying a subset of a specified set of attributes, gives rise to a complete lattice of concepts in this way.

As an example of an interesting and important result on fixed points which applies to the above lattice P, we mention that if $f\colon P \to P$ is an order preserving map (i.e. $p \leq q$ implies $f(p) \leq f(q)$) then the set $P^f = \{p \in P \mid f(p) = p\}$ of fixed points of f is itself a complete lattice when ordered by restriction of $\leq$, and in particular it is non-empty. The Knaster-Tarski theorem asserts that this holds for any complete lattice P

The course will use the text “Introduction to lattices and order” by Davey and Priestly. Formal mathematical prerequisites will be minimal, and the instructor will make every effort to ensure the course is accessible to all students taking it while still challenging for those with stronger mathematical backgrounds. For instance, though some examples and discussion in the text assume knowledge of elementary abstract algebra, special cases and alternative motivations will be discussed in class for students without this background. Students, especially non-math majors, who are potentially interested in the course but who lack its formal prerequisites are encouraged to contact the instructor to determine if taking it would be reasonable in their case.