Multiple choice.

(1) \( [x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \) where \( c_1 b_1 + c_2 b_2 = x \). So \( c_1 [2 1]^T + c_2 [-1 1]^T = [4 5]^T \) where \( T \) denotes transpose. That is, \( 2c_1 - c_2 = 4 \), \( c + c_2 = 5 \). Adding, \( 3c_1 = 9 \), \( c_1 = 3 \), so \( c_2 = 2 \) and \( [x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) (or solve by row-reducing the augmented matrix of the system).

(2) The product of the complex eigenvalues (with their multiplicities) is the determinant, which is \( 2 \cdot 2 - (-2) \cdot 2 = 8 \). Or: the characteristic polynomial is \( \lambda^2 - \text{tr}(A)\lambda + \det(A) \) where \( A \) is the matrix. This is \( \lambda^2 - 4\lambda + 8 \) with roots \( 2 \pm 2i \) (by quadratic formula) and product \( (2 + 2i)(2 - 2i) = 2^2 + 2^2 = 8 \).

(3) The orthogonal complement of a set \( A \) of vectors is the set of all vectors orthogonal to each vector in \( A \); it is always a subspace. Upper half plane is not closed under multiplication by negative scalars. Union of two (distinct) lines is not closed under addition. If a \( 2 \times 2 \) matrix has two distinct real eigenvalues, the set of all eigenvectors is a union of two (distinct) lines and so is not closed under addition. Obviously, only one of these is a subspace.

(4) Row-reduce:

\[
\begin{bmatrix}
-2 & 4 & 1 & 4 \\
-3 & 6 & -1 & 1 \\
1 & -2 & 2 & 3 \\
2 & -4 & 5 & 9
\end{bmatrix}
\]

The pivot rows of the last matrix (first, second, third) give the desired basis (note rows are written as column vectors in the answers).

(5) Let \( \mathcal{E} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \} \) be the standard basis. Then \( P = P_{\mathcal{E} \leftarrow B} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \). Note \( A \) is the \( \mathcal{E} \)-matrix of the linear transformation, so \( P^{-1}AP = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix} \). This simplifies to \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).
(6) Start row-reducing: 
\[
\begin{bmatrix}
1 & 3 & -2 & 5 & 3 \\
0 & 1 & -1 & 2 & 1 \\
2 & 1 & 1 & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & -2 & 5 & 3 \\
0 & 1 & -1 & 2 & 1 \\
0 & 5 & -5 & 8 & 3 \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & -2 & 5 & 3 \\
0 & 1 & -1 & 2 & 1 \\
0 & 0 & 0 & -2 & 2 \\
\end{bmatrix}
\]
This is in echelon form (though not row-reduced) and has 3 pivot positions; the row-reduced echelon form will have (the same) 3 pivot positions so the rank is 3.

(7) \( u \cdot u = \|u\|^2 \), and this is the only correct statement.

(8) Let \( u = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \), \( v = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \). The projection is \( x = \frac{u \cdot v}{v \cdot v} v \). Here, \( u \cdot v = 1 \cdot (-4) + 7 \cdot 2 = 10 \), \( v \cdot v = (-4)^2 + 2^2 = 20 \) and \( x = \frac{1}{2} v = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \).

(9) If \( \det(A) = 0 \), then \( A \) is not invertible, \( A \) has rank less than \( n \), \( \det(AB) = \det(A) \det(B) = 0 \), \( \lambda = 0 \) is a root of the characteristic polynomial \( \det(A - \lambda I) \) and hence is an eigenvalue of \( A \), and the columns of \( A \) don’t span \( \mathbb{R}^n \).

(10) \( P_{E\leftarrow B} = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} \) and \( P_{E\leftarrow C} = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} \). So \( P_{C\leftarrow B} = P_{C\leftarrow E} P_{E\leftarrow B} = \)
\[
(\begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix})^{-1} \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 24 & -8 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}.
\]

Partial credit.

(11)(a) \( A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \). The characteristic polynomial is \( \det(A - \lambda I) \) which is det
\[
\begin{bmatrix}
3 - \lambda & 0 & 0 \\
0 & 1 - \lambda & 2 \\
0 & 2 & 1 - \lambda \\
\end{bmatrix}
= (3 - \lambda) \det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix}.
\]
This equals \( (3 - \lambda)((1 - \lambda)^2 - 4) = -(\lambda - 3)^2(\lambda + 1) \). Its roots, the eigenvalues of \( A \), are \( \lambda = 3, 3, -1 \). Now \( A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \). A column vector \( [x, y, z]^T \) is in the nullspace of \( A - 3I \) if \( x \) is arbitrary and \( y = z \). Hence \([1, 0, 0]^T\) and \([0, 1, 1]^T\) forms a basis for the 3-eigenspace.

Similarly, \( A - (-1)I = A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \). A vector \( [x, y, z]^T \) is in the nullspace of \( A + I \) if \( x = 0 \) and \( y = -z \). Hence \([0, -1, 1]^T\) forms a basis for the -1-eigenspace.
(b) Let
\[
P = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{bmatrix}
\]
have the eigenvectors as columns and
\[
D = \begin{bmatrix}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]
have the corresponding eigenvalues as diagonal entries.

Then \(AP = PD\) and \(A = PDP^{-1}\).

(12)(a) Gram-Schmidt produces from a linearly independent set \(v_1, \ldots, v_n\) an orthogonal set \(w_1, \ldots, w_n\) where \(w_1 = v_1\) and
\[
w_i = v_i - \frac{v_i \cdot w_1}{w_1 \cdot w_1} w_1 - \ldots - \frac{v_i \cdot w_{i-1}}{w_{i-1} \cdot w_{i-1}} w_{i-1}
\]
for \(i > 1\). Here,
\[
w_1 = \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix},
\quad w_2 = \begin{bmatrix}
-1 \\
1 \\
-1
\end{bmatrix} - \begin{bmatrix}
4 \\
2 \\
0
\end{bmatrix} = \begin{bmatrix}
-1 \\
0 \\
-1
\end{bmatrix},
\]
\[
w_3 = \begin{bmatrix}
-2 \\
-2 \\
4
\end{bmatrix} - \begin{bmatrix}
0 \\
2 \\
2
\end{bmatrix} = \begin{bmatrix}
1 \\
-3 \\
-1
\end{bmatrix}.
\]

(13)(a) \(u_1 \cdot u_2 = 3 \cdot (-4) + 4 \cdot (-3) + 0 \cdot 0 = 0\)

(b)
\[
w = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{30}{25} u_1 + \frac{-15}{25} u_2 = \begin{bmatrix}
6 \\
3 \\
0
\end{bmatrix}
\]

(c) \(z = y - w = \begin{bmatrix}
0 \\
0 \\
-2
\end{bmatrix}\)

(d) \(\|y\|^2 = 6^2 + (-2)^2 = 49\), \(\|w\|^2 = 6^2 + 3^2 + 0^2 = 45\) and \(\|z\|^2 = 0^2 + 0^2 + (-2)^2 = 4\). One has \(\|y\|^2 = \|z\|^2 + \|w\|^2\) by Pythagoras’ theorem, since the difference \(z = y - w\) between \(y\) and its projection \(w\) on the span of \(u_1\) and \(u_2\) is orthogonal to each vector (such as \(w\)) in that span.