1. In $\mathbb{R}^4$, find the distance of the vector $\mathbf{y}$ to the subspace W spanned by the **orthogonal** vectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 4 \end{bmatrix}.$$

(a) $\sqrt{15}$ (b) $\sqrt{12}$ (c) $\sqrt{14}$ (d) $\sqrt{11}$ (e) $\sqrt{17}$

Solution. Observe that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal basis for W. Therefore, the orthogonal projection of the vector $\mathbf{y}$ on to W is

$$\widehat{\mathbf{y}} = \text{proj}_{\mathbf{W}}\mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{x}_1}{\mathbf{x}_1 \bullet \mathbf{x}_1}\mathbf{x}_1 + \frac{\mathbf{y} \bullet \mathbf{x}_2}{\mathbf{x}_2 \bullet \mathbf{x}_2}\mathbf{x}_2 + \frac{\mathbf{y} \bullet \mathbf{x}_3}{\mathbf{x}_3 \bullet \mathbf{x}_3}\mathbf{x}_3 = \frac{6}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{10}{15}\begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix} + \frac{-2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}.$$

Since,

$$\mathbf{y} - \widehat{\mathbf{y}} = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 2 \end{bmatrix}$$

we have that the distance of the vector $\mathbf{y}$ to the subspace W is equal to

$$\|\mathbf{y} - \widehat{\mathbf{y}}\| = \sqrt{(-1)^2 + 3^2 + (-1)^2 + 2^2} = \sqrt{15}.$$

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2. Find a **least squares solution** to the system

$$\begin{bmatrix} 1 & -1 & -5 \\ 6 & 1 & 0 \\ 1 & -5 & 1 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9 \end{bmatrix}.$$

Note that the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of the coefficient matrix A form an **orthogonal** basis for $\text{Col}\, A$.

(a) $\begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$ (b) $\begin{bmatrix} -2/3 \\ 0 \\ 1/3 \end{bmatrix}$ (c) $\begin{bmatrix} 2/3 \\ 5/3 \\ 1/3 \end{bmatrix}$ (d) $\begin{bmatrix} 2/3 \\ -5/3 \\ -1/3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution. Since the columns $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of the coefficient matrix A form an orthogonal basis for $W = \text{Col}\, A$, we have that the orthogonal projection of the vector $\mathbf{b}$ (the right-hand side of the system) on to W is

$$\widehat{\mathbf{b}} = \text{proj}_{\mathbf{W}}\mathbf{b} = \frac{\mathbf{b} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \frac{\mathbf{b} \bullet \mathbf{a}_3}{\mathbf{a}_3 \bullet \mathbf{a}_3}\mathbf{a}_3 = \frac{36}{54}\mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \frac{9}{27}\mathbf{a}_3 = A\begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}.$$

Therefore $\widehat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$ is a least squares solution to the given system.

3. Figure 1 shows the direction field for the differential equation $\frac{dy}{dt} = f(y)$, where $f(y)$ is a polynomial of third degree.

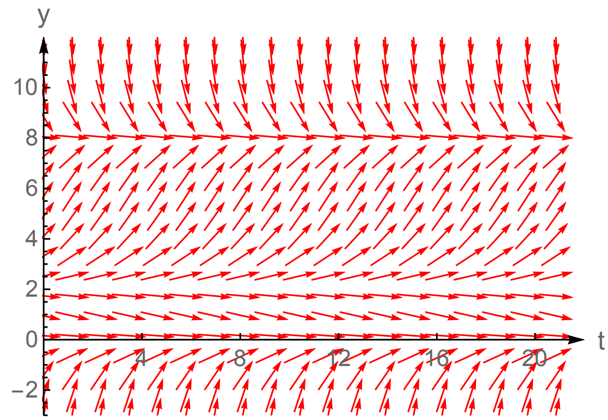

Figure 1

Which of the following statements is **false**?

(a) The solution with initial value $y(0) = 7.9$ is increasing and becomes equal to 8 in **finite** time.
(b) The **equilibrium** solutions of this differential equations are $y = 8$, $y = 2$ and $y = 0$.
(c) $y = 8$ and $y = 0$ are asymtotically **stable** solutions.
(d) $y = 2$ is an **unstable** solution.
(e) The solution with initial value $y(0) = 1.9$ is decreasing and going to zero as t goes to **infinity**.

Solution. Looking at this direction field we see that all statements are true, except the statement: "The solution with initial value $y(0) = 7.9$ is increasing and becomes equal to 8 in finite time." In fact, the solution with initial value $y(0) = 7.9$ can not meet the equilibrium solution $y(t) = 8$ at some time $t_0 > 0$ because then we would have two solutions to the equation $\frac{dy}{dt} = f(y)$ with initial data $y(t_0) = 8$, contradicting the basic theorem about existence and uniqueness of solution when $f(y)$ is continuously differentiable (see textbok).

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4. Which of the follow differential equations has the direction field shown in Figure 1 above?

(a) $\frac{dy}{dt} = -y(1 - \frac{y}{2})(1 - \frac{y}{8})$ (b) $\frac{dy}{dt} = (2 - y)(y - 8)(y - 4)$ (c) $\frac{dy}{dt} = -(2 - y)^2(8 - y)$
(d) $\frac{dy}{dt} = y(1 - \frac{y}{2})(1 - \frac{y}{8})$ (e) $\frac{dy}{dt} = (2 - y)^2(8 - y)$

Solution. Looking at the direction field in Figure 1, we see that only the differential equation $\frac{dy}{dt} = -y(1 - 0.5y)(1 - 0.125y)$ has the equilibrium solutions $y = 0, 2, 8$ and the slopes of the arrows are consistent with the sign of the function $f(y) = -y(1 - 0.5y)(1 - 0.125y)$

5. A tank, with a capacity of 1000 gallons, initially contains 100 gallons of brine with a concentration of 0.2 lb salt per gallon. A solution containing 0.4 lb of salt per gallon is pumped into the tank at the rate of 5 gal/min, and the well-stirred mixture flows out of the tank at the rate of 2 gal/min. Write the initial value problem (i.e. a differential equation and an initial condition) needed to find the amount $S(t)$ of salt in the tank at any time t (before the tank is full). **DO NOT SOLVE.**

(a) $\frac{dS}{dt} + \frac{2}{100+3t}S = 2, \;\; S(0) = 20.$ (b) $\frac{dS}{dt} + \frac{1}{100+2t}S = 2, \;\; S(0) = 20.$

(c) $\frac{dS}{dt} + \frac{2}{100+5t}S = 2, \;\; S(0) = 20.$ (d) $\frac{dS}{dt} + \frac{2}{100}S = 2, \;\; S(0) = 20.$

(e) $\frac{dS}{dt} + \frac{2}{100-3t}S = 2, \;\; S(0) = 100.$

Solution. The quantity $S(t)$ satisfies the differential equation

$$\frac{dS}{dt} = 5 \cdot (0.4) - 2 \cdot \frac{S(t)}{100 + (5-2)t} \quad \text{or} \quad \boxed{\frac{dS}{dt} + \frac{2}{100+3t}S = 2}.$$

We also have $\boxed{S(0) = 20}$.

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6. If $y(x)$ is the solution to the differential equation $xy' + 5y = x^2$, $x > 0$, such that $\lim_{x\to 0} y(x) = 0$, then compute $y(14)$.

(a) 28 (b) 17 (c) 5 (d) 14 (e) $\frac{1}{14}$

Solution. First, we write the DE in the form: $y' + \frac{5}{x}y = x$. Then, we compute the integrating factor $\mu(x) = e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$. Now, multiplying the last DE by x^5 we obtain $(x^5y)' = x^6$. Furthermore, integrating it we obtain $x^5y = \frac{1}{7}x^7 + c$ or $y = \frac{1}{7}x^2 + \frac{c}{x^5}$. Finally, using the condition $\lim_{x\to 0} y(x) = 0$ we get $c = 0$ and therefore $\boxed{y = \frac{1}{7}x^2}$. This gives $y(14) = 28$.

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7. For what values of r is $y(t) = e^{rt}$ a solution to $y'' - 3y' - 10y = 0$?

(a) $r = 5$ or $r = -2$ (b) $r = 5$ or $r = 2$ (c) $r = 2$ or $r = -3$ (d) $r = -3$ only

(e) $r = -3$ or $r = 2$

Solution. If $y = e^{rt}$, then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting into the ODE $y'' - 3y' - 10y = 0$ gives $r^2e^{rt} - 3re^{rt} - 10e^{rt} = 0$ i.e. $(r^2 - 3r - 10)e^{rt} = 0$. So we get a solution e^{rt} when $r^2 - 3r - 10 = 0$ i.e. $(r-5)(r+2) = 0$ i.e. $r = 5$ or $r = -2$.

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8. A rich donor gifts an amount A_0 to ND for student scholarships. The ND investment office invests this amount in an account earning annual interest of 5% compounded continuously and makes withdrawals continuously at the rate of 10 million dollars per year, without the amount in the account changing (i.e. it is always equal to A_0). Find the initial amount A_0.

(a) 200 million dollars (b) This is impossible. (c) 100 million dollars

(d) 50 million dollars (e) 500 million dollars

Solution. If we denote by $A(t)$ the amount (in millions of dollars) in the account at any time t, then it satisfies the differential equation

$$\frac{dA}{dt} = 0.05A - 10.$$

Since $\frac{dA}{dt} = 0$ we must have $0.05A - 10 = 0$ or $A = \frac{10}{0.05} = \boxed{200}$ millon dolars.

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9. Find the interval of existence of the solution to the following initial value problem:

$$x\frac{dy}{dx} + (y+1)^2 = 0,\ x > 0, \quad y(1) = 4.$$

(a) $(e^{-0.2}, \infty)$ (b) $(0, \infty)$ (c) $(1, \infty)$ (d) $(\ln 4, \infty)$ (e) $(0.2, \infty)$

Solution. Separating variables first and then integrating gives

$$\frac{dy}{(y+1)^2} = -\frac{dx}{x} \quad \text{or} \quad \int (y+1)^{-2}dy = -\int \frac{dx}{x} \quad \text{or} \quad \frac{1}{y+1} = \ln x + c.$$

Letting $x = 1$ and $y = 4$ we get $0.2 = c$, which gives the solution

$$\boxed{y(x) = \frac{1}{\ln x + 0.2} - 1.}$$

Now, we observe that the denominator in the solution formula becomes zero when $\ln x = -0.2$ or $\boxed{x = e^{-0.2}}$. Then, the solution becomes infinity (blows up). Thus, the existence interval of the solution to the given initial value problem is $(e^{-0.2}, \infty)$.

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10. (11 pts) Apply the Gram-Schmidt process to find an orthogonal basis for the column space of the following matrix

$$\begin{bmatrix} 3 & -7 & 8 \\ -1 & 5 & -2 \\ 1 & 1 & 1 \\ 3 & -5 & 1 \end{bmatrix}$$

You need not normalize your basis.

Solution. Denoting by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ the three column vectors of the matrix and applying the Gram-Schmidt process we obtain the following orthogonal vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$:

$$\mathbf{u}_1 = \mathbf{x}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} -7 \\ 5 \\ 1 \\ -5 \end{bmatrix} - \frac{-40}{20}\begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix},$$

and

$$\mathbf{u}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{x}_3 \bullet \mathbf{u}_2}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 8 \\ -2 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{20}\begin{bmatrix} 3 \\ -1 \\ 1 \\ 3 \end{bmatrix} - \frac{-10}{20}\begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

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11. (12 pts)

(1) Applying the fundamental ODE theorem for linear first order differential equations, one concludes that the following initial value problem

$$(1+t^4)\frac{dy}{dt} + 4t^3 y = 6t - 2te^{-t^2},\ y(0) = 9,$$

has a unique solution for $-\infty < t < \infty$. Find this solution.

(2) While the initial value problem (1) has a unique solution, show that the following initial value problem

$$\frac{dy}{dt} = y^{1/3},\ y(0) = 0,$$

has more than one solution by finding at least two of them.

Solution.

(1) The differential equation can be written as $\frac{d}{dt}\left[(1+t^4)y\right] = 6t - 2te^{-t^2}$. Therefore, $(1+t^4)y = 3t^2 + e^{-t^2} + c$ or $y(t) = \dfrac{3t^2 + e^{-t^2} + c}{1+t^4}$. Since $9 = y(0) = 1 + c$ we obtain that $\boxed{y(t) = \dfrac{3t^2 + e^{-t^2} + 8}{1+t^4}}$.

(2) First, we observe that $y_1 = 0$ is a solution. Then, separating variables gives $y^{-1/3}dy = dt$. Furthermore, integrating we get $\frac{3}{2}y^{2/3}dy = t + c$. Letting $t = 0$ and $y = 0$ to this formula gives $c = 0$. Finally, solving for y we obtain $y = \pm\left(\frac{2}{3}t\right)^{3/2}$, which gives the following two solutions

$$y_2(t) = \begin{cases} \left(\frac{2}{3}t\right)^{3/2}, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad \text{and} \quad y_3(t) = \begin{cases} -\left(\frac{2}{3}t\right)^{3/2}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Thus, we found three! different solutions to the the initial value problem $\frac{dy}{dt} = y^{1/3},\ y(0) = 0$. In fact, it has infinitely many solutions (see textbook).

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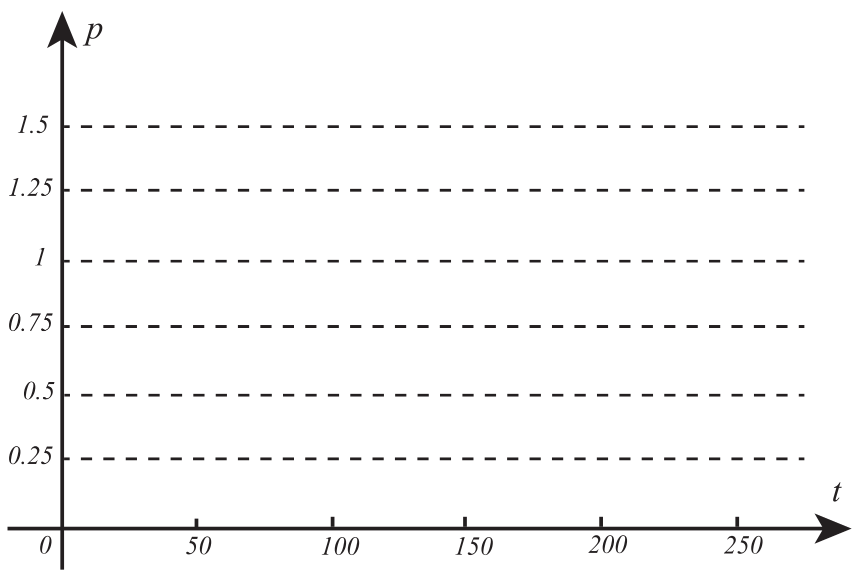
12. (14 pts) Assume that a region's population $p(t)$ (in billions) at any time t (in years) is modeled by the differential equation

$$\frac{dp}{dt} = 0.02p(1-p).$$

(1) Solve this differential equation with initial value $p(0) = p_0 > 0$ to find an explicit formula for $p(t)$.

(2) Use the formula in (1) to compute $\lim_{t\to\infty} p(t)$.

(3) Sketch the solution curves that correspond to the initial values $p(0) = 0.25$, $p(0) = 1$, and $p(0) = 1.5$, clearly showing where they are increasing/decreasing.



Solution. (1) Separating variables we have $\dfrac{dp}{p(1-p)} = 0.02dt$. Then, writing $\dfrac{1}{p(1-p)} = \dfrac{1}{p} + \dfrac{1}{1-p}$, and integrating gives $\ln|p| - \ln|1-p| = 0.02t + c$ or $\ln\dfrac{|p|}{|1-p|} = 0.02t + c$ or $\dfrac{p}{1-p} = \pm e^c e^{0.02t}$. Finally, letting $C = \pm e^c$ we get the following general solution in implicit form $\dfrac{p}{1-p} = Ce^{0.02t}$. Letting $p = p_0$ and $t = 0$, this formula gives $\dfrac{p_0}{1-p_0} = C$. Substituting this C into the general solution we get the formula $\dfrac{p}{1-p} = \dfrac{p_0}{1-p_0}e^{0.02t}$, which solved for p gives the desired solution to the given initial value problem

$$\boxed{p(t) = \frac{p_0}{p_0 + (1-p_0)e^{-0.02t}}.}$$

(2) From this formula in (1) we see that $\boxed{\lim_{t\to\infty} p(t) = 1.}$

(3) The solution curve with $p(0) = 1$ is the equilibrium solution $p = 1$. Using the direction fields we see that the solution with $p(0) = 1.5$ is decreasing towards the equilibrium solution $p = 1$, while the solution with $p(0) = 0.25$ is increasing towards the equilibrium solution $p = 1$. Below, are the graphs of these solution curves and the direction field of our ODE.