STRATIFIED EXACT CATEGORIES AND HIGHEST WEIGHT REPRESENTATIONS

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ABSTRACT. We define stratified exact categories, which are a class of exact categories abstracting very general features of the category of modules with a Verma flag in a highest weight category. For any small stratified exact category, the associated abelian category of left-exact contravariant functors to abelian groups has highest weight structure in some weak general sense. Much of the paper is devoted to giving constructions of stratified exact categories and functors between them, and describing conditions which give rise to stronger properties of the associated categories and functors. We discuss only the most general properties of standard objects (projectives, injectives, highest weight objects, tilting modules etc), and various natural functors (base change, ungrading functors, translation and projective functors etc) in this setting. The formal results here are a part of a project to construct and study certain representation theories associated to Coxeter groups (conjecturally related to Kazdhan-Lusztig polynomials) and to fans of polyhedral cones (conjecturally related to h-vectors of polytopal complexes), and to clarify their analogies to and relationships with highest weight representation categories arising in Lie theory.

The Kazdhan-Lusztig polynomials $P_{x,y}$ defined in [35] (and their variants) have, for crystallographic Coxeter groups, important applications in many areas of Lie theory. It has been conjectured in loc cit that these polynomials $P_{x,y}$ have non-negative coefficients for an arbitrary Coxeter group. The conjecture is known to be true for crystallographic Coxeter groups, using an interpretation of Kazdhan-Lusztig polynomials as Poincaré series of local intersection cohomology of Schubert varieties (see e.g. [36] and [29]) or closely related interpretations in representation theory.

To study the conjecture in general, we have initiated (in [18], [22], [21], [23]) the study of certain highest weight representation categories naturally associated to data consisting (essentially) of a reflection representation of a Coxeter system (W, S) on a space V and a possibly non-standard system of positive roots of W on V. In graded characteristic zero versions of these representation categories, the multiplicities of graded simple modules as composition factors of Verma modules are conjecturally given by coefficients of Kazdhan-Lusztig polynomials (we call this the Kazdhan-Lusztig conjecture, in view of its close relationship to the proven Kazdhan-Lusztig conjecture on Verma module multiplicities in semisimple complex Lie algebras ([35], [7], [4], [3])). We have also begun the study of closely

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analogous representation theories associated to (possibly non-rational) fans of polyhedral cones in a Euclidean space, with similar conjectural applications to the study of g-vectors of convex polytopes and h-vectors of polytopal complexes [51]. These representation categories have an interesting and combinatorially rich theory, and several natural questions and conjectures about Kac-Moody Lie algebras, quantum groups etc and associated geometry are suggested by analogy with results and conjectures about these representation categories.

A definition and some of the most basic properties of some of these representation categories was surveyed with incomplete proofs in [23]. In this paper we provide, with proofs, some extensions of the formal part of the results there which are important for the examples; notably, we allow infinite weight posets to be able to obtain more "global" results on structure constants of Iwahori-Hecke algebras (cf. [23, 5.6]), we give the results more generally for application to the integral versions of representation theories associated to crystallographic Coxeter groups and to the study of polytopal complexes, and we make the role of exact categories more explicit in order to improve (at least slightly) the functoriality of the constructions.

The representation categories of interest are constructed in roughly the following way. Call an exact category with a family of "standard" objects satisfying the usual Ext-vanishing conditions of Verma modules in highest weight categories, and in which every object has a "good filtration" by these standard objects, a stratified exact category; thus, such categories are a general analogue of the category of modules with a "Verma flag" in a highest weight category. Now associated to a stratified exact category, one has an associated abelian category of left exact contravariant functors from it to abelian groups; this has highest weight structure in some very weak general sense. Such representation categories can typically be realized as a category of "diagonalizable modules" over a "diagonalizable ring," the description being formally similar to the definition of category $\mathcal{O}$ of modules for a Kac-Moody Lie algebra. We are interested in these categories over commutative rings and in closely related highest weight categories over fields obtained by base change from such module categories.

Accordingly, much of the paper is concerned with giving a number of constructions of stratified exact categories and functors between the corresponding abelian categories, and with describing simple conditions on a stratified exact category which give rise to desirable features of the representation categories and functors which may be constructed from them. We have collected general facts which are useful for the study of the motivating examples from Coxeter groups, polyhedral cones and Lie theory but which are independent of the context. Most of the results and methods are variants, generalizations or special cases (in some cases, all three) of well known ones from Lie theory or general algebra, but we have recorded them here since the existing references seem inadequate for our intended applications. The constructions and basic results we give are extremely general and thus very widely applicable. However, though this greatly facilitates the construction and study of the motivating examples, none of the deeper phenomena and conjectures which make those examples particularly interesting hold in the generality of this paper; moreover, we have no particular reason to expect that the deeper properties of the motivating examples can be proved within the rather restrictive (not very functorial) context of this paper.

The first section of the paper provides an overview. It lists the results which are most useful for the study of the intended examples and informally discusses these results in relation to some of those examples. At the end, it mentions a few of the deeper properties which are conjectured or known to hold in some of those examples; the detailed discussion of the examples is deferred to future papers. Sections 2–16 contain the proofs of the general facts mentioned in the first section and of additional related results. There are three appendices. Appendix A explains poset terminology used throughout this paper. Appendix B describes some facts and terminology concerning exact categories. Appendix C collects some terminology and general facts we shall use about categories with automorphisms, and their relationship with "diagonalizable" graded rings and modules; a much briefer discussion which should be adequate for many readers is given in 1.10.

1. Main results, examples, and conjectures

In this section, we describe the results in this paper. General background information and terminology concerning posets and exact categories can be found in Appendix A and Appendix B respectively. Terminology is mostly standard; one exception is that we shall find it very convenient to call a functor $F\colon C \to D$ between exact categories perfectly exact if it is full, faithful, exact, reflects exactness and has extension closed strict image, and to say that C is a perfectly exact subcategory of D if it is a subcategory of D and the inclusion functor is perfectly exact.

1.1. Stratified exact categories.

Let $\mathcal{C}$ be an exact category and let $\{\mathcal{C}_x\}_{x\in\Omega}$ be a family of strict full additive subcategories $\mathcal{C}_x$ of $\mathcal{C}$ indexed by an interval finite poset Ω. For any subset Γ of Ω, let $\mathcal{C}_\Gamma$ denote the smallest extension closed (additive) subcategory of $\mathcal{C}$ containing $\mathcal{C}_x$ for $x \in \Gamma$, regarded as a perfectly exact subcategory of $\mathcal{C}$.

Definition. We say that $(\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})$ is a stratified exact category if the following conditions all hold:

(i) $\operatorname{Hom}(M, N) = 0$ if M is in $\mathcal{C}_x$, N is in $\mathcal{C}_y$ and $x \not\leq y$

(ii) $\mathcal{C}_\Omega = \mathcal{C}$

(iii) for $x \in \Omega$, any object M of $\mathcal{C}_x$ is projective in $\mathcal{C}_{\not> x}$

When the conditions hold, we shall abuse terminology and call $\mathcal{C}$ itself a stratified exact category (with strata $\mathcal{C}_x$ and weight poset Ω).

Remarks. Condition (iii) could be replaced by the requirement that if L is in in $\mathcal{C}_x$, N is in $\mathcal{C}_y$ and $y \not< x$, then $\operatorname{Ext}^1_{\mathcal{C}}(N, L) = 0$ i.e. for such L, N, any short exact sequence

$$0 \to L \to M \to N \to 0 \tag{1.1.1}$$

in $\mathcal{C}$ is split exact. It follows that if $\mathcal{C}$ is stratified, then the opposite category $\mathcal{C}^{\mathrm{op}}$ is a stratified exact category with strata $\mathcal{C}_x^{\mathrm{op}}$ and weight poset Ω^{op} (we identify the opposite poset Ω^{op} with Ω as a set), and $\mathcal{C}_\Gamma$ is stratified with weight poset Γ for any $\Gamma \subseteq \Omega$. Moreover, $\mathcal{C}_x = \mathcal{C}_{\{x\}}$ is a split exact category for each $x \in \Omega$ (i.e. its exact sequences are all split exact sequences).

1.2. Let $\mathcal{B}$ be an exact category and $\mathcal{C}_x$, for $x \in \Omega$, be perfectly exact, strict subcategories of $\mathcal{B}$. We assume that $\operatorname{Hom}(M, N) = 0$ if M is in $\mathcal{C}_x$, N is in $\mathcal{C}_y$ and $x \not\leq y$. Suppose also that for each $x \in \Omega$, there exists a Serre subcategory $\mathcal{B}_x$ of $\mathcal{B}$, containing $\mathcal{C}_y$ for all $y \not> x$, such that every object of $\mathcal{C}_x$ is projective in $\mathcal{B}_x$. Define $\mathcal{C}$ to be the smallest extension-closed full additive category of $\mathcal{B}$ containing the objects of $\mathcal{C}_y$ for all $y \in \Omega$, regarded as a perfectly exact subcategory of $\mathcal{B}$.

The following result is proved in 3.8 (note part (a) is immediate form the definitions).

Lemma. (a) *The family $(\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})$ is a stratified exact category.*

(b) *Assume idempotents split in $\mathcal{C}_x$ for all $x \in \Omega$. Then any short exact sequence $0 \to M' \to M \to M'' \to 0$ in B with M, M'' objects of $\mathcal{C}$, is an exact sequence in $\mathcal{C}$ (in particular, M' is in $\mathcal{C}$). Thus, $\mathcal{C}$ is a perfectly exact subcategory of $\mathcal{B}$.*

Remarks. Of course any stratified exact category $\mathcal{C}$ arises from this construction, in a trivial way with $\mathcal{B} = \mathcal{C}$. If $\mathcal{C}$ is small, it also arises from this construction taking for $\mathcal{B}$ the abelian category $\mathcal{C}^*$ of left exact contravariant functors from $\mathcal{C}$ to abelian groups, as we see subsequently.

1.3. Suppose above that $\mathcal{B}$ is a highest weight (abelian) category with weight poset Ω over a field, such as category $\mathcal{O}$ for a semisimple complex Lie algebra or Kac-Moody Lie algebra, or the category of finite dimensional rational G-modules for a semisimple algebraic group over an algebraically closed field of prime characteristic; see e.g. [10] or [32] for general frameworks. One may take $\mathcal{C}_x$ to be the category of objects which are direct sums of universal highest weight modules with highest weight x (also known as Verma or Weyl modules) and $\mathcal{B}_x$ to be the full subcategory of objects all of whose irreducible subquotients are indexed by highest weights which are not greater than x. Then the category $\mathcal{C}$ above becomes the category of objects with a finite "good filtration" (by Verma modules). In such situations, one can often reconstruct $\mathcal{B}$ or a closely related abelian category as a natural subcategory of $\mathcal{C}^*$. For example, if Ω is finite and $\mathcal{B}$ has a projective generator P in $\mathcal{C}$, one typically has $\mathcal{B} \cong \mathcal{A}$-Modfg and $\mathcal{C}^* \cong \mathcal{A}$-Mod where $\mathcal{A} = \operatorname{End}(P)^{\mathrm{op}}$; this idea goes back at least to Bernstein-Gelfand-Gelfand, who applied it to blocks of $\mathcal{O}$ for a semisimple complex Lie algebra. In more general situations, when one works with "thickened" versions of highest weight categories over general commutative rings instead of fields and allows infinite weight posets, there is a large abelian subcategory $\mathcal{C}^\dagger$ of $\mathcal{C}^*$ which typically is similarly related to $\mathcal{B}$; $\mathcal{C}^\dagger$ usually has a natural realization as a subcategory $\mathcal{E}$ of the category $\mathcal{A}$-Mod of "diagonalizable" modules for a "diagonalizable" ring $\mathcal{A}$ subject to additional conditions formally resembling those in the definition of category $\mathcal{O}$ for a Kac-Moody Lie algebra. See 1.11 for an example and 1.20 for a general result in this vein.

In contrast to the usual situation in Lie theory, in most of our intended applications to Coxeter groups and polyhedral cones (see 1.12 and 1.13), one does not have an a priori description of the highest weight category $\mathcal{B}$ of interest. In these cases, we have to give an independent construction of $\mathcal{C}$ and define $\mathcal{B}$ as the abelian subcategory $\mathcal{B} := \mathcal{C}^\dagger$ of $\mathcal{C}^*$ (more precisely, we construct the thickened versions over commutative rings in this way, realize them as module categories and obtain the desired highest weight categories over fields by "base change"). The examples of stratified exact categories $\mathcal{C}$ of interest to us for the intended applications are generally constructed using a faithful exact functor $F\colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is typically a “familiar” abelian or exact category (such as the category graded modules over a a polynomial ring over a field, or filtered modules over the T-equivariant K-theory ring of a flag variety) but where F does not have the favorable properties of the inclusion $\mathcal{C} \to \mathcal{B}$ in 1.2. In fact, F typically does not reflect exactness or have extension closed strict image, and may not even be full. General methods of constructing exact category embeddings $F\colon \mathcal{C} \to \mathcal{D}$ of this type will be described below for a larger class of “weakly stratified exact categories” $\mathcal{C}$; if $\Omega = \{\bullet\}$, a stratified exact category is a split exact category, whereas a weakly stratified exact category is an exact category.

1.4. **Weakly stratified exact categories.** We begin again with an exact category $\mathcal{D}$ and a family $\{\mathcal{D}_x\}_{x\in\Omega}$ of strict, full additive subcategories $\mathcal{D}_x$ indexed by an interval finite poset Ω.

We consider the following two conditions on this data:

(i) $\mathrm{Hom}(M, N) = 0$ if M is in $\mathcal{D}_x$, N is in $\mathcal{D}_y$ and $x \not\leq y$

(ii) if L is in in $\mathcal{D}_x$, N is in $\mathcal{D}_y$, $x \not\leq y$ and $y \not\leq x$, then $\mathrm{Ext}^1_{\mathcal{D}}(N, L) = 0$.

Lemma. *Assume that* 1.4(i) *and* 1.4(ii) *hold.*

(a) *There is a full additive subcategory $\mathcal{C}$ of $\mathcal{D}$ consisting of the objects M in $\mathcal{D}$ which possess a filtration $M = M^0 \supseteq M^1 \supseteq \ldots \supseteq M^n = 0$ with M^{i-1}/M^i in $\mathcal{D}_{x_i}$ for some $x_1, \ldots, x_n$ in Ω (depending on M) such that $x_i \leq x_j$ implies $i \leq j$.*

(b) *Fix such a filtration for each M in $\mathcal{C}$, and set $M(x_i) := M^{i-1}/M^i$ and $M(x) := 0$ if $x \in \Omega$ with $x \neq x_i$ for any i. Then up to isomorphism, $M(x)$ is independent of the choice of the filtration, and there is a natural additive functor $\tau_x\colon \mathcal{C} \to \mathcal{D}_x$ with $\tau_x(M) = M(x)$ for M in $\mathcal{C}$.*

(c) *More generally, let Γ be a locally closed subset of Ω (i.e. $x, y \in \Gamma$ implies $[x, y] \subseteq \Gamma$). Then there is a natural additive functor $\sigma_\Gamma\colon \mathcal{C} \to \mathcal{C}$, denoted $M \mapsto M(\Gamma)$, such that $M(\Gamma)$ is an admissible subquotient object of M and $\tau_x(M) \cong \tau_x\bigl(M(\Gamma)\bigr)$ if $x \in \Gamma$, and $\tau_x\bigl(M(\Gamma)\bigr) = 0$ if $x \notin \Gamma$.*

(d) *If Γ and Λ are locally closed subsets of Ω such that Γ is a coideal of Λ, there is a natural short exact sequence $0 \to M(\Gamma) \to M(\Lambda) \to M(\Lambda \setminus \Gamma) \to 0$ in $\mathcal{D}$ for each object M of $\mathcal{C}$.*

This is proved in Section 2, where we also provide more canonical descriptions of $\mathcal{C}$ and $M(\Gamma)$ and establish additional facts on filtrations as in (a) (which are just analogues, appropriate to the weaker conditions assumed here, of Verma flags in highest weight categories).

For any locally closed subset Γ of Ω, let $\mathcal{C}_\Gamma$ be the full additive subcategory of objects M of $\mathcal{C}$ with $M(x) = 0$ for $x \notin \Gamma$. Let $\iota_\Gamma\colon \mathcal{C}_\Gamma \to \mathcal{C}$ be the inclusion, and $\tau_\Gamma\colon \mathcal{C} \to \mathcal{C}_\Gamma$ denote the restriction of σ_Γ.

1.5. Now we can define weakly stratified exact categories.

Definition. We shall say that $\mathcal{D}$ is a weakly stratified exact category (with strata $\{\mathcal{D}_x\}_{x\in\Omega}$ and (interval finite) weight poset Ω) if the following conditions hold:

(i) $\mathcal{D}$ is an exact category,

(ii) the $\mathcal{D}_x$ for $x \in \Omega$ are perfectly exact, strict subcategories of $\mathcal{D}$ satisfying 1.4(i) and 1.4(ii) above

(iii) every object of $\mathcal{D}$ is in the additive category $\mathcal{C}$ as defined above i.e. $\mathcal{C} = \mathcal{D}$ as additive categories.

(iv) for any locally closed subset Γ of Ω, the truncation functor σ_Γ is exact.

Remarks. Condition (iv) could be replaced by the requirement that $\tau_x \colon \mathcal{C} \to \mathcal{D}_x$ is exact for all x, or replaced by the requirement that a sequence (1.1.1) in $\mathcal{D}$ is exact iff $gf = 0$ and each sequence

$$0 \to L(x) \xrightarrow{\tau_x(f)} M(x) \xrightarrow{\tau_x(g)} N(x) \to 0 \tag{1.5.1}$$

with $x \in \Omega$ is exact in $\mathcal{D}_x$ (this follows using the 9-lemma and Lemma 1.4).

If (i)–(iv) hold, then for any locally closed $\Gamma \subseteq \Omega$, $\mathcal{D}_\Gamma := \mathcal{C}_\Gamma$ is a perfectly exact subcategory of $\mathcal{D}$ and is itself naturally a weakly stratified exact category with weight poset Γ and strata $\mathcal{D}_x$ for $x \in \Gamma$.

1.6. The following lemma is proved in 3.9.

Proposition. *Let $\mathcal{D}$ be an exact category with a family $\{\mathcal{D}_x\}_{x\in\Omega}$ of perfectly exact strict subcategories indexed by a locally finite poset Ω.*

(a) *If $(\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Omega})$ is a weakly stratified exact category, then* $\mathrm{Ext}^1_{\mathcal{D}}(L, N) = 0$ *if L is in $\mathcal{D}_x$, N in $\mathcal{D}_y$ and $x \not\leq y$.*

(b) *If $(\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Omega})$ is a weakly stratified exact category, then for any locally closed subset Γ of Ω and any objects M, N of $\mathcal{D}_\Gamma$, the natural map*

$$\mathrm{Ext}^i_{\mathcal{D}_\Gamma}(M, N) \to \mathrm{Ext}^i_{\mathcal{D}}(M, N)$$

is an isomorphism.

(c) *If $(\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Omega})$ is a stratified exact category, then* $\mathrm{Ext}^i_{\mathcal{D}}(N, L) = 0$ *for all $i \geq 1$ if N is in $\mathcal{D}_x$ and L is in $\mathcal{D}_y$ with $x \not< y$.*

(d) *$(\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Omega})$ is a stratified exact category iff it is a weakly stratified exact category and each stratum $\mathcal{D}_x$ is a split exact category.*

1.7. **Construction of weakly stratified exact categories.** The following construction of weakly stratified exact categories is a more general version of the construction sketched in [23, 1.1.–1.5].

Proposition. *Let $\mathcal{D}$ be an exact category and $\mathcal{D}_x$ for $x \in \Omega$ be strict, full additive subcategories of $\mathcal{D}$ which are each endowed with a structure of exact category so the inclusion $\mathcal{D}_x \to \mathcal{D}$ is exact. Assume* 1.4(i) *and* 1.4(ii) *hold, so one may define the category $\mathcal{C}$ and functors $\tau_x \colon \mathcal{C} \to \mathcal{D}_x$ as in* 1.4. *Define a sequence* (1.1.1) *in $\mathcal{C}$ to be exact if $gf = 0$ and for each x,* (1.5.1) *is a short exact sequence in $\mathcal{D}_x$.*

Then $(\mathcal{C}, \{\mathcal{D}_x\}_{x\in\Omega})$ is a weakly stratified exact category.

More generally still, in Sections 3 and 5, we shall give a construction which, to exact categories $\mathcal{D}$, $\mathcal{D}_x$ for $x \in \Omega$ and a family $\{F_x \colon \mathcal{D}_x \to \mathcal{D}\}_{x\in\Omega}$ of exact functors, associates a weakly statified exact category $\mathcal{C}^0$ with strata $\mathcal{C}^0_x \cong \mathcal{D}_x$ (as exact categories) and an exact functor $F \colon \mathcal{C}^0 \to \mathcal{D}$. One may view $\mathcal{C}^0$ as having been obtained by gluing together the strata $\mathcal{D}_x$ allowing extensions from their strict images $F_x(\mathcal{D}_x)$ in $\mathcal{D}$ which are "compatible" with the poset Ω. In the case where the strict images of the F_x satisfy 1.4(i) and 1.4(ii), we denote the weakly stratified exact category we construct as $\mathcal{C}^0 = \mathcal{C}^0[\{F_x \colon \mathcal{D}_x \to \mathcal{D}\}_{x\in\Omega}]$ (the construction in this case is given in Section 3); if in addition all F_x are full, faithful, exact inclusion functors, then the functor F induces an equivalence of weakly stratified exact categories between $\mathcal{C}^0$ and $\mathcal{C}$ as defined in the above proposition. The weakly stratified exact category we associate to general functors $\{F_x\}_{x\in\Omega}$ is $\mathcal{C}^0[\{j_xF_x\colon \mathcal{D}_x \to \mathcal{D}^{I^{\mathrm{op}}}\}_{x\in\Omega}]$ where $\mathcal{D}^{I^{\mathrm{op}}}$ is an exact category of "sheaves" on Ω and $j_x\colon \mathcal{D} \to \mathcal{D}^{I^{\mathrm{op}}}$ sends an object M of $\mathcal{D}$ to the corresponding "skyscraper sheaf" with global sections M and support equal to the closure $\{\, y \in \Omega | y \geq x \,\}$ of x (see 5.9 for precise details).

1.8. **Construction of representation categories.** If C is a svelte exact category (i.e. skeletally small), we replace C by an equivalent small exact category C_0 and define C^* to be the abelian category of left exact contravariant functors from C to abelian groups (see Appendix B for more details here and below). There is a natural perfectly exact functor (Gabriel-Quillen embedding) $\phi_C\colon C \to C^*$ given by $\phi_C(M) = \mathrm{Hom}_C(?, M)$.

In general, if D is another such (small) exact category and $F\colon C \to D$ is a right exact functor, F induces a left exact functor $F^*\colon D^* \to C^*$. The functor F^* has a left adjoint which is a right exact functor $F_*\colon C^* \to D^*$ extending F in the sense that $F_*\phi_C \cong \phi_D F$. The correspondence $F \mapsto F_*$ preserves composites and adjoints.

Now let $\mathcal{C}$ be a svelte weakly stratified exact category. For any open subset (ideal) Γ of Ω, the functor $\tau_\Gamma^*\colon \mathcal{C}_\Gamma^* \to \mathcal{C}^*$ is perfectly exact. Define $\mathcal{C}^\dagger$ to be the full abelian subcategory of $\mathcal{C}^*$ consisting of objects which belong to the strict image of τ_Γ^* for some f.g. ideal Γ of Ω (see 4.1).

1.9. Some general properties of $\mathcal{C}^*$ and $\mathcal{C}^\dagger$ for a stratified exact category $\mathcal{C}$, are given in Section 4. Module-theoretic variants of many of these facts are described later in this section; these variants require slightly stronger assumptions on $\mathcal{C}$ which hold in most of the intended applications.

Before proceeding with the discussion of these results, we introduce the class of diagonalizable rings and their diagonalizable modules which are used in their module-theoretic formulation (see Appendix C) and mention three examples which have motivated the considerations of this paper; many of the subsequent results will be briefly discussed in relation to these examples.

1.10. **Diagonalizable rings.** By a G-graded J-diagonalizable ring A, we mean a (possibly non-unital) G-graded ring $A = \oplus_{g\in G} A_g$ (G a group) with a specified set $\{e_j\}_{j\in J}$ of orthogonal homogeneous idempotents such that $A = \oplus_{j,k\in J}\, e_j A e_k$. If $B = \oplus_{j,k\in J} f_j B f_k$ is another such ring, a ring homomorphism $A \to B$ is required to map $e_j \mapsto f_j$ for all j. We call a G-graded A-module $M = \oplus_{g\in G} M_g$ diagonalizable if $M = \oplus_{j\in J}\, e_j M$. Let A-mod denote the category of diagonalizable graded A-modules, with homogeneous A-module homomorphisms of degree 1_G as morphisms. We generally call A itself simply a G-graded ring, omitting explicit mention of the set J unless necessary to avoid confusion; if the grading G is fixed in the discussion, as it often is, we may even call A just a ring. Similarly, the objects of A-mod will hereafter be called graded A-modules or even just A-modules. All limits, colimits etc of G-graded J-diagonalizable rings (resp., modules) are taken in the category of G-graded J-diagonalizable rings (resp., modules). There is a natural action of G as a group of automorphisms of A-mod by grading shift. For a graded A-module M, rad A (resp., Rad A) denotes the graded (resp., ungraded) radical of M i.e. the intersection of the family of all graded (resp., ungraded) maximal submodules of M.

We call an additive, (resp., abelian or exact) category C with a given action of a group G by automorphisms as additive (resp., abelian or exact category) an additive (resp., abelian or exact) category over G. We call the automorphisms translations of C and generally denote them as $\{T_g\}_{g\in G}$; we also often write $T_gM = M\langle g^{-1}\rangle$. A functor $F\colon C \to D$ between additive categories over G is said to be a functor over G if it is compatible with translations, in the sense that $FT_g = T_gF$ for all F.

Let C be an additive category over G. For a family $\mathbf{M}$ of objects in C, $\operatorname{Add}\mathbf{M}$ (resp., $\operatorname{add}\mathbf{M}$) denotes the full additive subcategory of all direct summands of finite direct sums of copies of (resp., translates of) objects of $\mathbf{M}$. For M, N in C, we define the G-graded $\mathbb{Z}$-module $\hom_C(M,N)$ with $\hom_C(M,N)_g = \operatorname{Hom}_C(M, T_gN)$. Similarly, define $\operatorname{ext}^i(M,N)$ if C is an abelian or exact category and the T_g are exact functors. Now if $\mathbf{M} = \{M_j\}_{j\in J}$, then there is a naturally associated J-diagonalizable, G-graded ring $A = \operatorname{end}(\mathbf{M})^{\mathrm{op}}$ with $e_jAe_k = \hom(M_j, M_k)$ and multiplication by composition. Moreover, there is a natural functor $\hom(\mathbf{M}, ?)\colon C \to A\text{-mod}$ with $e_j\hom(\mathbf{M},N) = \hom(M_j,N)$ for N in C. Similarly, one defines $\hom(?,\mathbf{M})$, and defines $\operatorname{ext}^i(\mathbf{M},?)$, $\operatorname{ext}^i(?,\mathbf{M})$ if C is exact.

1.11. **Example 1: Categories from Kac-Moody Lie algebras.** Let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ be a symmetrizable Kac-Moody Lie algebra with enveloping algebra $U = U(\mathfrak{g})$ (unexplained notation is as in [34] unless otherwise indicated). Partially order $\mathfrak{h}^*$ by setting $\mu \leq \lambda$ if $\lambda - \mu \in Q^+ := \oplus_{\alpha\in\Pi}\mathbb{N}\alpha$. Let $S = U(\mathfrak{h})$ (a polynomial ring on a basis of $\mathfrak{h}$) and A be one of the following S-algebras: $A = S$, or A is the localization $S_{\mathfrak{m}_0}$ of S at the maximal ideal $\mathfrak{m}_0$ of polynomials in S with zero constant term, or $A = S/\mathfrak{m}_0 \cong \mathbb{C}$. Let $\mathfrak{m}$ denote the maximal ideal of A if $A = \mathbb{C}$ or $A = S_{\mathfrak{m}_0}$, and let $\mathfrak{m} = \mathfrak{m}_0$ if $A = S$. Let $\pi\colon S \to A$ be the structural morphism.

Define the Anderson-Jantzen-Soergel category (cf. [1]) $\mathcal{M}_A$ of (U,A)-bimodules M with an $\mathfrak{h}^*$-gradation $M = \oplus_{\lambda\in\mathfrak{h}^*}M_\lambda$ as right A-module such that $hm = m\pi(h + \langle\lambda,h\rangle)$ for all $m \in M_\lambda$, $h \in \mathfrak{h} \subseteq S \subseteq U$ and $U_\lambda M_\mu \subseteq M_{\lambda+\mu}$ where U_λ is the λ-weightspace of $U(\mathfrak{g})$ under the adjoint action of $\mathfrak{h}$; morphisms are (U,A)-bimodule homomorphisms respecting the grading. Then $\mathcal{M}_A$ is an abelian category; we consider also its full abelian subcategories $\mathcal{M}'_A \supseteq \mathcal{M}''_A$ with $\mathcal{M}'_A$ consisting of modules M such that there exist $\lambda_1,\ldots,\lambda_n \in \mathfrak{h}^*$ such that $M_\lambda \neq 0$ unless $\lambda \leq \lambda_i$ for some i and $\mathcal{M}''_A$ consisting of objects M of $\mathcal{M}'_A$ such that each M_λ is a f.g. A-module. Then $\mathcal{M}''_{\mathbb{C}}$ is the usual category $\mathcal{O}$ for $\mathfrak{g}$; we regard $\mathcal{M}'_A$ and $\mathcal{M}''_A$ as "thickened versions" of $\mathcal{O}$ over A (the former without finiteness conditions).

Let $\mathfrak{b}^+ := \mathfrak{n}^+ + \mathfrak{h}$. For any $\lambda \in \mathfrak{h}^*$, let A^λ be a $(U(\mathfrak{b}^+),A)$-bimodule equal to A as right A-module, with $\mathfrak{n}^+(A_\lambda) = 0$ and $hm = m\pi(h + \langle\lambda,h\rangle)$ for all $m \in A^\lambda$ and $h \in \mathfrak{h}$. The (U,A)-bimodule $N_\lambda := Z_A(\lambda) := U \otimes_{U(\mathfrak{b}^+)} A^\lambda$ can be naturally regarded as an object of $\mathcal{M}''_A$ with $1_U \times 1_A \in Z_A(\lambda)_\lambda$ (cf [1]). Thus, $Z_{\mathbb{C}}(\lambda)$ is the usual Verma module of highest weight λ.

Define $\mathcal{D} = \mathcal{M}'_A$ and for $x \in \Omega$, let $\mathcal{D}_x := \operatorname{Add}\{Z_A(x)\}$. One can show that $\operatorname{End}_{\mathcal{D}}(Z_A(\Lambda)) \cong A$ naturally, and that 1.4(i) and 1.4(ii) hold (in fact, if $A = S$ or $A = S_{\mathfrak{m}_0}$ then $\operatorname{Hom}_{\mathcal{D}}(Z_A(\lambda), Z_A(\mu)) = 0$ unless $\lambda = \mu$).

The "blocks" of $\mathcal{M}'_A$ over A are defined to be the equivalence classes for the finest equivalence relation $\equiv$ on $\mathfrak{h}^*$ such that $\lambda \equiv \mu$ if $\operatorname{Ext}^i_{\mathcal{M}'_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$ for some $i \leq 1$. For general reasons, the blocks for $A = \mathbb{C}$ and $A = S_{\mathfrak{m}_0}$ are both the same as the blocks of $\mathcal{O}$ which were explicitly computed in [13]; in that case, one has for each block Ω a full subcategory $\mathcal{O}_\Omega$ of $\mathcal{O}$ consisting of all modules all of whose irreducible subquotient modules have their highest weights in Ω, and $\mathcal{O}$ is determined by these block subcategories. (Over $A = S$, the blocks are quite different.)

One would like, for a fixed block Ω of $\mathcal{O}$, to construct a thickened version of the block category $\mathcal{O}_\Omega$ over $A = S_{\mathfrak{m}_0}$, or, more subtly, over $A = S$. In order to do this, one can proceed as follows. Give Ω the induced order as a subset of $\mathfrak{h}^*$. Let $\mathcal{C} = \mathcal{C}_A$ be the smallest extension closed additive subcategory of $\mathcal{D}$ containing the objects $Z_A(x)$ for all $x \in \Omega$, regarded as a perfectly exact subcategory of $\mathcal{D}$. It is easy to show (e.g. using 1.2) that $\mathcal{C}_A$ is equal to the stratified exact category associated by the construction 1.7 to $\mathcal{D}$ and its additive subcategories $\mathcal{D}_x$ (regarded as split exact categories) for $x \in \Omega$. The category $\mathcal{E} := \mathcal{C}_A^\dagger$ may be regarded as an analogue of $\mathcal{M}'_A$ associated to the block Ω. It contains (for fixed A) full abelian subcategories $\mathcal{E}_{\text{fin}} \subseteq \mathcal{E}_{\text{fg}}$ satisfying additional finiteness conditions (see 1.30). If $A = \mathbb{C}$, it can be shown using 1.20 that $\mathcal{E}_{\text{fin}} = \mathcal{E}_{\text{fg}}$ is equivalent to the full subcategory of $\mathcal{O}_\Omega$ consisting of objects M such that there exist $\lambda_1, \ldots, \lambda_n \in \Omega$ such that if $M_\mu \neq 0$, then $\lambda_i - \mu \in Q^+$ for some i. For each of the three possible choices of A we allow, $\mathcal{E}_{\text{fin}}$ may be regarded as an analogue (corresponding to the weight poset $\Omega \subseteq \mathfrak{h}^*$) of $\mathcal{M}''_A$ (which corresponds to the weight poset $\mathfrak{h}^*$).

1.12. **Example 2: Categories from Coxeter groups.** We describe here one natural family of representation theories (over the real numbers) associated to Coxeter groups, from among several mentioned in [23]. We allow infinite weight posets and "singular infinitesimal character on both sides" since these situations (cf. [23, 5.6.]) are responsible for many features of our treatment here. Another class of examples which have motivated the general setup considered in this paper are various integral and prime characteristic versions of some of these examples (associated to crystallographic reflection representations), but we make only sporadic mention of these.

Let (W, R) be a (f.g. for simplicity) Coxeter system in its (standard, for definiteness) reflection representation (see [6] or [31]) on a real finite-dimensional vector space V. For the exposition here, if the associated bilinear form is degenerate (e.g. for an affine Weyl group) we need to enlarge V and extend the form to a non-degenerate symmetric bilinear form on V.

Consider W in either Chevalley-Bruhat order or reverse Chevalley-Bruhat order; accordingly, we define $l\colon W \to \mathbb{Z}$ by $l = l_0$ or $l = -l_0$ where l_0 is the standard length function on W (one could also use orders associated to more general length functions as in [19], see [23]). Let S be the symmetric algebra of V over $\mathbb{R}$ graded in even degrees so $S_2 = V$; the W action on V extends naturally to an action of W as a group of graded $\mathbb{R}$-algebra automorphisms of S. For any finite standard parabolic subgroup W_L of W, let S^L denote the corresponding ring of W_L-invariants on S (it is well known that S^L is a graded polynomial ring). Fix finite standard parabolic subgroups W_J and W_K of W, and let Ω be the set of minimal length (W_J, W_K) double coset representatives in W with respect to l, given the order induced as a subset of W by $\leq$.

Let $\mathcal{D}$ be the abelian category of graded $S^J \otimes_{\mathbb{R}} S^K$-modules, with homogeneous maps of degree zero respecting the module structure as morphisms. The group $G = \mathbb{Z}$ acts as a group of automorphisms $M \mapsto M\langle n\rangle$, $n \in \mathbb{Z}$ of $\mathcal{D}$ by grading shifts, where $(M\langle n\rangle)_m = M_{m-n}$. Define for each $d \in \Omega$ the graded module $N_d := S^{d^{-1}Jd \cap K}\langle l(d)\rangle$ in $\mathcal{D}$ with $S^J \otimes_{\mathbb{R}} S^K$ action given by $(a \otimes b)(n) = nbd^{-1}(a)$ for $a \in S^J$, $b \in S^K$ and $n \in S^{d^{-1}Jd \cap K}$, where $nbd^{-1}(a)$ is just the product in $S^{d^{-1}Jd \cap K}$. For $d \in \Omega$ set $\mathcal{D}_d := \operatorname{add} N_d$.

Here, one has $\mathrm{end}(N_d) \cong S^{d^{-1}Jd\cap K}$ for each $d \in \Omega$, and the conditions 1.4(i) and 1.4(ii) hold (in fact, $\mathrm{hom}(N_x, N_y) = 0$ for $x \neq y$ in Ω); however, in contrast to Example 1, one may have $\mathrm{ext}^1(N_x, N_y) \neq 0$ with $y \leq x$ (cf. [23] or [17] for explicit formulae in the case $K = \emptyset$). Using 1.7, one may define the associated stratified exact category $\mathcal{C}$ and abelian categories $\mathcal{C}^\dagger \cong \mathcal{E} \supseteq \mathcal{E}_{\mathrm{fg}} \supseteq \mathcal{E}_{\mathrm{fin}}$ which are conjecturally closely analogous to the corresponding categories in Example 1 with $A = S$, but with graded structure. To obtain analogues in this setting over $\mathbb{C}$ of the representation theories over $A = \mathbb{C}$ in Example 1, one applies certain base change and forgetful functors (forgetting grading) to those over S, as discussed subsequently.

If it is necessary to indicate dependence of $\mathcal{C}$ on J, K, l we write it as ${}^J\mathcal{C}_l^K$.

Remarks. If W is finite, one could work equivalently with graded $S^J \otimes_{S^W} S^K$-modules instead of graded $S^J \otimes_{\mathbb{R}} S^K$-modules.

1.13. **Example 3: Categories from fans of polyhedral cones.** To partly motivate the more general constructions in Sections 3 and 5, we give a natural example in which the construction in Proposition 1.7 is not quite adequate.

Let V be a finite-dimensional $\mathbb{R}$-vector space, and let S be the symmetric algebra of V over $\mathbb{R}$, graded so $S_2 = V$. Let $\mathcal{D}$ be the category over $\mathbb{Z}$ of graded $S \otimes_{\mathbb{R}} S$-modules, with translations by grading shifts.

By a fan Ω in V, we mean a set Ω of polyhedral cones in V such that any face of a cone in the fan is also a cone in the fan, and the intersection of any two cones of the fan is a face of both of those two cones. Fix a (finite for simplicity) fan Ω in V, and partially order Ω by inclusion. For $x \in \Omega$, define $l(x)$ as the dimension of the linear span of x.

Now assume further that V is a Euclidean space i.e. V is endowed with an inner product (a positive definite $\mathbb{R}$-valued symmetric bilinear form). For $x \in V$, let s_x be the orthogonal reflection in x i.e. the linear map on V with the linear span of x (resp., the orthogonal complement of x) as its 1-eigenspace (resp., -1-eigenspace), and extend s_x to a graded $\mathbb{R}$-algebra automorphism of S. For $x \in \Omega$, define N_x in $\mathcal{D}$ to be equal to $S\langle l(d)\rangle$ as graded $\mathbb{R}$-vector space, with $S \otimes_{\mathbb{R}} S$-module structure given by $(a \otimes b)n = nbs_x(a)$ (product in S) for $a, b \in S$ and $n \in N_x$. Let $\mathcal{D}_x = \mathrm{add}\, N_x$.

Now one has $\mathrm{end}(N_x) \cong S$ for $x \in \Omega$; if Ω consists of all faces of some fixed polyhedral cone, then the conditions 1.4(i) and 1.4(ii) hold (in fact, $\mathrm{hom}(N_x, N_y) = 0$ unless $x = y$). For a general fan, neither 1.4(i) nor 1.4(ii) hold, but one can use 5.9 to define $\mathcal{C}$, $\mathcal{C}^*$, $\mathcal{C}^\dagger$ etc, which are conjecturally closely analogous to categories in the preceding two examples. Again, one can obtain analogues of the categories in Example 1 over $A = \mathbb{C}$ by applying base change and forgetful functors described later.

Remarks. In general, a "global" weakly stratified exact category $\mathcal{C}$ can be regarded as having been obtained by "gluing" the "local" categories $\mathcal{C}_{\leq y}$ for $y \in \Omega$; in the above example, observe that each $\mathcal{C}_{\leq y}$ is the stratified exact category associated to the face lattice of a single polyhedral cone and can therefore be constructed using just Proposition 1.7.

It should not be difficult to show that, more generally, (weakly) stratified exact categories with weight posets forming an open cover of Ω can be glued to give a (weakly) stratified exact category with weight poset Ω, if they satisfy suitable compatibility conditions (analogous to those needed for gluing sheaves).

1.14. As already mentioned, the "representation categories" $\mathcal{C}^\dagger$ considered in Examples 1–3 above have realizations as full subcategories $\mathcal{E}$ of $\mathcal{A}$-mod, for an Ω-diagonalizable G-graded ring $\mathcal{A}$, where $G = \{1\}$ (resp., $G = \mathbb{Z}$) in Example 1 (resp., Examples 2 and 3). This is useful in applications for obtaining other similar representation categories (e.g. by base change or by forgetting the grading, or by imposing extra finiteness conditions similar to those imposed on $\mathcal{M}'_A$ to get $\mathcal{M}''_A$ in Example 1).

The ring $\mathcal{A}$ arises as $\operatorname{end}_{\hat{\mathcal{C}}}(\mathbf{P})^{\mathrm{op}}$ for a suitable family $\mathbf{P}$ of projective objects in a category $\hat{\mathcal{C}}$ of pro-objects of $\mathcal{C}$, as we shall now describe. For finite Ω, one may canonically identify $\hat{\mathcal{C}} \cong \mathcal{C}$.

1.15. **Pro-objects.** Fix a weakly stratified exact category $\mathcal{C}$ with strata $\mathcal{C}_x$ for $x \in \Omega$. The category $\hat{\mathcal{C}}$ in general is studied in Section 6 after some preparation in Section 5. We summarize the definition and the most important technical properties of $\hat{\mathcal{C}}$ below.

Let I_1 be the set of ideals Γ of Ω such that $\Gamma \subseteq \Gamma'$ for some finitely-generated ideal Γ' of Ω. Let I_0 be the set of ideals of the form $\{\, y | y \leq x \,\}$ for some $x \in \Omega$, Let $I_0 \subseteq I \subseteq I_1$ and regard I, I_0 and I_1 as posets ordered by inclusion. Note that $\Omega \in I_1$ for $l = -l_0$ in Example 2 or if Ω is finite (e.g. in Example 3).

Let I be as above and define a category $\hat{\mathcal{C}}_I$ as the full subcategory of inverse systems $\{Q_\Lambda\}_{\Lambda \in I}$ of objects of $\mathcal{C}$ such that $Q_\Lambda(x) = 0$ for $x \notin \Lambda$, and for $\Lambda \supseteq \Sigma$ in I, the restriction map $Q_\Lambda \to Q_\Sigma$ is an admissible epimorphism in $\mathcal{C}$ with kernel $\sigma_{\Lambda \setminus \Sigma}(Q_\Lambda)$.

For locally closed subsets Γ of Ω, there is a "truncation functor" $\hat{\sigma}_{\Gamma,I} \colon \hat{\mathcal{C}}_I \to \hat{\mathcal{C}}_I$ defined on objects by $\{Q_\Lambda\}_{\Lambda \in I} \mapsto \{Q_\Lambda(\Gamma)\}_{\Lambda \in I}$. There is also a natural functor $\theta_I \colon \mathcal{C} \to \hat{\mathcal{C}}_I$ mapping an object Q of $\mathcal{C}$ to the inverse system $\{Q(\Lambda)\}_{\Lambda \in I}$, with the unique restriction maps which are admissible epimorphisms in $\mathcal{C}$ compatible with the canonical epimorphisms $Q \to Q(\Lambda)$ for $\Lambda \in I$.

Lemma. (a) *$\hat{\mathcal{C}}_I$ has a natural structure of exact category.*
(b) *The forgetful functor $F \colon \{Q_\Lambda\}_{\Lambda \in I} \mapsto \{Q_\Lambda\}_{\Lambda \in I_0}$ is an equivalence of exact categories $\hat{\mathcal{C}}_I \to \hat{\mathcal{C}}_{I_0}$, satisfying $F\theta_I \cong \theta_{I_0}$ and $F\hat{\sigma}_{\Lambda,I} \cong \hat{\sigma}_{\Lambda,I_0}F$. Hence we write simply $\hat{\mathcal{C}}_I = \hat{\mathcal{C}}$, $\hat{\sigma}_{\Gamma,I} = \hat{\sigma}_\Gamma$ and $\theta_I = \theta$.*
(c) *The functor θ is perfectly exact, and $\theta\sigma_\Gamma \cong \hat{\sigma}_\Gamma\theta$ for any locally closed subset Γ of Ω.*
(d) *If Γ is a locally closed subset of Ω which generates an ideal $\Gamma' \in I_1$, there is a natural exact functor $\rho_\Gamma \colon \hat{\mathcal{C}} \to \mathcal{C}$ which we denote $Q \mapsto Q(\Gamma)$, determined by $\{Q_\Lambda\}_{\Lambda \in I_1} \mapsto \tau_\Gamma Q_{\Gamma'}$. Moreover, $\rho_\Gamma\theta \cong \sigma_\Gamma$.*
(e) *If $\Gamma \in I_1$, then θ is an equivalence of exact categories.*

We will write an object M of $\hat{\mathcal{C}}$ as $M = \{M_\Gamma\}_{\Gamma \in I}$, where we choose $I = I_0$ or $I = I_1$ as convenience dictates. Frequently, we regard $\mathcal{C}$ as a subcategory of $\hat{\mathcal{C}}$ by means of θ, and we identify $\mathcal{C} = \hat{\mathcal{C}}$ if $\Omega \in I_1$.

Remarks. One may think of $\hat{\mathcal{C}}$ as the "inverse limit" of the $\mathcal{C}_\Lambda$ for $\Lambda \in I$ with respect to the truncation functors $\tau_\Sigma \colon \mathcal{C}_\Lambda \to \mathcal{C}_\Sigma$ for $\Lambda \supseteq \Sigma$ in I. Objects of $\hat{\mathcal{C}}_{I_1}$ are analogues of flabby sheaves of objects of $\mathcal{C}$, in the sense that for the "presheaf" $\{Q_U\}_{U \in I_1}$ in $\hat{\mathcal{C}}_{I_1}$, the sheaf condition holds for finite coverings of $U \in I_1$ by objects of I_1. The equivalence $\hat{\mathcal{C}}_{I_1} \cong \hat{\mathcal{C}}_{I_0}$ is technically important for Zorn's lemma arguments involving $\hat{\mathcal{C}}$, and corresponds to the usual local description of sheaves by their stalks.

1.16. **Bistable functors.** In Section 6, we consider various "stability properties" of a right exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between stratified exact categories. These hold automatically if the weight posets of $\mathcal{C}$ and $\mathcal{D}$ are finite. The strongest of these conditions, which we call bistability, implies F_* and F^* restrict to functors $F_\dagger$ and $F^\dagger$ between $\mathcal{C}^\dagger$ and $\mathcal{D}^\dagger$; a weaker condition than bistability, which we call stability backwards for f.g ideals, has the following consequences.

Theorem. *Let $\mathcal{C}^i$ be weakly stratified exact categories. Suppose that $F, F'\colon \mathcal{C}^1 \to \mathcal{C}^2$ and $G\colon \mathcal{C}^2 \to \mathcal{C}^1$ are right exact functors which are stable backwards for f.g. ideals and $\epsilon\colon F \to F'$ is a natural transformation.*

(a) *F can be naturally extended to a right exact functor $\hat{F}\colon \hat{\mathcal{C}}^1 \to \hat{\mathcal{C}}^2$ so that for an object M in $\hat{\mathcal{C}}^1$, $(\hat{F}M)_\Gamma := \varprojlim_{\Lambda \in I^1} \sigma_\Gamma F(M_\Lambda)$. If F is exact, so is $\hat{F}$.*

(b) *ϵ induces in a natural way a natural transformation $\hat{\epsilon}\colon \hat{F} \to \hat{F}'$*

(c) *If G is right adjoint to F, then $\hat{G}\colon \hat{\mathcal{C}}^2 \to \hat{\mathcal{C}}^1$ is a right adjoint to $\hat{F}$.*

One uses this in applications of Example 2 to the combinatorial study of the generic Iwahori-Hecke algebra, for which it is necessary to show that certain functors $\hat{F}$ preserve projective objects of $\hat{\mathcal{C}}$.

1.17. **Standard objects N.** We now fix a stratified exact category $\mathcal{C}$ over a group G (i.e. $\mathcal{C}$ is stratified exact category with a compatible structure of exact category over G such that the translations preserve each stratum $\mathcal{C}_x$ for $x \in \Omega$).

Suppose $\mathcal{C}$ is svelte. Fix once and for all for each $x \in \Omega$ a family $\mathbf{N}_x := \{N_{x,i}\}_{i \in I_x}$ of objects of $\mathcal{C}_x$ such that $\mathcal{C}_x = \operatorname{add} \mathbf{N}_x$ (we call $\mathbf{N}_x$ a family of standard objects of weight x in $\mathcal{C}$; in many, but not all, applications there is a natural choice of $\mathbf{N}_x$). We define the G-graded unital ring $R_{x,i} := \operatorname{end}(N_{x,i})^{\mathrm{op}}$ and the G-graded (diagonalizable) ring $R_x := \operatorname{end}(\mathbf{N}_x)^{\mathrm{op}}$, so $e_i R_x e_j = \hom(N_{x,i}, N_{x,j})$ for $i, j \in I_x$. Define the graded Jacobson radicals $J_{x,i} := \operatorname{rad} R_{x,i}$ and $J_x := \operatorname{rad} R_x$.

Generally, we will write $\{N_{x,i}\}_i$ for $\{N_{x,i}\}_{i \in I_x}$, $\{N_{x,i}\}_{x,i}$ for $\{N_{x,i}\}_{x \in \Omega, i \in I_x}$ and similarly for other families indexed by pairs (x, i). Sometimes we write $e_i \in R_x$ as $e_{x,i}$; if $\{I_x\}$ is a singleton, as in Examples 1–3, we may suppress the $i \in I_x$ from notation, writing $N_{x,i}$ as N_x, $e_{x,i}$ as e_x or e etc.

In Example 1, G is trivial and $R_x = A$ (ungraded ring) for all x. In Example 2 (resp., Example 3), $G = \mathbb{Z}$ acting by degree shifts and $R_x = S^{x^{-1}Jx \cap K}$ (resp., $R_x = S$). Note all these rings are Noetherian (even as ungraded rings).

1.18. **Projective pro-objects.** Let $\mathcal{P}$ (resp., $\hat{\mathcal{P}}$) denote the split exact category of projective objects in $\mathcal{C}$ (resp., $\hat{\mathcal{C}}$); we study them in Section 7. The main facts are as follows. A convergent direct sum in $\hat{\mathcal{C}}$ is defined to be one of the form $M = \oplus_j M_j$ such that for each $\Gamma \in I_1$, $M_{j,\Gamma}$ is non-zero for only finitely many j (such a sum exists for any such M_j and is also the product of the M_j in $\hat{\mathcal{C}}$; see 6.5). By a standard family of projectives in $\hat{\mathcal{C}}$, we mean a family $\mathbf{P} := \{P_{x,i}\}_{x \in \Omega, i \in I_x}$ of projective objects of $\hat{\mathcal{C}}$ such that $P_{x,i}(x) \cong N_{x,i}$ and $P_{x,i}(y) = 0$ unless $y \geq x$. We define the support of an object M of $\hat{\mathcal{C}}$ to be the closure of $\{\, x \in \Omega \mid M(x) \neq 0 \,\}$ (i.e. the coideal it generates).

Theorem. (a) *An object P of $\hat{\mathcal{C}}$ is projective iff for each $x \in \Omega$, the natural map $g_{P,x}\colon \operatorname{hom}_{\mathcal{C}}(P(x), \mathbf{N}_x) \to \operatorname{ext}^1_{\mathcal{C}}(P(< x), \mathbf{N}_x)$ induced by the short exact sequence $0 \to P(x) \to P(\leq x) \to P(< x) \to 0$ in $\mathcal{C}$ is an epimorphism.*

(b) *A standard family of projective objects of $\hat{\mathcal{C}}$ exists iff $\hat{\mathcal{C}}$ has enough projective objects in the usual sense, or equivalently if each $\mathcal{C}_\Gamma$ for $\Gamma \in I$ has enough projectives. Then we say $\mathcal{C}$ has enough projective pro-objects.*

(c) *There are enough projective pro-objects of $\mathcal{C}$ if $\operatorname{ext}^1_{\mathcal{C}}(N_{y,i}, \mathbf{N}_x)$ is a graded Noetherian right R_x-module for all $x > y$ in Ω and i.*

(d) *If $\mathbf{P}$ is a standard family of projectives, each object M of $\hat{\mathcal{C}}$ has a projective resolution $P^\bullet \to M \to 0$ by projective objects which are direct summands of convergent direct sums of translates of objects of $\mathbf{P}$. In fact, let Γ_0 be the support of M and recursively define Γ_{i+1} to be Γ_i with all of its minimal elements deleted (or $\Gamma_{i+1} = \emptyset$ if $\Gamma_i = \emptyset$). Then $\cap_i \Gamma_i = \emptyset$ (so $\Gamma_i = \emptyset$ for large i if $\Omega \in I_1$) and one can choose such a resolution so the support of P^i is contained in Γ_i.*

Projective objects are constructed essentially by taking iterated "maximal non-split extensions," extending a method used to construct projective objects in suitable highest weight categories over fields in [9, (5.9)] and [3, 3.2.1]. In situations like Example 1, there are often alternative constructions of projectives using "truncated induced modules;" these go back to Bernstein-Gelfand-Gelfand in the case of category $\mathcal{O}$ for a semisimple complex Lie algebra and have been used for infinite weight posets in [47] and over more general commutative rings than fields in [24] and [1], for instance. Using (c) above, it is not difficult to see that $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ both have enough projective pro-objects in Examples 1–3.

Remarks. For any M in $\hat{\mathcal{C}}$ and $x \in \Omega$, $\operatorname{hom}_{\mathcal{C}}(M(x), \mathbf{N}_x)$ is a f.g. projective right R_x-module so in particular, if P in $\hat{\mathcal{C}}$ is projective, then $\operatorname{ext}^1_{\mathcal{C}}(P(< x), \mathbf{N}_x)$ is f.g. as right R_x-module.

1.19. $\mathcal{C}^\dagger$ **as a module category.** Assume for the remainder of Section 1 that the stratified exact category $\mathcal{C}$ has enough projective pro-objects. Fix a standard family $\mathbf{P}$ of projectives of $\hat{\mathcal{P}}$. We write $P_{x,i} = \{P_{x,i,\Gamma}\}_{\Gamma \in I_1}$ and define the family $\mathbf{P}_\Gamma := \{P_{x,i,\Gamma}\}_{x,i}$ of objects of $\mathcal{C}$. Define $\mathcal{A}_\Gamma := \operatorname{end}_{\mathcal{C}}(\mathbf{P}_\Gamma)^{\mathrm{op}}$ and $\mathcal{A} := \operatorname{end}_{\hat{\mathcal{C}}}(\mathbf{P})^{\mathrm{op}}$. The family $\mathcal{A}_\Gamma$ is a family of quotient rings of $\mathcal{A}$, and $\mathcal{A}$ identifies with the inverse limit of this family (in the category of G-graded diagonalizable rings $A = \oplus_{x,i,y,j}\, e_{x,i} A e_{y,j}$).

Let $\mathcal{E}$ denote the full abelian subcategory of $\mathcal{A}$-mod consisting of graded $\mathcal{A}$-modules which are $\mathcal{A}_\Gamma$ modules for some $\Gamma \in I_1$, and let $\hat{\mathcal{E}}$ be the full subcategory of $\mathcal{A}$-mod consisting of graded $\mathcal{A}$-modules which are the direct limit (i.e. union) of their submodules in $\mathcal{E}$.

In Section 8, we prove the following result (actually under much weaker hypotheses than those above).

Theorem. (a) *There is an equivalence of abelian categories $\mathcal{C}^* \cong \hat{\mathcal{E}}$ under which the Yoneda functor $\phi_C\colon \mathcal{C} \to \mathcal{C}^*$ given by $M \mapsto \operatorname{Hom}_{\mathcal{C}}(?, M)$ corresponds to the composite functor $\varphi = \operatorname{hom}_{\hat{\mathcal{C}}}(\mathbf{P}, \theta ?)\colon \mathcal{C} \to \hat{\mathcal{E}}$.*

(b) *This equivalence restricts to equivalences of abelian categories $\mathcal{C}^\dagger \cong \mathcal{E}$ and $\mathcal{C}^*_\Gamma \cong \mathcal{A}_\Gamma$-mod for $\Gamma \in I_1$.*

(c) *If $\Gamma \in I_1$, the composite $\mathcal{C}_\Gamma \to \mathcal{C}^*_\Gamma \cong \mathcal{A}_\Gamma$-mod restricts to a perfectly exact functor, from the split exact category of projective objects of $\mathcal{C}_\Gamma$ to the split* *exact category of f.g. graded projective $\mathcal{A}_\Gamma$-modules; moreover, the restriction is an equivalence if idempotents split in $\mathcal{C}$*

1.20. We wish to describe a result which, under suitable assumptions, enables one to identify $\mathcal{B} \cong \mathcal{E} \cong \mathcal{C}^\dagger$ in situations like that of 1.2.

Suppose that $\mathcal{C}$ in 1.19 is a perfectly exact subcategory over G of an abelian category $\mathcal{B}$ over G. Assume that there is a given directed family (with respect to inclusion) of Serre subcategories $\{\mathcal{B}_\Lambda\}_{\Lambda\in I_1}$ of $\mathcal{B}$ over G such that

(i) $\mathcal{B}$ is the directed union $\mathcal{B} = \varinjlim_{\Lambda\in I_1} \mathcal{B}_\Lambda$.
(ii) each $\mathcal{B}_\Lambda$ has arbitrary coproducts
(iii) for each $\Lambda \in I_1$, each object of $\mathbf{P}_\Lambda$ is projective in $\mathcal{B}_\Lambda$
(iv) every object of $\mathcal{B}_\Lambda$ is a quotient of a (possibly infinite) direct sum of copies of translates of objects in $\mathbf{P}_\Lambda$
(v) each functor $\hom(P, ?)$ for P in $\mathbf{P}_\Lambda$ preserves infinite direct sums in $\mathcal{B}_\Lambda$

Proposition. *Let $\iota\colon \mathcal{C} \to \mathcal{B}$ be the inclusion. Then there is an equivalence of categories $F\colon \mathcal{B} \to \mathcal{E}$ satisfying $F\iota \cong \varphi\colon \mathcal{C} \to \mathcal{E}$ under which the strict image of $\mathcal{B}_\Gamma$ is $\mathcal{A}_\Gamma$-*mod*. Conversely, if one identifies $\mathcal{C}$ with its strict image under φ in $\mathcal{E}$, the above conditions are satisfied with $\mathcal{B} = \mathcal{E}$ and $\mathcal{B}_\Gamma = \mathcal{A}_\Gamma$-*mod *for $\Gamma \in I_1$.*

We give an example related to Example 1. Let $\mathcal{D}_A$ denote the smallest extension closed additive subcategory of $\mathcal{M}'_A$ containing all $Z_A(\lambda)$ for $\lambda \in \mathfrak{h}^*$, regarded as a perfectly exact subcategory of $\mathcal{M}'_A$. Then $\mathcal{D}_A$ is a stratified exact category with weight poset $\mathfrak{h}^*$ (ordered as in Example 1) and strata $\operatorname{Add} Z_A(x)$. The proposition can be used to show that $\mathcal{D}_A^\dagger \cong \mathcal{M}'_A$.

1.21. **Comparison of Extensions.** The following basic lemma comparing extensions in $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{A}_\Gamma$-mod is proved in 9.2 (cf [15, Statement 3], [10, 3.9], 1.6(b) and 4.5).

Lemma. (a) *For M, N in $\mathcal{C}$, the natural maps induced by φ give isomorphisms $\operatorname{ext}^i_{\mathcal{C}}(M,N) \cong \operatorname{ext}^i_{\mathcal{E}}(\varphi(M),\varphi(N))$*
(b) *If $\Gamma \in I_1$ and M, N are graded $\mathcal{A}_\Gamma$-modules, then the natural maps give isomorphisms $\operatorname{ext}^i_{\mathcal{A}_\Gamma\text{-mod}}(M,N) \cong \operatorname{ext}^i_{\mathcal{E}}(M,N)$.*

1.22. **Left Stratified Rings.** In Section 10, we characterize the rings which arise by the construction of 1.19 applied to stratified exact categories, as follows. We take $I = I_1$.

Consider a diagonalizable G-graded ring $A = \oplus_{x,x'\in\Omega} \oplus_{i,i'\in I_x} e_{x,i}Ae_{x',i'}$ with an inverse system $\{A(\Gamma)\}_{\Gamma\in I}$ of quotient rings satisfying the following conditions:

(i) $e_{x,i}A(\Gamma)e_{x',i'} = 0$ unless $x, x' \in \Gamma$
(ii) For $\Gamma \subseteq \Lambda$ in I, the kernel V of the epimorphism $A(\Lambda) \to A(\Gamma)$ is the two-sided ideal of $A(\Lambda)$ generated by $\{e_{x,i}\}_{x\in\Lambda\setminus\Gamma,i}$, and for each y and j, $Ve_{y,j}$ has a finite filtration as graded A-module with successive subquotients in $\operatorname{add} A(\leq z)e_{z,k}$ for various $z \geq y$ and k with $z \in \Lambda \setminus \Gamma$.
(iii) $A \cong \varprojlim_{\Gamma\in I} A(\Gamma)$ as diagonalizable graded ring.

We say that the pair $(A, \{A(\Gamma)\}_{\Gamma\in I})$ is left stratified (or that A is a left stratified ring with weight poset Ω and standard quotients $\{A(\Gamma)\}_{\Gamma\in I}$) if these conditions hold. If A^{op} is left-stratified with weight poset Ω and standard quotients $\{A(\Gamma)^{\mathrm{op}}\}$, we say that A is right stratified by $\{A(\Gamma)\}$. Finally, we say that A is stratified by $\{A(\Gamma)\}$ if it is left and right stratified by $\{A(\Gamma)\}$.

Proposition. *A pair* $(A, \{A(\Gamma)\}_{\Gamma \in I})$ *is left stratified iff it is isomorphic to a pair* $(\mathcal{A}, \{\mathcal{A}(\Gamma)\}_{\Gamma \in I})$ *obtained from a standard family of projective pro-objects of a stratified exact category* $(\mathcal{C}, \{\mathcal{C}_x\}_{x \in \Omega})$ *as in* 1.19*. Moreover, if this holds then* $(A, \{A(\Gamma)\})$ *is stratified iff in addition each module* $\oplus_i e_{y,j} A(\leq x) e_{x,i}$ *is a f.g. projective right module for the ring* $\oplus_{i,i'} e_{x,i} A(\leq x) e_{x,i'}$*.*

We shall say that $\mathcal{C}$ is a strongly stratified exact category if it has enough projective pro-objects and $\mathcal{A}$ is a stratified ring (this condition is independent of the choice of the $\mathbf{N}_x$ and $\mathbf{P}$). Note $\mathcal{C}$ is strongly stratified iff $\mathcal{C}_\Gamma$ is strongly stratified for each finite locally closed subset Γ of Ω.

Remarks. The class of left stratified rings is not a particularly natural one, but it is adequate (and in fact very convenient) for our applications.

1.23. In studying the motivating examples of stratified exact categories $\mathcal{C}$, strong stratification of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ is often an important point of the theory. In Example 1, $\mathcal{C}_{\mathbb{C}}$ and $\mathcal{C}_{\mathbb{C}}^{\mathrm{op}}$ are (trivially) strongly stratified. If $A = S_{\mathfrak{m}_0}$, $\mathcal{C}_A$ can be shown to be strongly stratified but I do not know if $\mathcal{C}_A^{\mathrm{op}}$ is strongly stratified. We conjecture that (resp., ask if) strong stratification of $\mathcal{C}_S$ (resp., $\mathcal{C}_S^{\mathrm{op}}$) holds in Example 1.

A basic fact which will be proved in subsequent papers is that $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are both strongly stratified in Examples 2 and 3 (see [23, 3.5] for a sketch of a special case in Example 2). In general, the existence of a suitable duality (contravariant equivalence) on the subcategory of projective objects of $\hat{\mathcal{C}}$ would imply that $\mathcal{C}$ is stratified, and such a duality is known or conjectured to exist (but often not easy to construct) in many natural situations (e.g. its existence is known in Examples 2–3 and in Example 1 with $A = \mathbb{C}$ or $A = S_{\mathfrak{m}_0}$, but is only conjectural in Example 1 with $A = S$).

1.24. Stratified rings are related to the integral quasi-hereditary k-algebras [11] and to the cellular algebras of [28]. They are also related to classes of algebras considered by Konig [48] and to algebras defined recently by Zong-Zhu Lin and Jie Du; over fields they are related to the BGG algebras of [32]. We make no attempt to discuss the relationships thoroughly, but in Section 11, we discuss some simple conditions under which stratified rings in our sense are integral quasi-hereditary algebras or cellular algebras. As a special case of the facts there, one can show that the (unital) stratified rings associated to Example 3 are, as ungraded ring, integral quasihereditary (with respect to the natural sequence of defining ideals) and cellular as $S \otimes_{\mathbb{R}} \mathbb{R}$-algebra or $\mathbb{R} \otimes_{\mathbb{R}} S$-algebra. In Example 2, replacing Ω by a finite locally closed subset Γ of Ω and $\mathcal{C} = {}^J \mathcal{C}_l^K$ by its full subcategory $\mathcal{C}_\Gamma$ for the construction of $\mathcal{A}$ (so as to obtain a unital ring), the resulting stratified ring is integral quasi-hereditary and cellular as ungraded $S^J \otimes_{\mathbb{R}} \mathbb{R}$-algebra (resp., $\mathbb{R} \otimes_{\mathbb{R}} S^K$-algebra) if $J = \emptyset$ (resp., $K = \emptyset$), but not in general.

1.25. **$\boldsymbol{\Delta}$-modules and $\boldsymbol{\nabla}$-modules.** We now define important families $\boldsymbol{\Delta}_x := \{\Delta_{x,i}\}_i$, $\boldsymbol{\nabla}_x := \{\nabla_{x,i}\}_i$ of objects of $\mathcal{E}$. Let Γ be an ideal of Ω with x as maximal element, for example $\Gamma := \{ y \mid y \leq x \}$. The "$\boldsymbol{\Delta}$-modules" (corresponding to the choice of standard objects $\{N_{x,i}\}_{x,i}$) are defined by

$$\Delta_{x,i} := \varphi(N_{x,i}) \cong \hom_{\mathcal{C}}(\mathbf{P}_\Gamma, N_{x,i})$$

and the corresponding $\boldsymbol{\nabla}$-modules are defined by

$$\nabla_{x,i} = \hom_{\mathcal{C}}(\mathbf{P}(x), N_{x,i})$$

where $\mathbf{P}(x) := \{P_{y,j}(x)\}_{y,j}$ (note $\hom(\mathbf{P}(x), M)$ is a $\mathcal{A}$-module for M in $\mathcal{C}$ by the natural homomorphism $\operatorname{end}(\mathbf{P})^{\mathrm{op}} \to \operatorname{end}(\mathbf{P}(x))^{\mathrm{op}}$).

Write $\Delta_x = \oplus_i \Delta_{x,i}$ and $\nabla_x = \oplus_i \nabla_{x,i}$, $\boldsymbol{\Delta} := \{\Delta_{x,i}\}_{x,i}$, $\boldsymbol{\nabla} := \{\boldsymbol{\nabla}_{x,i}\}_{x,i}$. There is a natural family of maps $\alpha_x := \{\alpha_{x,i}\}_i$ where $\alpha_{x,i} \colon \Delta_{x,i} \to \nabla_{x,i}$ is the evident map $\hom(P_{x,i}(\leq x), N_{x,i}) \to \hom(P_{x,i}(x), N_{x,i})$. To indicate dependence on $\mathcal{A}$, we write if necessary $\boldsymbol{\Delta}^{\mathcal{A}}$, $\boldsymbol{\nabla}^{\mathcal{A}}$ etc.

Remarks. As objects of the functor category $\mathcal{C}^*$, one has $\Delta_{x,i} = \hom(?, N_{x,i}) = \hom(\sigma_{\leq x}?, N_{x,i})$, $\nabla_{x,i} = \hom(\sigma_x ?, N_{x,i})$ and $\alpha_{x,i}$ is induced by the canonical natural transformation $\sigma_x \to \sigma_{\leq x}$.

1.26. In Section 9, we prove the following ext vanishing properties of these modules (similar to those of Verma and coVerma modules [10], [32]; cf also 1.41).

Proposition. (a) *There are graded ring isomorphisms* $R_x = \operatorname{end}_{\mathcal{E}}(\mathbf{N}_x)^{\mathrm{op}} = \operatorname{end}_{\mathcal{E}}(\boldsymbol{\Delta}_x)^{\mathrm{op}} = \operatorname{end}_{\mathcal{E}}(\boldsymbol{\nabla}_x)^{\mathrm{op}}$ *of rings, and* (R_x, R_x)*-bimodule isomorphisms* $\hom_{\mathcal{E}}(\boldsymbol{\Delta}_x, \boldsymbol{\Delta}_x) \xrightarrow{\cong} \hom_{\mathcal{E}}(\boldsymbol{\Delta}_x, \boldsymbol{\nabla}_x) \xleftarrow{\cong} \hom_{\mathcal{E}}(\boldsymbol{\nabla}_x, \boldsymbol{\nabla}_x) \cong R_x$ *induced by* α_x.

(b) *One has* $\operatorname{ext}^p_{\mathcal{E}}(\boldsymbol{\Delta}_x, \boldsymbol{\Delta}_y) = 0$ *unless either* $x < y$*, or* $x = y$ *and* $p = 0$.

(c) *One has* $\operatorname{ext}^p_{\mathcal{E}}(\boldsymbol{\nabla}_x, \boldsymbol{\nabla}_y) = 0$ *unless either* $x > y$*, or else* $x = y$ *and* $p = 0$.

(d) *One has* $\operatorname{ext}^p_{\mathcal{E}}(\boldsymbol{\Delta}_x, \boldsymbol{\nabla}_y) = 0$ *unless* $x = y$ *and* $p = 0$.

Note that $e_{y,j}\nabla_x \cong \hom(P_{y,j}(x), \mathbf{N}_x)$ is a f.g. projective right R_x-module. On the other hand, $e_{y,j}\Delta_x = \operatorname{Hom}(P_{y,j,\leq x}, \mathbf{N}_x)$ is a f.g. projective R_x-module for all x, y and j iff $\mathcal{A}$ is a stratified ring (see 10.2).

1.27. We define also the following right $\mathcal{A}$-modules. Let $\Gamma \in I_1$ with x as maximal element. Define the $(R_x, \mathcal{A}_\Gamma)$-bimodule

$$\Delta_x^{\mathcal{A}^{\mathrm{op}}} := \hom(\mathbf{N}_x, \mathbf{P}_\Gamma) \cong \oplus_j e_{x,j}\mathcal{A}_\Gamma \cong \hom(\mathbf{N}_x, \mathbf{P}(x)).$$

Define also the right $\mathcal{A}$-module $\Delta_{x,i}^{\mathcal{A}^{\mathrm{op}}} := e_{x,i}\mathcal{A}_\Gamma$, and the families $\boldsymbol{\Delta}_x^{\mathcal{A}^{\mathrm{op}}} := \{\boldsymbol{\Delta}_{x,i}^{\mathcal{A}^{\mathrm{op}}}\}_i$, $\boldsymbol{\Delta}^{\mathcal{A}^{\mathrm{op}}} := \{\boldsymbol{\Delta}_{x,i}^{\mathcal{A}^{\mathrm{op}}}\}_{x,i}$ of right $\mathcal{A}$-modules.

Similarly, define the right $\mathcal{A}$-module

$$\nabla_{x,i}^{\mathcal{A}^{\mathrm{op}}} := \oplus_{y,j} \hom_{R_x^{\mathrm{op}}}(e_{y,j}\Delta_x, e_i R_x),$$

the $(R_x, \mathcal{A})$-bimodule $\nabla_x^{\mathcal{A}^{\mathrm{op}}} = \oplus_i \nabla_{x,i}^{\mathcal{A}^{\mathrm{op}}}$ and the families $\boldsymbol{\nabla}_x^{\mathcal{A}^{\mathrm{op}}} := \{\nabla_{x,i}^{\mathcal{A}^{\mathrm{op}}}\}_i$, $\boldsymbol{\nabla}^{\mathcal{A}^{\mathrm{op}}} := \{\nabla_{x,i}^{\mathcal{A}^{\mathrm{op}}}\}_{x,i}$ of right $\mathcal{A}$-modules.

If $\mathcal{A}$ is stratified, the modules defined above are the analogues for $\mathcal{A}^{\mathrm{op}}$ of $\boldsymbol{\Delta}_x$, $\boldsymbol{\nabla}_x$ etc for $\mathcal{A}$ (see 10.2).

1.28. **Tilting modules.** Suppose in this subsection that Ω is finite. We assume that $\mathcal{C}$ and its dual stratified exact category $\mathcal{C}^{\mathrm{op}}$ both have sufficiently many projective objects. Choose standard families $\mathbf{P}$ of projectives of $\mathcal{C}$ and $\mathbf{Q}^{\mathrm{op}}$ of projectives of $\mathcal{C}^{\mathrm{op}}$. Here, $\mathbf{Q} = \{Q_{x,i}\}_{x,i}$ is a family of injective objects of $\mathcal{C}$ with $Q_{x,i}(x) \cong N_{x,i}$ and $Q_{x,i}(y) = 0$ unless $y \leq x$ (we call such a family a standard family of injectives of $\mathcal{C}$). One has equivalences $\mathcal{C}^* \cong \mathcal{A}$-mod and $(\mathcal{C}^{\mathrm{op}})^* \cong \mathcal{B}^{\mathrm{op}}$-mod where $\mathcal{B} = \operatorname{end}_{\mathcal{C}}(\mathbf{Q})^{\mathrm{op}}$ and $\mathcal{B}^{\mathrm{op}}$ is a left stratified ring over G^{op} with weight poset Ω^{op}. Set $\mathbf{T} = \{\mathcal{T}_{x,i}\}_{x,i}$ where $\mathcal{T}_{x,i} = \varphi(Q_{x,i})$, so $\operatorname{end}_{\mathcal{A}}(\mathbf{T})^{\mathrm{op}} \cong \mathcal{B}$. In Section 12, we prove the following generalization of part of Ringel's results [46] on quasi-hereditary algebras.

Theorem. (a) $\mathbf{T}$ *is a full family of tilting modules for* $\mathcal{A}$ *(cf.* C.13*). Hence the right derived functor* $R\big(\hom_{\mathcal{A}}(\mathbf{T}, ?)\big)$ *induces an equivalence of triangulated* *categories from the bounded derived category $D^b(\mathcal{A}\text{-mod})$ to $D^b(\mathcal{B}\text{-mod})$, with the left derived functor $\mathcal{T}\otimes^L_{\mathcal{B}}?$ as inverse equivalence.*

(b) *Suppose that $\mathcal{B}$ is a strongly stratified ring. Then* $\operatorname{add}\mathbf{T}$ *is the full additive subcategory of* $\mathcal{A}$-mod *consisting of objects which have both a finite filtration with objects of* $\operatorname{add}\boldsymbol{\Delta}_x$ *for various x as successive quotients, and a finite filtration with objects of* $\operatorname{add}\boldsymbol{\nabla}_x$ *for various x as successive subquotients.*

By duality, the analogous statements also hold with $\mathcal{C}$ replaced by $\mathcal{C}^{\mathrm{op}}$.

To discuss Examples 1 and 2 in relation to this result, in this subsection only we replace in those examples the (possibly infinite) weight poset Ω by a finite locally closed subset Γ of Ω and $\mathcal{C}$ by its full subcategory $\mathcal{C}_\Gamma$. Then $\mathcal{A}$ and $\mathcal{B}$ are defined in all Examples 1–3. In Examples 2 and 3, all hypotheses of the theorem hold for $\mathcal{C}$ or if $\mathcal{C}$ is replaced by $\mathcal{C}^{\mathrm{op}}$. The hypotheses also hold trivially for $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ in Example 1 with $A=\mathbb{C}$. For $A=S_{\mathfrak{m}_0}$ in Example 1, the hypotheses hold for $\mathcal{C}^{\mathrm{op}}$ but I don't know if they hold for $\mathcal{C}$. For $A=S$ in Example 1, I don't know if the hypotheses hold for either $\mathcal{C}$ or $\mathcal{C}^{\mathrm{op}}$.

1.29. **Blocks.** In Section 16, we consider "blocks" in a split stratified exact category $\mathcal{C}$; they are subsets of Ω, defined in a way similar to that for $\mathcal{O}$ in Example 1, and there are similar decompositions of $\mathcal{C}$ and $\mathcal{E}$ into block subcategories.

1.30. **Finiteness conditions.** It is useful at times to consider subcategories of $\mathcal{E}$ with extra finiteness conditions: $\mathcal{E}_{\mathrm{fg}}$ (resp., $\mathcal{E}_{\mathrm{fin}}$) denotes the Serre subcategory of $\mathcal{E}$ consisting of all modules M in $\mathcal{E}$ for which each weight space $e_{x,i}M$ is Noetherian (resp., is Artinian and Noetherian i.e. has a composition series) as graded $e_{x,i}\mathcal{A}e_{x,i}$-module (or, equivalently as graded $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module if M is a $\mathcal{A}_\Gamma$-module for $\Gamma\in I_1$). Let $\mathcal{E}_{\mathrm{wfin}}$ be the Serre subcategory of $\mathcal{E}$ consisting of all M in $\mathcal{E}$ with the property that for each x,i,g, $e_{x,i}M_g$ has a composition series as $e_{x,i}\mathcal{A}_{1_G}e_{x,i}$-module. It is easy to see that $\mathcal{E}_{\mathrm{fin}}\subseteq\mathcal{E}_{\mathrm{fg}}$ and $\mathcal{E}_{\mathrm{fin}}\subseteq\mathcal{E}_{\mathrm{wfin}}$ (inclusions of full subcategories).

1.31. **k-structure.** If k is a commutative unital Z-graded ring (with Z a subgroup of the center of G), regard it as a G-graded ring with $k_g=0$ if $g\notin Z$. Assume that $\mathcal{C}$ is a k-category over G (i.e. for each $M,N\in\mathcal{C}$, $\hom(M,N)$ has a given graded k-module structure compatible with its graded abelian group structure and such that composition is k-bilinear). Then the associated left-stratified ring $\mathcal{A}$ has a natural structure of graded k-algebra. We record the following simple facts about this situation.

Lemma. *Assume that k is a commutative Noetherian (resp., Artinian) graded ring and each space* $\hom_{\mathcal{C}}(N_{x,i},N_{y,j})$ *is f.g. as k-module. Then*

(a) *If M, N are objects of $\mathcal{C}$, then $\operatorname{ext}^i_{\mathcal{C}}(M,N)$ is a f.g. k-module for each i.*
(b) *The category $\mathcal{E}_{\mathrm{fg}}$ contains the strict image of φ, and it contains $\Delta_{x,i}$, $\mathcal{A}_\Gamma e_{x,i}$ for $\Gamma\in I$, $\nabla_{x,i}$ and, if Ω is finite and $\mathcal{C}$ has enough injectives, $\mathcal{E}_{\mathrm{fg}}$ contains the tilting modules $\mathcal{T}e_{x,i}$.*
(c) *$\mathcal{E}_{\mathrm{fg}}$ consists of all objects of $\mathcal{E}$ for which each weight space $e_{x,i}M$ is f.g. as k-module.*
(d) *If k is Artinian, $\mathcal{E}_{\mathrm{fg}}=\mathcal{E}_{\mathrm{fin}}$.*
(e) *If Ω and each index set I_x for $x\in\Omega$ is finite, then $\mathcal{A}$ is f.g. as k-module, and hence is a left and right Noetherian (resp., Artinian) unital G-graded*

ring, while $\mathcal{E}_{\rm fg}$ coincides with the full subcategory of graded Noetherian $\mathcal{A}$-modules. In this case, $\operatorname{ext}^i_{\mathcal{E}}(M,N)$ is a f.g. k-module for any M, N in $\mathcal{E}_{\rm fg}$.

The categories $\mathcal{C}$ in Examples 1–3 obviously have k-structure for various Noetherian rings k (e.g. $k = A$ in Example 1, $k = S^J \otimes_{\mathbb{R}} S^K$ or $k = S^J$ or $k = S^K$ in Example 2, $k = S \otimes_{\mathbb{R}} S$ or $k = \mathbb{R} \otimes_{\mathbb{R}} S$ or $k = S \otimes_{\mathbb{R}} \mathbb{R}$ in Example 3).

1.32. The existence of k-structure is important in the examples for constructing additional representation categories from $\mathcal{C}^\dagger$ by "base change." In order to give the details of this, we first indicate how a stratified exact category $\mathcal{C}_0$ is naturally associated to the left stratified ring A in the proof of 1.22 in 10.1. Let $\mathcal{D}$ be the full subcategory of A-mod consisting of modules which are A_Γ-modules for some $\Gamma \in I_1$ and define $\mathcal{D}_x = \operatorname{add}\{A_{\leq x} e_{x,i}\}_i$. Then the $\mathcal{D}_x$ satisfy the conditions 1.4(i) and 1.4(ii); hence one has an associated split exact category $\mathcal{C}_0$ over G obtained by the construction 1.7 (actually, using 1.2 for instance, $\mathcal{C}_0$ is the smallest extension closed subcategory of $\mathcal{D}$ containing all $\mathcal{D}_x$, and is a perfectly exact subcategory of $\mathcal{D}$). In the category $\hat{\mathcal{C}}_0$ of pro-objects of $\mathcal{C}_0$, one has a standard family $\mathbf{P} := \{P_{x,i}\}_{x,i}$ of projective objects with $P_{x,i,\Gamma} = A(\Gamma)e_{x,i}$, and one may identify $A = \operatorname{end}(\mathbf{P})^{\rm op}$ naturally with $A(\Gamma) = \operatorname{end}(\mathbf{P}(\Gamma))^{\rm op}$ for $\Gamma \in I$.

With this choice of $\mathbf{P}$, $\varphi\colon \mathcal{C}_0 \to A$-mod is just the inclusion, and $\mathcal{C}_0^\dagger$ naturally identifies with $\mathcal{D}$. If $A = \mathcal{A}$ is the left stratified ring associated to a stratified exact category $\mathcal{C}$ by 1.19, then $\mathcal{C}_0$ is naturally equivalent to the Karoubianization of $\mathcal{C}$ (see B.11); in particular, $\mathcal{C} \cong \mathcal{C}_0$ if idempotents split in $\mathcal{C}$.

1.33. **Flat base change.** We first mention a type of base change which is very useful in the study of the motivating examples. Suppose that $\mathcal{C}$ arises from the construction 1.7 where $\mathcal{D} = A$-mod for some graded k-algebra A and $\mathcal{D}_x$ are full additive subcategories satisfying the conditions 1.4(i) and 1.4(ii). If k' is a k-flat commutative Z-graded k-algebra, one can under suitable conditions construct an analogue of $\mathcal{C}$ over k' by applying 1.7 to the full additive subcategories $\mathcal{D}'_x := \operatorname{add}\{\, k' \otimes_k M | M \text{ in } \mathcal{D}_x \,\}$ of $k' \otimes_k A$-mod. We consider such flat base-change in 13.3. This technique can sometimes be used to prove results by reducing to the case where blocks are of smaller "rank" and the situation is sufficiently simple as to be amenable to direct calculation; for example, this technique is used to establish the factorizations of determinants of certain "Shapovalov" bilinear forms associated to Example 2 (see [23, 7.3]) and Example 3.

1.34. **Good base change.** We say that $\mathcal{C}$ has good base change over k if $\mathcal{C}$ is strongly stratified and each ring R_x is k-projective; then each Δ_x and ∇_x is k-projective and hence k-flat. We say that $\mathcal{C}$ has very good base change over k if $\mathcal{C}$ and $\mathcal{C}^{\rm op}$ are both strongly stratified and each ring R_x is projective as k-module.

1.35. Let k be a commutative Z-graded ring. Identify the stratified exact k-category $\mathcal{C}$ over G with a full subcategory of $\mathcal{A}$-mod using φ, so $N_{x,i} = \Delta_{x,i} = \mathcal{A}_{\leq x} e_{x,i}$. In 13.4, we prove the following.

Theorem. *Assume Δ_x is k-flat for all x, and let k' be any Z-graded commutative k-algebra. Let $\mathcal{A}' := \varprojlim_{\Lambda \in I} \mathcal{A}'_\Gamma$ where $\mathcal{A}'_\Gamma := k' \otimes_k \mathcal{A}_\Gamma$.*

(a) *$\mathcal{A}'$ is a left stratified ring with weight poset Ω and standard quotients $\mathcal{A}'_\Gamma$. Moreover, $\mathcal{A}'$ is stratified if $\mathcal{A}$ is stratified.*

(b) *Let* $\mathcal{C}' = k' \otimes_k \mathcal{C}$ *denote the exact category associated to the stratified exact ring* $\mathcal{A}'$ *as in* 1.32*. Then base change* $k' \otimes_k ?$ *induces a bistable exact functor* $F : \mathcal{C} \to \mathcal{C}'$ *satisfying* $F\sigma_\Sigma \cong \sigma'_\Sigma F$*, where for locally closed* $\Sigma \subseteq \Omega$*,* σ'_Σ *denotes truncation to* Σ *in* $\mathcal{C}'$*.*

(c) $\hat{F} : \hat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ *preserves projective objects*

(d) *Identify* $\mathcal{C}^\dagger$ *and* $(\mathcal{C}')^\dagger$ *with subcategories* $\mathcal{E}$*,* $\mathcal{E}'$ *of* $\mathcal{A}$-mod *and* $\mathcal{A}'$-mod *as in* 1.19*. Then the functor* $F_\dagger : \mathcal{E} \to \mathcal{E}'$ *is induced by the base change functors* $k' \otimes_k ? : \mathcal{A}_\Gamma\text{-mod} \to \mathcal{A}'_\Gamma\text{-mod}$

(e) *Let* $N'_{x,i} = \Delta'_{x,i} := k' \otimes_k \Delta_{x,i}$ *and* $\nabla'_{x,i} := k' \otimes_k \nabla_{x,i}$*. Then corresponding to choices of standard objects* $\mathbf{N}'_x := \{N'_{x,i}\}_i$ *in* $\mathcal{C}'$*,* $\mathbf{\Delta}'_x := \{\Delta'_{x,i}\}_i$ *is the family of* $\mathbf{\Delta}$*-modules of weight* x *and* $\mathbf{\nabla}'_x := \{\nabla'_{x,i}\}_i$ *is the corresponding family of* $\mathbf{\nabla}$*-modules in* $\mathcal{E}'$*. Moreover,* $\operatorname{end}(\mathbf{\Delta}'_x)^{\mathrm{op}} \cong k' \otimes_k R_x$ *naturally as well.*

(f) *Suppose* Ω *is finite and* $\mathcal{C}$ *has very good base change over* k*. Choose a standard family of injectives* $\mathbf{Q} = \{Q_{x,i}\}_{x,i}$ *as in* 1.28*; under our identifications the corresponding family of tilting modules is* $\mathbf{T} := \{\mathcal{T}e_{x,i}\}_{x,i} = \{Q_{x,i}\}_{x,i}$*. Then* $\mathbf{Q}' := \{Q'_{x,i}\}$ *is a standard family of injective objects of* $\mathcal{C}'$ *(see* 1.28*) so the corresponding family* $\mathbf{T}' = \{\mathcal{T}'_{x,i}\}_{x,i}$ *of tilting* $\mathcal{A}'$ *modules identifies with* $\mathbf{Q}'$*, and one has* $\operatorname{end}(\mathbf{T}')^{\mathrm{op}} \cong k' \otimes_k \operatorname{end}(\mathbf{T})^{\mathrm{op}}$ *naturally.*

Recall that $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ in Examples 2 and 3 are strongly stratified. One therefore has very good base change over $k = \mathbb{R} \otimes_{\mathbb{R}} S$ or over $k = S \otimes_{\mathbb{R}} \mathbb{R}$ in Example 3. In Example 2, one has very good base change over $k = \mathbb{R} \otimes_{\mathbb{R}} S^K$ or over $k = S^J \otimes_{\mathbb{R}} \mathbb{R}$ (e.g. since each ring of invariants $S^{d^{-1}Jd \cap K}$ is f.g. free over S^K, as is well known). In either Example 2 or Example 3, if $k \cong S$ above then the natural map $k/k_{>0} \cong \mathbb{R} \hookrightarrow \mathbb{C}$ makes $k' = \mathbb{C}$ into a k-algebra. Let $\mathcal{D} := k' \otimes_k \mathcal{C}$ and $\mathcal{E}' \cong \mathcal{D}^\dagger$; then the corresponding category $\mathcal{E}'_{\mathrm{fin}}$ is conjecturally closely analogous to (a graded analogue of) a block of $\mathcal{O}$ as in Example 1.

1.36. **Base change of adjoint functors.** In Example 2 and related integral variants, the categories one obtains by base change from $\mathcal{C}$ are related to one another by various adjoint functors which are closely analogous to the translation functors and projective functors (see [5]) for the blocks of $\mathcal{O}$ for a semisimple complex Lie algebra. The construction of translation functors in situations like Example 2 but with finite weight posets is indicated in [23, Section 3, 5.6]; the construction there can be extended to infinite weight posets using 1.8 and 1.16 in place of [23, 1.9]. The following general fact then often enables one to construct analogues of translation functors and projective functors for categories which are obtained from those examples by base change.

Theorem. *Suppose that* $F^i : \mathcal{C}^i \to \mathcal{C}^{i+1}$ *are bistable exact* k*-functors over* G *between stratified exact* k*-categories* $\mathcal{C}^i$ *over* G *with* F^i *left adjoint to* F^{i+1} *for all* $i \in \mathbb{Z}$ *(so* $\mathcal{C}^i = \mathcal{C}^{i+2}$*). Assume* Δ^i_x *is* k*-flat for all* $x \in \Omega^i$*, for each* $i \in \mathbb{Z}$*. Identify* $\mathcal{C}^i$ *(resp.,* $\mathcal{D}^i := k' \otimes_k \mathcal{C}^i$*) as subcategories of* $\mathcal{A}^i$-mod *(resp.,* $\mathcal{A}^{i\prime}$-mod*) for the associated left stratified rings and let* $L^i : \mathcal{C}^i \to \mathcal{D}^i$ *denote the natural exact functor induced by* $k' \otimes_k ?$ *where* k' *is a graded commutative* k*-algebra.*

Then there exist natural bistable exact functors $H^i = k' \otimes_k F^i : \mathcal{D}^i \to \mathcal{D}^{i+1}$ *with* $H^{i+1} L^i \cong L^{i+1} F^i$ *and with* H^i *left adjoint to* H^{i+1} *for all* i*. One has then corresponding families of adjoint functors* $\hat{H}^i : \hat{D}^i \to \hat{D}^{i+1}$*,* $H^i_* : \hat{\mathcal{E}}^i \to \hat{\mathcal{E}}^{i+1}$ *and* $H^i_\dagger : \mathcal{D}^{i\,\dagger} \to \mathcal{D}^{i+1\,\dagger}$*. Moreover, if* $K^i : \mathcal{C}^i \to \mathcal{C}^{i+1}$ *are functors satisfying the same* *conditions as the F^i, a natural transformation $\eta\colon F^i \to K^i$ induces in a natural way a natural transformation $k' \otimes_k \eta\colon k' \otimes_k F^i \to k' \otimes_k K^i$.*

1.37. **Ungrading functors.** There is a conjectural equivalence [23, Conjecture 8] (one of a larger family mentioned in [23, 9.13]) between a category $\mathcal{C}$ like those in Example 1 (over $A = S$, in case the block Ω contains the "sum of fundamental weights" ρ) with an ungraded analogue of the category $\mathcal{C} = {}^{\emptyset}\mathcal{C}^{\emptyset}_{-l_0}$ in Example 2. This equivalence would imply equivalences of corresponding categories over $\mathbb{C}$ arising by base change; if W is a finite Weyl group, the equivalence over $A = \mathbb{C}$ is actually a theorem (it follows from [50]). Such conjectural equivalences suggest the very interesting question of whether $\mathcal{C}$ (and thus $\mathcal{C}^{\dagger}$) over $A = S$ and $A = \mathbb{C}$ in Example 1 have natural graded versions in general. Immediately below, we discuss the much easier passage from a graded to ungraded representation theory and in 1.47 we collect some simple but useful general facts which facilitate comparison of the graded and ungraded representation theories under conditions which are known or expected to hold in many natural situations such as those mentioned above.

Suppose C is an exact category over G and D is an exact category. A functor $F\colon C \to D$ will be called an ungrading functor (resp., weak ungrading functor) if $FT_g = F$ for all translations T_g, $g \in G$ of C, and the natural maps

$$\mathrm{ext}^i_C(M,N) \to \mathrm{Ext}^i_D(FM, FN)$$

are isomorphisms for all M, N in C and i (resp., $i \leq 1$). For instance, if A is a G-graded (diagonalizable) ring which is (left) Noetherian even as ungraded diagonalizable ring, then the forgetful functor A-modfg $\to$ A-Modfg is well known to be an ungrading functor.

There is a similar ungrading functor naturally associated to any left stratified ring A as follows.

Suppose that A is a left stratified G-graded ring with weight poset Ω and admissible quotients $\mathcal{A}_\Gamma$ for $\Gamma \in I_1$. One obtains a left stratified ungraded ring $A' := \varprojlim_{\Gamma \in I_1^{\mathrm{op}}} A'(\Gamma)$ with admissible quotients A'_Γ equal to the ungraded rings underlying $A(\Gamma)$; moreover, A' is stratified iff A is stratified.

Let $\mathcal{C}$, $\mathcal{C}'$ denote the stratified exact categories associated by 1.32 to A and A' respectively. The forgetful functors (forgetting G-grading) $A(\Gamma)$-mod $\to$ $A(\Gamma)'$-Mod induce exact functors $F\colon \mathcal{C} \to \mathcal{C}'$ (trivially bistable) and $F_{\dagger}\colon \mathcal{C}^{\dagger} \to \mathcal{C}'^{\dagger}$. It is easy to see that F is an ungrading functor as defined above (e.g. by calculating $\mathrm{ext}^i(M,N)$ in $\mathcal{C}_\Gamma$ for some $\Gamma \in I_1$ using a projective resolution in $\mathcal{C}_\Gamma$).

Remarks. Using 13.5, one can give a similar result to 1.36 about "forgetting the grading" on adjoint functors. There is also an analogue of 1.35 for ungrading functors, which is essentially trivial.

1.38. **Grothendieck Groups.** We define as usual the Grothendieck group $\mathrm{K}_0(C)$ (B.16) of a svelte exact category C over G. We then regard $\mathrm{K}_0(C)$ as a module over the integral group ring $\mathbb{Z}[G]$ with $g[M]_C = [T_g M]_C$ for M in C. In particular, we may define Grothendieck groups of $\mathcal{C}$, $\mathcal{P}$, $\hat{\mathcal{C}}$, $\hat{\mathcal{P}}$, $\mathcal{C}_\Gamma$ etc. We also define a "completion" $\widehat{\mathrm{K}}_0(\mathcal{C})$ of $\mathrm{K}_0(\mathcal{C})$. Namely, $\widehat{\mathrm{K}}_0(\mathcal{C})$ is the $\mathbb{Z}[G]$-submodule of $\prod_{x\in\Omega} \mathrm{K}_0(\mathcal{C}_x)$ consisting of all families $\{a_x\}_{x\in\Omega}$ with $a_x \in \mathrm{K}_0(\mathcal{C}_x)$ such that for each $\Gamma \in I_1$, there are only finitely many $x \in \Gamma$ with $a_x \neq 0$. There is a natural $\mathbb{Z}[G]$-module homomorphism $\pi\colon \mathrm{K}_0(\hat{\mathcal{C}}) \to \widehat{\mathrm{K}}_0(\mathcal{C})$ given by $[M] \mapsto \{[M(x)]_{\mathcal{C}_x}\}_{x\in\Omega}$, and a natural $\mathbb{Z}[G]$-module homomorphism $\iota\colon \mathrm{K}_0(\hat{\mathcal{P}}) \to \mathrm{K}_0(\hat{\mathcal{C}})$ induced by the inclusion $\hat{\mathcal{P}} \to \hat{\mathcal{C}}$.

The main facts about these Grothendieck groups are listed below and proved in Section 14.

Theorem. (a) *The exact functors* $\tau_x \colon \mathcal{C} \to \mathcal{C}_x$ *define an isomorphism* $\mathrm{K}_0(\mathcal{C}) \cong \oplus_{x\in\Omega} \mathrm{K}_0(\mathcal{C}_x)$ *given by* $[M]_{\mathcal{C}} \mapsto ([M(x)]_{\mathcal{C}_x})_{x\in\Omega}$ *for* M *in* $\mathcal{C}$*.*

(b) *If* $\Omega \in I_1$*, the inclusion of* $\mathcal{P}$ *in* $\mathcal{C}$ *induces an isomorphism* $\mathrm{K}_0(\mathcal{P}) \cong \mathrm{K}_0(\mathcal{C})$*.*

(c) *If* Ω *is finite and* $\mathcal{A}$ *is Noetherian and of finite left graded global dimension, the inclusion of* $\mathrm{add}\,_{\mathcal{A}}\mathcal{A}$ *in* $\mathcal{A}$-modfg *induces an isomorphism of their Grothendieck groups. Then one has natural isomorphisms*

$$\mathrm{K}_0(\mathcal{C}) \cong \mathrm{K}_0(\mathcal{P}) \cong \mathrm{K}_0(\mathcal{A}\text{-modfg})$$

provided idempotents split in $\mathcal{C}$*.*

(d) *If* Ω *is finite, then* $\mathcal{A}$ *has finite left graded global dimension if it is stratified and for each* $x \in \Omega$*,* R_x *has finite left graded global dimension.*

(e) *The composite map* $\mathrm{K}_0(\hat{\mathcal{P}}) \xrightarrow{\iota} \mathrm{K}_0(\hat{\mathcal{C}}) \xrightarrow{\pi} \widehat{\mathrm{K}}_0(\mathcal{C})$ *is a* $\mathbb{Z}[G]$*-module isomorphism. In particular,* ι *(resp.,* π*) is a split monomorphism (resp., split epimorphism) and* $\mathrm{K}_0(\hat{\mathcal{P}}) \cong \widehat{\mathrm{K}}_0(\mathcal{C})$*.*

This result is important for its applications to the combinatorial study of positivity properties of Iwahori-Hecke algebras. In Example 2 (with $J = K = \emptyset$) the Grothendieck group of $\mathrm{K}_0(\mathcal{C})$ (resp., $\widehat{\mathrm{K}}_0(\mathcal{C})$) identifies naturally with the left regular module of the Iwahori-Hecke algebra $\mathcal{H}$ of W over $\mathbb{Z}[v, v^{-1}]$ (resp., a completion of the left regular module essentially as in [19, Section 4]) with the $\mathcal{H}$-action provided by projective functors acting on $\mathcal{C}$ (resp.,$\hat{\mathcal{C}}$). In these examples, N_x is (resp., P_x can be chosen to be) indecomposable in $\mathcal{C}$ (resp., $\hat{\mathcal{C}}$), as follows from results below, and their classes in the Grothendieck group provide a standard basis (resp. "basis") of the Grothendieck group with respect to which the action of the projective functors is given by infinite matrices with entries in $\mathbb{N}[v, v^{-1}]$ (these "bases" depend in a highly non-trivial way on the chosen length function l). Many conjectures (see e.g. [20, Conjecture 3] and [19, 4.19(2)]) on positivity properties of structure constants of Iwahori-Hecke algebras would follow provided these bases identify with standard, combinatorially defined "Kazdhan-Lusztig bases" of the Grothendieck group; the identification of these bases is essentially the content of the Kazdhan-Lusztig conjecture for Example 2, as discussed subsequently (see 1.49). For crystallographic reflection representations, one obtains "bases" with similar properties for each prime p, from the corresponding characteristic p representation theories (these "bases" depend on p, l and the chosen crystallographic reflection representation of W).

Remarks. The results of Section 4 in conjunction with Matlis theory of injective modules over (graded) commutative Noetherian rings permit a fairly good description of injectives in $\mathcal{C}^*$ whenever the R_x are commutative, unital graded Noetherian rings (e.g. in Examples 1–3). It is possible that several of the results we mention involving projective objects of $\hat{\mathcal{C}}$, such as the above applications of translation and projective functors to the combinatorics of Iwahori-Hecke algebras, could be reformulated perhaps more naturally (at least in those examples) in terms of injectives in $\mathcal{C}^*$.

1.39. **Local Rings.** We now discuss conditions under which the $N_{x,i}$ and $P_{x,i}$ in general can be chosen indecomposable, and some of their other implications. The following fact (see C.20) plays an important role in these results.

Lemma. *Let* $\mathbf{M} = \{M_i\}_i$ *be a family of objects with local endomorphism rings* $\operatorname{End}(M_i)^{\mathrm{op}}$ *in an additive category* C *with group* G *of automorphisms, such that for all* i, j, g *one has* $M_i \not\cong M_j\langle g\rangle$ *unless* $i = j$ *and* $g = 1_G$ *(we will say for short that* $\mathbf{M}$ *is a Krull-Schmidt family (over* G*) in* C *when these conditions hold). Set* $A := \operatorname{End}(\mathbf{M})^{\mathrm{op}}$*. Then*

(a) *finitely-generated* A*-modules have projective covers (in fact,* A *is a basic semiperfect ring as defined in* C.20*)*
(b) $e_i(A/\operatorname{rad} A)_g e_j = 0$ *unless* $i = j$ *and* $g = 1_G$
(c) $e_i(A/\operatorname{rad} A)_{1_G} e_i \cong \operatorname{End}(M_i)^{\mathrm{op}}/\operatorname{Rad}\operatorname{End}(M_i)^{\mathrm{op}}$*. In particular* $e_iAe_i = \operatorname{end}(M_i)^{\mathrm{op}}$ *is a graded local ring with trivially graded residue ring*
(d) *Idempotents split in the split exact category* $\operatorname{add}\mathbf{M}$*, and every object of* $\operatorname{add}\mathbf{M}$ *is isomorphic to a direct sum of objects* $M_i\langle g\rangle$ *with uniquely determined finite multiplicities.*

Remarks. The notions of Krull Schmidt families, basic semisimple rings, basic semiperfect rings and graded local rings with trivially graded residue ring are somewhat special but are well suited to our intended applications. The results we shall describe below under the assumption that each $\mathbf{N}_x$ is a Krull-Schmidt family can be extended, mutatis mutandis, to the slightly more general situation in which $\mathcal{C}_x = \operatorname{add}\mathbf{N}_x$ with all $\operatorname{End}(N_{x,i})^{\mathrm{op}}$ local, using C.22; the main difference is that the subgroups $G_{x,i}$ of G consisting of elements $g \in G$ satisfying $N_{x,i} \cong N_{x,i}\langle g\rangle$ may be non-trivial, as a result of which isomorphism classes of various objects $\Delta_{x,i}$, $P_{x,i}$, $\nabla_{x,i}$, $L_{x,i}$ etc must be parametrized differently and certain G-graded groups appearing in the theory need no longer be trivially graded in general.

1.40. **Indecomposable projective objects.** Assume unless otherwise indicated until 1.48 that each family $\mathbf{N}_x$ with $x \in \Omega$ is a Krull-Schmidt family (note that this hypothesis holds in Examples 2–3 and it also holds in Example 1 if $A = \mathbb{C}$ or $A = S_{\mathfrak{m}_0}$). It follows that idempotents split in $\mathcal{C}_x$ for all x and hence idempotents split in $\mathcal{C}$ (see 3.10).

Note $\operatorname{ext}^1(P_{i,x}(< y), \mathbf{N}_y)$ is automatically a f.g. right R_y-module. By the construction of $\mathbf{P}$ in Section 7, we may (and do) assume without loss of generality that each object $P_{x,i}$ in $\mathbf{P}$ is chosen to satisfy the following equivalent conditions:

(i) $g_{P_{i,x},y}$ of 1.18 is a projective cover of $\operatorname{ext}^1(P_{i,x}(< y), \mathbf{N}_y)$ as right R_y-module.
(ii) the map $g_{\mathbf{P}_{y,i},x} \otimes_{R_x} \operatorname{Id}_{R_x/J_x}$ is an isomorphism for all $y < x$.

If Ω is finite and $\mathcal{C}$ has enough injectives, we assume also that the standard projectives $Q^{\mathrm{op}}_{x,i}$ in $\mathcal{C}^{\mathrm{op}}$ are chosen in the same way.

Theorem. (a) *Every object* M *of* $\hat{\mathcal{C}}$ *has a projective cover and hence a minimal projective resolution.*
(b) *The object* $P_{x,i}$ *is a projective cover of* $N_{x,i}$ *in* $\hat{\mathcal{C}}$*.*
(c) *Any projective object* P *in* $\hat{\mathcal{C}}$ *is a convergent direct sum with uniquely determined finite multiplicities of translates of objects in* $\mathbf{P}$*.*
(d) *Define the graded local ring* $S_{y,i} := \operatorname{end}(P_{y,i})^{\mathrm{op}}$*. The natural epimorphism* $\alpha\colon S_{y,i} \to \operatorname{end}(P_{y,i}(y))^{\mathrm{op}} = \operatorname{end}(N_{y,i})^{\mathrm{op}} = R_{y,i}$ *induces an isomorphism* $S_{y,i}/\operatorname{rad} S_{y,i} \cong R_{y,i}/J_{y,i}$ *of trivially graded division rings.*
(e) *There is an isomorphism* $\mathcal{A}/\operatorname{rad}\mathcal{A} \cong \oplus_{y,i} R_{y,i}/J_{y,i}$ *of trivially graded rings (on the right hand side, we have a decomposition as a direct sum of two sided ideals which are division rings).*

1.41. **Highest weight modules.** Let us say that a module M in $\mathcal{E}$ is a highest weight module of highest weight (x,i) and degree g if it is generated by an element v of $e_{x,i}M_g$ and satisfies $e_{y,j}M = 0$ for all $y > x$. Define the $\mathcal{A}$-module

$$\overline{\Delta}_{x,i} = \Delta_{x,i} \otimes_{R_{x,i}} R_{x,i}/J_{x,i} = \Delta_{x,i}/\Delta_{x,i}J_{x,i}.$$

Clearly, $\overline{\Delta}_{x,i}$ is a $\mathcal{A}_{\leq x}$-module, so it is in $\mathcal{E}$.

Theorem. (a) *The module $\Delta_{x,i}\langle g\rangle$ is a universal highest weight module of highest weight (x,i) and degree g, in the sense that any highest weight module of highest weight (x,i) and degree g is a quotient of $\Delta_{x,i}\langle g\rangle$*

(b) *$\Delta_{x,i}$ has a unique maximal graded submodule* rad $\Delta_{x,i}$*, with corresponding irreducible quotient module $L_{x,i}$.*

(c) *One has* $\mathrm{end}(L_{x,i})^{\mathrm{op}} \cong R_{x,i}/J_{x,i}$*, $L_{x,i} = (e_{x,i}L_{x,i})_{1_G}$ and $e_{y,j}L_{x,i} = 0$ if $(y,j) \neq (x,i)$.*

(d) *Any irreducible object in $\mathcal{E}$ is isomorphic to $L_{x,i}\langle g\rangle$ for uniquely determined x, i and g.*

(e) *Any simple subquotient of* rad $\Delta_{x,i}$ *is isomorphic to $L_{y,j}\langle g\rangle$ for some $y \leq x$, j and $g \in G$.*

(f) *$\overline{\Delta}_{x,i}$ has a unique maximal graded submodule* rad $\overline{\Delta}_{x,i} = \oplus_{(y,j)\neq(x,i)} e_{y,j}\overline{\Delta}_{x,i}$ *and $\overline{\Delta}_{x,i}/\mathrm{rad}\ \overline{\Delta}_{x,i} \cong L_{x,i}$.*

(g) *Any simple subquotient of* rad $\overline{\Delta_{x,i}}$ *is isomorphic to $L_{y,j}\langle g\rangle$ for some $g \in G$ and $(y,j) \neq (x,i)$ with $y \leq x$.*

(h) $\mathrm{end}(\overline{\Delta}_{x,i})^{\mathrm{op}} \cong R_{x,i}/J_{x,i}$.

1.42. **Indecomposability.** The following lemma records the useful fact that under our assumptions, the various families of objects we have constructed consist of indecomposable objects, pairwise non-isomorphic up to degree shift, with local endomorphism rings.

Lemma. *The following families of objects are Krull-Schmidt families:*

(a) *$\{P_{x,i}\}_{x,i}$ in $\hat{\mathcal{C}}$ and equivalently $\{\mathcal{A}e_{x,i}\}_{x,i}$ in* $\mathcal{A}$-mod

(b) *$\{\Delta_{x,i}\}_{x,i}$ $\{\nabla_{x,i}\}_{x,i}$, $\{\overline{\Delta}_{x,i}\}_{x,i}$, $\{\mathcal{A}_\Gamma e_{x,i}\}_{x,i}$ for $\Gamma \in I_1$, and $\{L_{x,i}\}_{x,i}$ in $\mathcal{E}$.*

(c) *$\{Q_{x,i}\}_{x,i}$ in $\mathcal{C}$ and equivalently $\{\mathcal{T}e_{x,i}\}_{x,i}$ (the tilting modules) in $\mathcal{E}$ (if Ω is finite and $\mathcal{C}$ has enough injectives).*

1.43. **Composition factor multiplicity.** Recall $e_{x,i}\mathcal{A}e_{x,i}$ is a graded local ring with trivially graded residue ring, and any irreducible $e_{x,i}\mathcal{A}e_{x,i}$-module is isomorphic to $e_{x,i}L_{x,i}\langle h\rangle = L_{x,i}\langle h\rangle$ for a unique $h \in G$. Hence $e_{x,i}\mathcal{A}_{1_G}e_{x,i}$ is an ungraded local ring and $e_{x,i}L_{x,i}$ is its unique simple module.

For M in $\mathcal{E}_{\mathrm{wfin}}$, define the "composition factor multiplicity" of $L_{x,i}\langle g\rangle$ in M by

$$[M : L_{x,i}\langle g\rangle] := [e_{x,i}M \colon e_{x,i}L_{x,i}\langle g\rangle]$$

where on the right hand side we have the usual composition factor multiplicity of $e_{x,i}L_{x,i}\langle g\rangle$ in $e_{x,i}M_g$ as a $e_{x,i}\mathcal{A}e_{x,i}$-module. If M is in $\mathcal{E}_{\mathrm{fin}}$, then $\sum_g [M : L_{x,i}\langle g\rangle]$ is finite (equal to the length of $e_{x,i}M$ as graded $e_{x,i}\mathcal{A}e_{x,i}$-module).

Observe that $[M : L_{x,i}\langle g\rangle] = 0$ if $e_{x,i}M_g = 0$; in particular, $[L_{y,j}\langle h\rangle : L_{x,i}\langle g\rangle]$ is zero unless $x = y$, $i = j$ and $h = g$, in which case it equals 1. Given a short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{E}_{\mathrm{wfin}}$, one clearly has

$$[M : L_{x,i}\langle g\rangle] = [L : L_{x,i}\langle g\rangle] + [N : L_{x,i}\langle g\rangle].$$

If M in $\mathcal{E}_{\mathrm{wfin}}$ has a composition series then $[M : L_{x,i}\langle g\rangle]$ is just the usual composition factor multiplicity; a similar statement holds for "local" composition series, which are defined in a manner similar to that for category $\mathcal{O}$ (see 15.7).

1.44. **Brauer-Humphreys (BGG) Reciprocity.** Suppose that $\mathcal{C}$ has k-structure where k is a field. Let $A = \mathbb{Z}[G]$ be the integral group ring of G (with the elements of G as $\mathbb{Z}$-basis) and $\hat{A} = \prod_g \mathbb{Z}g$ be its completion consisting of arbitrary formal $\mathbb{Z}$-linear combinations of elements of G. We regard $\hat{A}$ as a left A-module in the natural way.

Define the following Poincaré series. Set $((P_{x,i} : N_{y,l})) := \sum_g n_{g,l}g \in A$ where $n_{g,l}$ is the number of times $T_gN_{y,l} = N_{y,l}\langle g^{-1}\rangle$ occurs as a direct summand of $P_{x,i}(y)$ in $\mathcal{C}_y$ (note this is well-defined by 1.39 and that for fixed x, i and y, $((P_{x,i} : N_{y,l}))$ is non-zero for only finitely many l). For a G-graded k-vector space $V = \oplus_g V_g$ with each V_g finite-dimensional, define $\{\{V\}\} := \sum_g \dim_k(V_g)\, g \in \hat{A}$. Finally, for M in $\mathcal{E}_{\mathrm{wfin}}$, define $[[M : L_{x,j}]] := \sum_g [M : L_{x,j}\langle g\rangle]\, g \in \hat{A}$.

Proposition. *If $(e_iR_xe_j)_g$ is finite-dimensional over k for all i, j, x and g then each $L_{x,i}$ is finite-dimensional over k, each $\nabla_{y,j}$ is in $\mathcal{E}_{\mathrm{wfin}}$ and*

$$\sum_l ((P_{x,i} : N_{y,l}))\{\{e_lR_ye_j\}\} = (\dim_k L_{x,i})[[\nabla_{y,j} : L_{x,i}]].$$

This is a version of so-called Brauer-Humphreys reciprocity or BGG reciprocity (see e.g. [10, 3.11] or [32]). The hypotheses are satisfied taking $k = \mathbb{R}$ in Example 2 and Example 3, for instance.

Remarks. We mention the formal fact 15.9 which shows how the multiplicities in 1.40(c) may be characterized in terms of "Shapovalov maps" $f_{p,y}$ which are closely related to the maps $g_{P,y}$ for projective P in Theorem 1.18(a).

1.45. **Preservation of indecomposability.** It is useful to have conditions under which functors $\hat{F}$, $F_\dagger$ etc induced by a base change or ungrading factor $F\colon \mathcal{C} \to \mathcal{D}$ between stratified exact categories with the same weight poset send indecomposable objects of various types (e.g. projectives, irreducibles, $\boldsymbol{\Delta}$-modules, $\boldsymbol{\nabla}$-modules, tilting modules etc) associated to $\mathcal{C}$ to corresponding indecomposable objects of the same type associated to $\mathcal{D}$. Equivalently, such results can be viewed as giving conditions under which indecomposable objects of each type associated to $\mathcal{D}$ "lift" to indecomposables of the same type associated to $\mathcal{C}$. We give two simple results in this vein below; they are proved in Section 17 along with some related facts.

1.46. **Indecomposability and base change.** Assume in this subsection that $\mathcal{C}$ has k-structure, where k is a commutative G-graded unital ring, that $k' = k/J$ is a graded quotient ring of k, and that each Δ_x for $\mathcal{C}$ is k-flat. We continue to assume that each Δ_x is a Krull-Schmidt family and that $P_{x,i}$ is chosen to be indecomposable.

Fix k' as above and define the functor $k'\otimes_k ? : \mathcal{E} \to k' \otimes_k \mathcal{E}$, which we denote by $M \mapsto M'$. Recall $\mathcal{E}' := k' \otimes_k \mathcal{E}$ is a category of $\mathcal{A}'$-modules where $\mathcal{A}' = \varprojlim_{\Gamma\in I} \mathcal{A}'_\Gamma$ and $\mathcal{A}'_\Gamma := k' \otimes \mathcal{A}_\Gamma$, and that $\mathcal{C}' := k' \otimes_k \mathcal{C}$ is a full subcategory of $\mathcal{E}'$. We have the associated category $\hat{\mathcal{C}}' := k' \otimes_k \hat{\mathcal{C}}$ of pro-objects of $\mathcal{C}'$ and the standard projective objects $P'_{x,i} := \{\mathcal{A}'_\Gamma e_{x,i}\}_{\Gamma\in I}$ in $\hat{\mathcal{C}}'$.

Lemma. *Suppose that for each $x \in \Omega$ and $i \in I_x$, $JR_{x,i} \subseteq \operatorname{rad} R_{x,i}$. Then*

(a) *The families of objects* $\{P'_{x,i}\}_{x,i}$ *in* $\hat{\mathcal{C}}'$ *and* $\{\Delta'_{x,i}\}_{x,i}$, $\{\nabla'_{x,i}\}_{x,i}$, $\{\overline{\Delta}'_{x,i}\}_{x,i}$, $\{\mathcal{A}_\Gamma e'_{x,i}\}_{x,i}$ *with* $\Gamma \in I_1$ *and* $\{L'_{x,i}\}_{x,i}$ *in* $\mathcal{E}'$ *are Krull-Schmidt families (as are the tilting modules* $\{\mathcal{T}'_{x,i}\}_{x,i}$ *if* Ω *is finite and* $\mathcal{C}$ *has very good base change over* k*).*

(b) *The modules* $L'_{x,i}\langle g\rangle$ *for all* x, i, g *form a full set of representatives of isomorphism classes of simple module in* $\mathcal{E}'$.

1.47. **Ungrading Functors and Indecomposability.** We continue to assume that the $\mathbf{\Delta}_x$ are Krull-Schmidt families and $P_{x,i}$ is chosen indecomposable. Let $\mathcal{A}' = \varprojlim_{\Gamma\in I_1} \mathcal{A}(\Gamma)'$ (limit as ungraded diagonalizable ring) where $\mathcal{A}(\Gamma)'$ is obtained by forgetting the grading on $\mathcal{A}(\Gamma)$. Let $\mathcal{C}'$ be the ungraded version of $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{C}'$ be the ungrading functor as constructed in 1.37. Recall that when we consider the underlying ungraded ring or module of a G-graded ring or module, we write rad (resp., Rad) to denote the radical as graded (resp., ungraded) module.

Proposition. *Assume that the natural isomorphism* $\operatorname{end}(\Delta_{x,i})^{\mathrm{op}} \to \operatorname{End}(F_\dagger\Delta_{x,i})^{\mathrm{op}}$ *(forgetting the grading on the domain) induces an isomorphism* $\operatorname{rad}\operatorname{end}(\Delta_{x,i})^{\mathrm{op}} \cong \operatorname{Rad}\operatorname{End}(F_\dagger\Delta_{x,i})^{\mathrm{op}}$ *for each* x *and* i *(this holds for instance if* $\operatorname{end}(N_{x,i})^{\mathrm{op}}$ *is a (left or right) graded Artinian ring).*

(a) *The family* $\{\hat{F}P_{x,i}\}_{x,i}$ *in* $\hat{\mathcal{C}}'$ *and also the families* $\{F_\dagger\Delta_{x,i}\}_{x,i}$, $\{F_\dagger\nabla_{x,i}\}_{x,i}$, $\{F_\dagger\overline{\Delta}_{x,i}\}_{x,i}$, $\{F_\dagger\mathcal{A}_\Gamma e_{x,i}\}_{x,i}$ *for* $\Gamma \in I_1$ *and* $\{F_\dagger L_{x,i}\}_{x,i}$ *in* $\mathcal{E}'$ *are all Krull-Schmidt families (as is the family of tilting modules* $\{F_\dagger\mathcal{T}_{x,i}\}_{x,i}$ *if* Ω *is finite and* $\mathcal{C}^{\mathrm{op}}$ *and* $\mathcal{C}$ *are strongly stratified).*

(b) *Any simple object of* $\mathcal{E}'$ *is isomorphic to* $F_\dagger L_{x,i}$ *for unique* x *and* i

(c) *For* $\Gamma \in I$, *one has* $\operatorname{rad}\mathcal{A}(\Gamma) = \operatorname{Rad}\mathcal{A}(\Gamma)$

(d) *If* M *is in* $\mathcal{E}$, *then* $F_\dagger M$ *is in* $\mathcal{E}'_{\mathrm{fin}}$ *iff* M *is in* $\mathcal{E}_{\mathrm{fin}}$. *In that case, one has* $\sum_{g\in G}[M : L_{x,i}\langle g\rangle] = [F_\dagger M : F_\dagger L_{x,i}]$.

(e) *For* M *in* $\mathcal{E}$, $\operatorname{Soc} F_\dagger M = F_\dagger \operatorname{soc} M$ *and* $\operatorname{Rad} F_\dagger M = \operatorname{rad} M$ *where* soc *and* rad *(resp.,* Soc *and* Rad *) denote the socle and radical in* $\mathcal{E}$ *(resp.,* $\mathcal{E}'$*).*

1.48. **Conjectures.** The most basic open problem in the theory described so far for Examples 2 and 3 is to determine the composition factor multiplicities $[\overline{\Delta}_x : L_y\langle g\rangle]$. There is an explicit Kazdhan-Lusztig conjecture which states how (in the characteristic zero theories we consider here), the $[\overline{\Delta}_x : L_y\langle g\rangle]$ should be determined by Kazdhan-Lusztig polynomials in Example 2 (resp., g-polynomials of face lattices of convex polytopes in Example 3). Using the reciprocity law 1.44 (and the duality constructed in subsequent papers) the problem of determining the multiplicities is equivalent to the determination of the Poincaré series $((P_y : N_x))$, and the Kazdhan-Lusztig conjecture can be reformulated as the statement that the "basis" of the Grothendieck group of $\hat{\mathcal{C}}$ corresponding to the projective objects P_x corresponds to an appropriate combinatorially defined "Kazdhan-Lusztig" basis.

We don't explicitly formulate these conjectures here, but mention that in Example 2 (resp., Example 3) the Kazdhan-Lusztig conjecture reduces trivially to the case where $J = K = \emptyset$ and the weight poset Ω is replaced by a finite interval of W in the order $\leq$ (resp., where Ω is the face lattice of a single polyhedral cone). In these cases, precise statements of the conjectures are given in [23, 4.3. Conjecture 1, 4.8].

1.49. We will show in subsequent papers that the Kazdhan-Lusztig conjecture in both Example 2 and Example 3 is equivalent to the following conjecture (cf. [50]):

Conjecture. In Example 2 or 3, the algebra $\mathcal{A}$ is positively graded, and $\mathcal{A}_0 = \oplus_{x\in\Omega}\mathbb{R}e_x$.

For finite Weyl groups in Example 2, the validity of Conjecture 1.49 follows using results sketched in [23] from the known Kazdhan-Lusztig conjecture for semisimple complex Lie algebras and [50]. We will prove the conjecture for Example 3 in subsequent papers in case Ω is the face lattice of a simplicial polytope; in that case, a key ingredient is the hard Lefschetz theorem for the polytope algebra of simple polytopes [39]. We will also prove the Kazdhan-Lusztig conjecture in Example 2 "generically" in the case $l = -l_0$ (we don't define "generically", but indicate that the result gives a direct proof of the Kazdhan-Lusztig conjecture for $l = -l_0$ if W is of types A_3 or B_3 or has no braid relations, and slightly more generally than the last, implies $[[\overline{\Delta}_x : L_y]]$ is as conjectured for $l = -l_0$ provided every z with $y \leq z \leq 1_W$ has a unique reduced expression).

1.50. Let us also state the following basic "Koszulity" conjecture, refining [23, Conjecture 4]. It remains open in both Examples 2 and 3 (although it follows from [3] in Example 2 if W is a finite Weyl group). Some of the conjecture's remarkable consequences were discussed in special cases in [21, 3.15].

Conjecture. In the minimal projective resolution $P^\bullet \to \theta(N_x) \to 0$ of N_x in $\hat{\mathcal{C}}$ in Example 2 or 3, P^j is a convergent direct sum of copies of objects $\{P_y\langle j\rangle\}$ with $y \in \Omega$.

1.51. Another important conjecture about these representation categories in Examples 2 and 3 asserts that "thickened principal series modules" are related by short exact sequences which may be regarded as thickened analogues of Duflo-Zelebenko four-term short exact sequences of principal series modules for semisimple complex Lie algebras (cf e.g. [33]); these conjectures will be proved for Example 3 in subsequent papers but remain open in general in Example 2. The most interesting conjectures about Examples 2 and 3 are certain Hodge-Lefschetz conjectures which apparently underlie 1.49 and 1.50 (they will be formulated precisely elsewhere). Roughly, certain natural objects arising in the study of category $\mathcal{C}$ in Examples 2 and 3 are conjectured to be Cohen-Macaulay self-dual graded modules over commutative graded rings; modulo appropriately chosen homogeneous systems of parameters, the resulting self-dual finite-dimensional graded vector spaces are conjectured to have natural structures of polarized graded spaces of Hodge-Lefschetz type in the sense of [49] (if one imposes suitable Hodge structures which are just Tate twists of trivial ones; the main point is that the spaces conjecturally satisfy a hard Lefschetz theorem and Riemann-Hodge inequalities). For example we conjecture this is true of $\mathrm{ext}^1(P_y(< x), N_x)\langle -1\rangle = (e_y\nabla_x/e_y\Delta_x)\langle -1\rangle$ as right $\mathbb{R}\otimes_{\mathbb{R}} S$-module in Example 2 (with $K = \emptyset$) or Example 3; a proof of this conjecture would immediately establish Conjecture 1.49. The conjectured Cohen-Macaulay self-duality would follow from properties of the duality on projective objects of $\hat{\mathcal{C}}$ which we construct in subsequent papers together with the conjectures on thickened principal series modules mentioned above. The hard Lefschetz theorem and Riemann-Hodge inequalities are undoubtably much deeper; at present, they have been proved only in Example 3 if Ω is the face lattice of a simplicial convex polytope, by reduction to results in the previously mentioned paper [39]. As another example, we ask if similar Hodge-Lefschetz conjectures hold for the tilting-module weight spaces $e_y\mathcal{T}_x$ as $S \otimes_{\mathbb{R}} S$-module in Example 2 with $J = K = \emptyset$ or in Example 3 (by the results of this paper together with the duality constructed in subsequent papers, these are known to be self dual f.g. graded free as $\mathbb{R} \otimes_{\mathbb{R}} S$-module or $S \otimes_{\mathbb{R}} \mathbb{R}$-module). Here, this conjecture is known to be true in Example 2 for finite Weyl groups for $y = e$, $w = w_0$ and $l = l_0$ by an (essentially trivial) reduction to the hard Lefschetz theorem and Riemann-Hodge inequalities for the cohomology ring of the associated flag variety. One is naturally led to speculate about whether there is some natural graded version of $\mathcal{C}$ over S in Example 1 which satisfies similar conjectures to those mentioned above.

1.52. For crystallographic Coxeter groups and fans of rational polyhedral cones, one expects close relationships between the categories associated to Examples 2 and 3 (and their variants) and interesting categories (of perverse sheaves, D-modules, mixed Hodge modules etc) associated to algebraic varieties such as flag and toric varieties. Such connections with geometric categories provide a possible route to proof of the Hodge-Lefschetz conjectures and Kazdhan-Lusztig conjectures in these special cases. For instance, it would be an interesting problem to establish a relationship between polarizations of graded spaces arising geometrically (e.g. in the theory of mixed Hodge modules) and the (conjectural) algebraically defined polarizations.

In general, however, there are no similarly rich geometric objects known to be associated to non-crystallographic groups and non-rational fans. This raises the question of whether there is some more extensive, functorial setting of categories which behave as if they are geometric origin and include (or are at least closely related to) both natural categories of perverse sheaves etc on algebraic varieties and interesting categories like those associated to Examples 2–3, but which may not be naturally associated to algebraic varieties in general. Whether discovery and study of such a setting proves to be necessary for the proof of the Hodge-Lefschetz conjectures in Examples 2–3 or not, the question of its existence is a natural and interesting one.

2. Filtrations

In this section we shall prove some basic facts on filtrations and truncation functors (especially Lemma 1.4) which play an important role in this paper. The proofs simply involve giving standard arguments from Lie theory involving Verma flags under weaker assumptions than usual. It would be possible to give direct proofs of 1.2 and 1.6 using the facts in this section, but we defer the proofs till the end of Section 3.

2.1. Let $D = (\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Omega})$ where Ω denotes a fixed interval finite poset with partial order $\leq$, $\mathcal{D}$ is an exact category and $\{\mathcal{D}_x\}_{x\in\Omega}$ is a family of full, strict additive subcategories of $\mathcal{D}$ satisfying 1.4(i) and 1.4(ii). We write Ext^i for $\mathrm{Ext}^i_{\mathcal{D}}$.

For any subset Γ of Ω, define

$$D_\Gamma := (\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Gamma}), \qquad D^{\mathrm{op}} := (\mathcal{D}^{\mathrm{op}}, \{\mathcal{D}_x^{\mathrm{op}}\}_{x\in\Omega^{\mathrm{op}}}).$$

Observe that D_Γ and D^{op} also satisfy the conditions above.

The following fact follows immediately from our assumptions using the long exact Ext-sequences, and will be used frequently.

2.1.1. Suppose that M, N in $\mathcal{D}$ have filtrations $M = M^0 \supseteq \cdots \supseteq M^m = 0$ and $N = N^0 \supseteq \cdots \supseteq N^n = 0$ with M^{i-1}/M^i in $\mathcal{D}_{x_i}$ and N^{j-1}/N^j in $\mathcal{D}_{y_j}$. If for all

i and j, $x_i \not\leq y_j$ (resp., x_i and y_j are not comparable in the order $\leq$ on Ω) then $\operatorname{Hom}(M,N)=0$ (resp., $\operatorname{Ext}^1(M,N)=0$).

Remarks. For simplicity of exposition, we will tacitly assume for the proofs of all results in this section that $\mathcal{D}$ is a perfectly exact subcategory of a fixed abelian category $\mathcal{B}$. The case of a general exact category $\mathcal{D}$ can be treated similarly by using B.7(d) and B.8.

2.2. We let $\mathcal{I}$ be the family of all coideals of Ω. Define a D-filtration $\{M(\Gamma)\}_{\Gamma\in\mathcal{I}}$ of an object M of $\mathcal{D}$ to be a family of admissible subobjects $M(\Gamma)$ of M satisfying the conditions below:

(i) $M(\emptyset)=0$ and $M(\Omega)=M$
(ii) $M(\Gamma)\subseteq M(\Lambda)$ if $\Gamma\subseteq\Lambda\in\mathcal{I}$
(iii) $M(\Gamma)/M(\Gamma\setminus\{x\})$ is in $\mathcal{D}_x$ if x is a minimal element of $\Gamma\in\mathcal{I}$
(iv) there exists a finite subset X of Ω such that for $\Gamma\subseteq\Lambda\in\mathcal{I}$, the canonical map $M(\Gamma)\to M(\Lambda)$ is an isomorphism whenever $\Gamma\cap X=\Lambda\cap X$.

Let $\mathcal{D}^0[D]$ denote the full additive subcategory of $\mathcal{D}$ consisting of objects which have a D-filtration (we will see later that $\mathcal{D}^0[D]$ coincides with $\mathcal{C}$ as defined in 1.4).

If $\Gamma\subseteq\Omega$, then $\mathcal{D}^0[D_\Gamma]$ is easily seen to be a full subcategory of $\mathcal{D}^0[D]$, using that coideals of Ω intersect Γ in coideals of Γ; moreover, by (iv), $\mathcal{D}^0[D]$ is the directed union of its subcategories $\mathcal{D}^0[D_\Gamma]$ where Γ ranges over the inclusion-ordered family of finite subsets of Ω.

Observe also that $\mathcal{D}^0[D^{\mathrm{op}}]=(\mathcal{D}^0[D])^{\mathrm{op}}$, with $M^{\mathrm{op}}(\Omega\setminus\Gamma)=(M/M(\Gamma))^{\mathrm{op}}$ for $\Gamma\in\mathcal{I}$ and M in $\mathcal{D}^0[D]$.

Lemma. *Let M, N be objects of $\mathcal{D}$.*

(a) *If M, N have D-filtrations $\{M(\Gamma)\}$ and $\{N(\Gamma)\}$ respectively, then any map $f\colon M\to N$ induces unique compatible maps of subobjects $M(\Gamma)\to N(\Gamma)$ for all $\Gamma\in\mathcal{I}$.*
(b) *A D-filtration of M is unique up to isomorphism if it exists.*
(c) *Let $D'=(\mathcal{D}_x,\{\mathcal{D}_x\}_{x\in\Omega'})$ where Ω' is the set Ω endowed with an interval finite partial order $\leq'$ refining $\leq$ (i.e. such that $x\leq y$ implies $x\leq' y$). Then $\mathcal{D}^0[D]=\mathcal{D}^0[D']$.*

Proof. For (a), observe that by definition $M(\Gamma)$ has a finite filtration with successive subquotients in $\mathcal{D}_x$ for various $x\in\Gamma$, while $N/N(\Gamma)$ has a finite filtration with successive subquotients in $\mathcal{D}_y$ for various $y\in\Omega\setminus\Gamma$. By 2.1.1, one gets

$$\operatorname{Hom}(M(\Gamma),N/N(\Gamma))=0, \tag{2.2.1}$$

proving (a). Then (b) follows from (a) by taking $M=N$ as objects of $\mathcal{D}$ but with possibly different D-filtrations, and f the identity map Id_M.

For the proof of (c), assume without loss of generality that Ω is finite. Since the coideals of Ω' form a subset of $\mathcal{I}$, it is clear that $\mathcal{D}^0[D']$ is a subcategory of $\mathcal{D}^0[D]$. Since $\leq'$ and $\leq$ have a common refinement which is a total order, we may also assume without loss of generality that $\leq'$ is a total order, say $\Omega=\{x_1,\dots,x_n\}$ where $x_1<'\dots<' x_n$ and $x_i<' x_j$ implies $i<j$; we call such a total order $\leq'$ on the set Ω a compatible total order. If, say, x_i and x_{i+1} are not comparable in $\leq$, one has another compatible total order $\leq''$ given by

$$x_1<''\dots<'' x_{i-i}<'' x_{i+1}<'' x_i<'' x_{i+2}<''\dots<'' x_n.$$

We say $\leq''$ is obtained from $\leq'$ by a swap (of x_i and x_{i+1}).

Now an object M of $\mathcal{D}$ is in $\mathcal{D}^0[D']$ iff there exists a filtration

$$M = M^0 \supseteq \ldots \supseteq M^n = 0$$

in $\mathcal{D}$ with M^{i-1}/M^i in $\mathcal{D}_{x_i}$ for $i = 1, \ldots, n$. Suppose given such a filtration. If x_i and x_{i+1} are not comparable in $\leq$, then the conditions on D imply that $\operatorname{ext}^1(M^{i-1}/M^i, M^i/M^{i+1}) = 0$ so $M^{i-1}/M^{i+1} \cong M^{i-1}/M^i \oplus M^i/M^{i+1}$. Thus M has a filtration

$$M = M^0 \supseteq \ldots \supseteq M^{i-1} \supseteq M' \supseteq M^{i+1} \supseteq \ldots \supseteq M^n = 0$$

with $M^{i-1}/M' \cong M^i/M^{i+1}$ and $M'/M^{i+1} \cong M^{i-1}/M^i$ showing M is in $\mathcal{D}^0[D'']$ where $D'' = (\mathcal{D}, \{\mathcal{D}_x\}_{x \in \Omega''})$ and Ω'' denotes Ω in the order $\leq''$. We now claim that

2.2.2. $\mathcal{D}^0[D']$ is independent of the choice of compatible total order Ω', and for each j, the subobject $M(\Gamma) := M^j$ of M depends only on the coideal $\Gamma := \{x_{j+1}, \ldots, x_n\}$ of Ω.

Since any totally ordered (by inclusion) family of coideals of Ω occurs as a family of coideals of some compatible total order $\leq'$ of Ω, it is clear that the subobjects $M(\Gamma)$ for $\Gamma \in \mathcal{I}$ so defined make M an object of $\mathcal{D}^0[D]$ as required.

Now the claim above follows from the following purely combinatorial assertion:

2.2.3. Given any two compatible total orders $\leq'$, $\leq''$ on the set Ω, there is a sequence $\leq' = \leq_0, \leq_1, \ldots, \leq_N = \leq''$ of compatible total orders of Ω such that $\leq_{i-1}$ and $\leq_i$ differ by a swap for each $i = 1, \ldots, N$. Moreover, if Γ is a coideal in both orders $\leq'$ and $\leq''$, one may choose the compatible total orders so Γ is a coideal of $\leq_i$ for all i.

To see this, let $\leq'$ be $x_1 <' \cdots <' x_n$ and $\leq''$ be $y_1 <'' \cdots <'' y_n$. The proof is by induction on n. Note that if $n > 0$, then x_n is a maximal element of Ω in $\leq$. If $x_n = y_i$, say, then y_i is not comparable in Ω with any of $y_{i+1}, \ldots, y_n$ and so by swapping y_i with $y_{i+1}, \ldots, y_n$ in turn we may assume without loss of generality that $x_n = y_n$. The inductive hypothesis gives a sequence of compatible total orderings of $\Omega \setminus \{x_n\}$ (in the partial order induced as a subset of Ω) with successive compatible orderings differing by a swap and containing both compatible total orders $x_1 <' \cdots <' x_{n-1}$ and $y_1 <'' \cdots <'' y_{n-1}$. Adding x_n as a maximum element of each of these total orders gives a sequence of compatible total orders of Ω as required, since x_n is maximal in Ω. □

2.3. By the definition in 1.4, an object of $\mathcal{D}$ belongs to $\mathcal{C}$ iff it is an object of $\mathcal{D}^0[D_{\Gamma'}]$ for some compatible total order Γ' of a finite subset Γ of Ω, where $\mathcal{D}[D_{\Gamma'}] = (\mathcal{D}, \{\mathcal{D}_x\}_{x \in \Gamma'})$. But $\mathcal{D}[D_{\Gamma'}] = \mathcal{D}[D_\Gamma]$, so $\mathcal{C} = \mathcal{D}^0[D]$, proving Lemma 1.4(a).

Note that for coideals $\Lambda \supseteq \Gamma$ of Ω and M in $\mathcal{D}^0[D]$, $M(\Lambda)/M(\Gamma)$ is in $\mathcal{D}^0[D_{\Lambda \setminus \Gamma}]$.

2.4. A commutative square in a category C is said to be bicartesian if it is both a pushout square and a pullback square in C.

Lemma. *For Γ, $\Sigma \in \mathcal{I}$ and an object M of $\mathcal{D}^0[D]$, the commutative square*

$$\begin{array}{ccc} M(\Sigma \cap \Gamma) & \longrightarrow & M(\Sigma) \\ \downarrow & & \downarrow \\ M(\Gamma) & \longrightarrow & M(\Sigma \cup \Gamma) \end{array}$$

is bicartesian in $\mathcal{D}$. Dually, the square obtained by replacing each $M(\Lambda)$ by $M/M(\Lambda)$ is bicartesian.

Proof. Consider first the case $\Gamma \cap \Sigma = \emptyset$. Then no element of Σ is comparable with any element of Γ. We have to show that the admissible monomorphisms $M(\Sigma) \to M(\Sigma \cup \Gamma)$ and $M(\Gamma) \to M(\Sigma \cup \Gamma)$ induce a direct sum decomposition of $M(\Gamma \cup \Sigma)$. By 2.1.1, $\mathrm{Ext}^1(M(\Sigma \cup \Gamma)/M(\Gamma), M(\Gamma)) = 0$. Hence the short exact sequence

$$0 \to M(\Gamma) \to M(\Sigma \cup \Gamma) \xrightarrow{j} M(\Sigma \cup \Gamma)/M(\Gamma) \to 0 \tag{2.4.1}$$

splits. But $M(\Sigma \cup \Gamma)/M(\Gamma)$ is in $\mathcal{D}^0[D_\Sigma]$, so by 2.2.1 a splitting map p for j factors as

$$p : M(\Sigma \cup \Gamma)/M(\Gamma) \xrightarrow{i} M(\Sigma) \xrightarrow{k} M(\Sigma \cup \Gamma)$$

with k the canonical admissible monomorphism. Also, since (2.4.1) splits and 2.1.1 gives $\mathrm{Hom}(M(\Sigma), M(\Gamma)) = 0$, it follows that k factors as

$$k : M(\Sigma) \xrightarrow{l} M(\Sigma \cup \Gamma)/M(\Gamma) \xrightarrow{p} M(\Sigma \cup \Gamma)$$

One readily checks that l is an inverse isomorphism to i, and that it induces the required decomposition

$$M(\Gamma \cup \Sigma) = M(\Gamma) \oplus M(\Sigma \cup \Gamma)/M(\Gamma) = M(\Gamma) \oplus M(\Sigma).$$

Now in general, with $\Gamma \cap \Sigma$ possibly non-empty, consider the following commutative diagram:

$$\begin{array}{ccccc}
M(\Sigma\cap\Gamma) & \longrightarrow & M(\Sigma\cap\Gamma)\oplus M(\Sigma\cap\Gamma) & \longrightarrow & M(\Sigma\cap\Gamma) \\
\| & & \downarrow & & \downarrow \\
M(\Sigma\cap\Gamma) & \longrightarrow & M(\Sigma)\oplus M(\Gamma) & \longrightarrow & M(\Sigma\cup\Gamma) \\
 & & \downarrow & & \downarrow \\
 & & M(\Sigma)/M(\Sigma\cap\Gamma)\oplus M(\Gamma)/M(\Sigma\cap\Gamma) & \xrightarrow{\cong} & M(\Sigma\cup\Gamma)/M(\Sigma\cap\Gamma)
\end{array}$$

Here, letting $i_{\Lambda,\Lambda'} \colon M(\Lambda) \to M(\Lambda')$ denote the canonical admissible monomorphism for $\Lambda \subseteq \Lambda' \in \mathcal{I}$, the left (resp., right) horizontal arrow in the middle row has components $(i_{\Gamma\cap\Sigma,\Gamma}, -i_{\Gamma\cap\Sigma,\Gamma})$ (resp., $(i_{\Sigma,\Sigma\cup\Gamma}, i_{\Gamma,\Sigma\cup\Gamma})$). The maps in the top and bottom rows are the induced ones, and the columns are obvious short exact sequences. The top row is clearly (split) exact; moreover, the isomorphism in the bottom line follows from the previously treated special case with Ω replaced by $\Omega \setminus (\Gamma \cap \Sigma)$ and M replaced by $M/M(\Gamma \cap \Sigma)$). By the 9-lemma, the middle row is a short exact sequence, and the given square is bicartesian as claimed. ☐

2.5. **Truncation functors.** Abbreviate $\mathcal{D}^0[D_\Gamma] = \mathcal{D}^0_\Gamma$ and $\mathcal{D}^0_\Omega = \mathcal{D}^0$.

Let M, N, Q be objects of $\mathcal{D}^0$ and $\Gamma \subseteq \Lambda \in \mathcal{I}$. By 2.2(a), we see that a map $M \to N$ of objects of $\mathcal{D}^0$ induces a unique map $M(\Lambda)/M(\Gamma) \to N(\Lambda)/N(\Gamma)$; thus, we have an additive functor $Q \mapsto Q(\Lambda)/Q(\Gamma)$. To prove Lemma 1.4(b)–(c), we will show for Q in $\mathcal{D}^0[D]$, that $Q(\Lambda)/Q(\Gamma)$ depends only on Q and $\Lambda \setminus \Gamma$, up to isomorphisms compatible with all these induced maps.

For any locally closed subset Σ of Ω, we define the admissible subquotient object $M(\Sigma) := M(\Sigma')/M(\Sigma' \setminus \Sigma)$ of M, where Σ' is the closure of Σ (this is compatible with the existing notation for $\Sigma \in \mathcal{I}$). As a special case of the above remarks, there is natural additive functor $\sigma_\Sigma \colon \mathcal{D}^0 \to \mathcal{D}^0$ given on objects by $Q \mapsto Q(\Sigma)$. Suppose now that $\Lambda \setminus \Gamma = \Sigma$. It is easy to see that $\Sigma' \cup \Gamma = \Lambda$ and $\Gamma \cap \Sigma' = \Sigma' \setminus \Sigma$. By 2.4, one has a canonical morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(\Sigma' \setminus \Sigma) & \longrightarrow & Q(\Sigma') & \longrightarrow & Q(\Sigma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Q(\Gamma) & \longrightarrow & Q(\Lambda) & \longrightarrow & Q(\Sigma) & \longrightarrow & 0 \end{array}$$

This gives a canonical isomorphism $Q(\Sigma) \cong Q(\Lambda)/Q(\Gamma)$ for all $\Gamma \subseteq \Lambda \in \mathcal{I}$ with $\Lambda \setminus \Gamma = \Sigma$; from the definition, this isomorphism is clearly compatible with the maps $f(\Sigma)\colon M(\Sigma) \to N(\Sigma)$ and $M(\Lambda)/M(\Gamma) \to N(\Lambda)/N(\Gamma)$ induced by a morphism $f\colon M \to N$.

This gives us our functor σ_Σ, which we call truncation to Σ; its strict image is obviously in $\mathcal{D}^0_\Sigma$, hence by restriction we get $\tau_\Sigma\colon \mathcal{D} \to \mathcal{D}^0_\Sigma$ and in particular $\tau_x := \tau_{\{x\}}\colon \mathcal{C} \to \mathcal{D}_x$ for $x \in \Omega$. The assertions 1.4(b)–(c) are clear from the construction. Observe that the full subcategory $\mathcal{C}_\Sigma$ of objects Q of $\mathcal{D}^0$ with $Q(x) = 0$ for $x \notin \Sigma$ coincides with $\mathcal{D}^0_\Sigma$.

Applying the above facts to to $Q(\Gamma)$ in $\mathcal{D}^0_\Gamma$ in place of Q, one sees that

2.5.1. For any locally closed subsets Γ and Σ of Ω such that Σ is a coideal of Γ there is a short exact sequence

$$0 \to Q(\Sigma) \to Q(\Gamma) \to Q(\Gamma \setminus \Sigma) \to 0 \tag{2.5.2}$$

in $\mathcal{D}$, and it is functorial in Q i.e. for $f\colon M \to N$ in $\mathcal{D}^0$, one has the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(\Sigma) & \longrightarrow & M(\Gamma) & \longrightarrow & M(\Gamma \setminus \Sigma) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_\Sigma(f)} & & \downarrow{\scriptstyle \sigma_\Gamma(f)} & & \downarrow{\scriptstyle \sigma_{\Gamma\setminus\Sigma}(f)} & & \\ 0 & \longrightarrow & N(\Sigma) & \longrightarrow & N(\Gamma) & \longrightarrow & N(\Gamma \setminus \Sigma) & \longrightarrow & 0 \end{array}$$

This proves 1.4(e). The following is just a restatement of 2.2.1 in this notation:

2.5.3. If Λ is an ideal of Ω, then $\operatorname{Hom}(M(\Omega \setminus \Lambda), N(\Lambda)) = 0$ for all M, N.

The verifications of the following facts are left to the reader. Let $\iota_\Gamma\colon \mathcal{D}^0_\Gamma \to \mathcal{D}^0$ denote the inclusion functor.

2.5.4. If Γ is an ideal (resp., coideal) of Ω then τ_Γ is left (resp., right) adjoint to ι_Γ and the counit $\tau_\Gamma\iota_\Gamma \to \mathrm{Id}$ (resp., unit $\mathrm{Id} \to \tau_\Gamma\iota_\Gamma$) is a natural isomorphism.

2.5.5. For two locally closed subsets Γ, Σ of Ω there is a natural isomorphism $\sigma_\Sigma\sigma_\Gamma \cong \sigma_{\Sigma\cap\Gamma}$.

2.6. We record the following facts for completeness.

Lemma. (a) *Idempotents split in $\mathcal{D}^0[D]$ if they split in $\mathcal{D}$ and they split in $\mathcal{D}_x$ for all $x \in \Omega$.*

(b) *If each category $\mathcal{D}_x$ is closed under the operation of taking direct summands in $\mathcal{D}$, then so is $\mathcal{D}^0[D]$.*

Proof. We prove (a), leaving the very similar proof of (b) to the reader. Let M be in $\mathcal{D}^0[D]$ and $e \in \operatorname{End}(M)$ be an idempotent. It is sufficient to show $\ker e$ is in $\mathcal{D}^0[D]$. Set $e' = \mathrm{Id}_M - e$. For each locally closed subset Γ of Ω, we have the idempotents $e(\Gamma) := \tau_\Gamma(e)$ and $e'(\Gamma) = \mathrm{Id}_{M(\Gamma)} - e(\Gamma)$ in $\operatorname{End}(M(\Gamma))$ by 2.5. Hence there is a corresponding direct sum decomposition $M(\Gamma) = \ker(e(\Gamma)) \oplus \ker(e'(\Gamma))$ in $\mathcal{D}$. Moreover, if Σ is a coideal of Γ then the short exact sequence in $\mathcal{D}$

$$0 \to M(\Sigma) \to M(\Gamma) \to M(\Gamma \setminus \Sigma) \to 0$$

canonically identifies with the direct sum of the two sequences

$$0 \to \ker(e''(\Sigma)) \to \ker(e''(\Gamma)) \to \ker(e''(\Gamma \setminus \Sigma)) \to 0$$

for $e'' = e$ and $e'' = e'$, so the latter are $\mathcal{D}$-exact. Since idempotents split in $\mathcal{D}_x$, it follows that $\ker(e(x))$ is in $\mathcal{D}_x$ for all $x \in \Omega$ and thus $\ker e$ is in $\mathcal{D}^0[D]$ (with $(\ker e)(\Gamma) \cong \ker(e(\Gamma)))$ as required. □

2.7. The following lemma will be used in the proof of 1.2.

Lemma. *If* $\mathrm{Ext}^1_{\mathcal{D}}(M, N) = 0$ *for* M *in* $\mathcal{D}_x$ *and* N *in* $\mathcal{D}_y$ *unless* $x < y$*, then* $\mathcal{D}^0[D]$ *is the full subcategory of objects* M *of* $\mathcal{D}$ *with a filtration* $M = M^0 \supseteq \ldots \supseteq M^m = 0$ *such that for each* i*,* M^{i-1}/M^i *is in* $\mathcal{D}_{y_i}$ *for some* $y_i \in \Omega$*.*

Proof. It is clearly sufficient to show that if $0 \to M'_x \to M \to M'' \to 0$ is an exact sequence in $\mathcal{D}$ with M'_x in $\mathcal{D}_x$ and M'' in $\mathcal{D}^0[D]$, then M is in $\mathcal{D}^0[D]$. Consider the commutative diagram

$$\begin{array}{ccccc}
M'_x & \longrightarrow & N & \longrightarrow & M''(\geq x) \\
\| & & \downarrow & & \downarrow \\
M'_x & \longrightarrow & M & \longrightarrow & M'' \\
 & & \downarrow & & \downarrow \\
 & & M''(\ngeq x) & = & M''(\ngeq x)
\end{array}$$

in $\mathcal{D}$. Here, the top right square is a pullback square, and the rows and columns (completed by zeros) are short exact sequences. By assumption, the top row is split exact, so N is in $\mathcal{D}^0[D_{\geq x}]$. The middle column shows M is in $\mathcal{D}^0[D]$. □

Remarks. If one assumes that each $\mathcal{D}_x$ is closed under extensions in $\mathcal{D}$, a similar argument shows one may replace "$x < y$" in the statement of the lemma by "$x \leq y$" (one replaces "$\geq x$", "$\ngeq x$" in the diagram by "$> x$", "$\ngtr x$").

3. Construction of stratified exact categories

In this section, we prove a more general version of the basic construction 1.7. We also give proofs of Lemma 1.2 and Proposition 1.6.

3.1. In this section, Ω denotes a interval finite poset and we consider a family $E = \{F_x : \mathcal{D}_x \to \mathcal{D}\}_{x\in\Omega}$ of exact functors from exact categories $\mathcal{D}_x$ into a fixed exact category $\mathcal{D}$. We define $E_\Gamma := \{F_x : \mathcal{D}_x \to \mathcal{D}\}_{x\in\Gamma}$ for any subset Γ of Ω (in the induced order), and $E^{\mathrm{op}} := \{F_x^{\mathrm{op}} : \mathcal{D}_x^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}\}_{x\in\Omega^{\mathrm{op}}}$ where $F_x^{\mathrm{op}} : \mathcal{D}_x^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

The following assumption will be in force throughout this section.

Assumption. $D = D[E] := (\mathcal{D}, \{F_x(\mathcal{D}_x)\}_{x\in\Omega})$ satisfies the conditions of 2.1, where $F_x(\mathcal{D}_x)$ denotes the strict image of F_x.

Observe that $D[E^{\mathrm{op}}] = (D[E])^{\mathrm{op}}$ and $D[E_\Gamma] = (D[E])_\Gamma$ then also satisfy the conditions of 2.1. As in the preceding section, define from D the category $\mathcal{D}^0 = \mathcal{D}^0[D]$ and the truncation functors $\sigma_\Gamma\colon \mathcal{D}^0 \to \mathcal{D}^0$ for Γ locally closed in Ω. We tacitly assume as before that $\mathcal{D}$ is a perfectly exact subcategory of an abelian category $\mathcal{B}$.

3.2. Form a new additive category $\mathcal{C}^0 = \mathcal{C}^0[E]$ as follows. An object M of $\mathcal{C}^0$ consists of an object M^0 of $\mathcal{D}^0$, a family $\{M_x\}_{x\in\Omega}$ with M_x an object of $\mathcal{D}_x$ and all but finitely many M_x zero, and a family $\{m_x\colon F_x(M_x) \xrightarrow{\cong} \sigma_x(M^0)\}_{x\in\Omega}$ of isomorphisms in $\mathcal{D}$. If N is another such object, a morphism $f\colon M \to N$ consists by definition of a morphism $f^0\colon M^0 \to N^0$ in $\mathcal{D}^0$ (equivalently, in $\mathcal{D}$) and a family of morphisms $\{F_x\colon M_x \to N_x\}_{x\in\Omega}$ such that $\sigma_x(f)m_x = n_x F_x(f_x)$. If $g\colon N \to P$ is another morphism, the composite gf is given by $(gf)^0 = g^0 f^0$ and $(gf)_x = g_x f_x$.

3.3. A sequence

$$0 \to M \xrightarrow{i} N \xrightarrow{j} P \to 0 \tag{3.3.1}$$

of objects and morphisms in $\mathcal{C}^0$ will be called a short exact sequence in $\mathcal{C}^0$ if the sequence $0 \to M^0 \to N^0 \to P^0 \to 0$ is exact in $\mathcal{D}$ and for all $x \in \Omega$ the sequence $0 \to M_x \to N_x \to P_x \to 0$ is exact in $\mathcal{D}_x$. We call i (resp., j) above an admissible epimorphism (we see later this makes $\mathcal{C}^0$ an exact category).

Observe that one has $\mathcal{C}^0[E^{\mathrm{op}}] = (\mathcal{C}^0[E])^{\mathrm{op}}$ canonically. We will often not prove (or even explicitly mention) the duals of the statements we make. Abbreviate $\mathcal{C}^0[E_\Gamma] = \mathcal{C}^0_\Gamma$ for a locally closed subset Γ of Ω.

Lemma. *A morphism $f\colon M \to N$ in $\mathcal{C}^0$ is an admissible epimorphism iff each $f_x\colon M_x \to N_x$ is an admissible epimorphism in $\mathcal{D}_x$. If $0 \to L \to M \to N \to 0$ is exact in $\mathcal{C}^0$, then the induced sequence $0 \to L^0(\Gamma) \to M^0(\Gamma) \to N^0(\Gamma) \to 0$ is exact in $\mathcal{D}$ for any locally closed subset Γ of Ω.*

Proof. Consider the first claim of the lemma. Suppose that $f\colon M \to N$ is a morphism in $\mathcal{C}^0$ and that for $x \in \Omega$, $0 \to L_x \to M_x \xrightarrow{f_x} N_x \to 0$ in $\mathcal{D}_x$ is exact in $\mathcal{D}_x$. For each finite locally closed subset Λ of Ω, construct the exact sequence $0 \to H_\Lambda \to M^0(\Lambda) \xrightarrow{\sigma_\Lambda f^0} N^0(\Lambda)$ in $\mathcal{B}$, and denote it by $A(\Lambda)$. We prove by induction on the cardinality of Λ that $A(\Lambda)$ is a short exact sequence in $\mathcal{D}$. If $\Lambda = \{x\}$, this follows from the exactness of F_x by considering the commutative diagram with short exact top row

$$\begin{array}{ccccc}
F_x(L_x) & \longrightarrow & F_x(M_x) & \longrightarrow & F_x(N_x) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
H_x & \longrightarrow & M^0(x) & \longrightarrow & N^0(x).
\end{array}$$

Now given a locally closed subset Λ of Ω and a coideal Γ of Λ with complementary ideal $\Sigma := \Lambda \setminus \Gamma$ such that $A(\Gamma)$ and $A(\Sigma)$ are exact in $\mathcal{D}$, consider the diagram

$$\begin{array}{ccccc} H_\Gamma & \longrightarrow & M^0(\Gamma) & \longrightarrow & N^0(\Gamma) \\ \downarrow & & \downarrow & & \downarrow \\ H_\Lambda & \longrightarrow & M^0(\Lambda) & \longrightarrow & N^0(\Lambda) \\ \downarrow & & \downarrow & & \downarrow \\ H_\Sigma & \longrightarrow & M^0(\Sigma) & \longrightarrow & N^0(\Sigma) \end{array}$$

in $\mathcal{D}$. Here, the top and bottom rows and the two rightmost columns are short exact sequences in $\mathcal{D}$ (and hence also in $\mathcal{B}$). The right map in the middle row is an epimorphism in $\mathcal{B}$ by the snake lemma. Hence the middle row is exact in $\mathcal{B}$. The first column is the (exact in $\mathcal{B}$ by the snake-lemma) sequence of kernels in $\mathcal{B}$ of the horizontal maps on the right. Since $\mathcal{D}$ is closed under extensions in $\mathcal{B}$, one sees that H_Λ is in $\mathcal{D}$. The middle row is therefore a short exact sequence in $\mathcal{D}$.

Now take Λ sufficiently large to include all x with $M_x \neq 0$ or $N_x \neq 0$ and set $L^0 := H_\Lambda$. One has $M^0(\Lambda) \cong M^0$ and $N^0(\Lambda) \cong N^0$. The above diagrams imply that L^0 is in $\mathcal{D}^0$, with $L^0(\Gamma) \cong H_\Gamma$. The isomorphisms $F_x(L_x) \cong H_x \cong L^0(x)$ define an object L of $\mathcal{C}^0$, and determine as required an evident short $\mathcal{C}^0$-exact sequence $0 \to L \to M \to N \to 0$.

The second claim of the lemma is clear from the above since one must have $L^0(\Lambda) \cong H_\Lambda$. □

3.4. We introduce some further notation. Given any object M of $\mathcal{C}^0$ and a locally closed subset Γ of Ω, define the object $M(\Gamma)$ of $\mathcal{C}^0[E_\Gamma]$ by $M(\Gamma)^0 = (M^0)(\Gamma)$ and $(M(\Gamma))_x = M_x$ with isomorphisms $F_x(M(\Gamma)_x) = F_x(M_x) \cong \sigma_x(M^0) \cong \sigma_x(M^0(\Gamma))$ for $x \in \Gamma$. The map $M \mapsto M(\Gamma)$ extends in an obvious way to a functor $\tau_\Gamma \colon \mathcal{C}^0 \to \mathcal{C}^0[E_\Gamma]$. Similarly, one has a functor $\iota_\Gamma \colon \mathcal{C}^0[E_\Gamma] \to \mathcal{C}^0$ defined on objects M of $\mathcal{C}^0$ as follows. Set $(\iota_\Gamma(M))^0 = M$, let $(\iota_\Gamma(M))_x = M_x$ if $x \in \Gamma$, and set $(\iota_\Gamma(M))_x = 0$ if $x \in \Omega \setminus \Gamma$. Finally, define $\sigma_\Gamma = \iota_\Gamma \tau_\Gamma \colon \mathcal{C}^0 \to \mathcal{C}^0$.

One can check from the definitions and the proof of 3.3 that

3.4.1. σ_Γ, τ_Γ and ι_Γ have the same properties 2.5.1–2.5.5 as those of the similarly denoted functors considered in 2.5, and, moreover, they are all exact (i.e. preserve short exact sequences).

Remarks. It will be clear after the proof of the following result that the functors here may be regarded as instances of the functors considered in 2.5.

3.5. We shall deduce 1.7 from the following more general fact.

Theorem. *With $\mathcal{C}^0$-exact sequences as the short exact sequences, $\mathcal{C}^0$ is an exact category. Idempotents split in $\mathcal{C}^0$ if they split in $\mathcal{D}$ and each $\mathcal{D}_x$*

Proof. We verify the axioms B.2 for an exact category. For the proof, we regard $\mathcal{C}^0[E_\Gamma]$ as a full subcategory of $\mathcal{C}^0$ via the functor ι_Γ (which is full and faithful, and preserves and reflects the "short exact sequences"). We thus have $\tau_\Gamma(M) = \sigma_\Gamma(M) = M(\Gamma)$ for M in $\mathcal{C}^0$. Note that $\mathcal{C}^0$ is the directed union of its subcategories $\mathcal{C}^0[E_\Gamma]$ over the finite locally closed subsets Γ of Ω. We may therefore assume without loss of generality that Ω is finite.

It is clear that the class of short exact sequences of $\mathcal{C}^0$ contains the split exact sequences and is closed under taking isomorphic sequences. Admissible epimorphisms are closed under composition by 3.3. Given a short exact sequence (3.3.1), it is easy to verify that $i = \ker j$. Indeed, given a morphism $f : L \to N$ with $jf = 0$, the unique maps $g^0 \colon N^0 \to M^0$ and $g_x \colon N_x \to M_x$ with $f^0 = i^0 g^0$ and $f_x = i_x g_x$ define the unique map $g : N \to M$ with $f = ig$.

Now we verify that the pullback of an admissible epimorphism $N \to P$ via a map $Q \to P$ is an admissible epimorphism. Form for $x \in \Omega$ and any locally closed subset Λ of Ω the pullback squares

$$\begin{array}{ccc} R_x & \longrightarrow & Q_x \\ \downarrow & & \downarrow \\ N_x & \longrightarrow & P_x \end{array} \qquad\qquad \begin{array}{ccc} R_\Lambda & \longrightarrow & Q^0(\Lambda) \\ \downarrow & & \downarrow \\ N^0(\Lambda) & \longrightarrow & P^0(\Lambda) \end{array}$$

in $\mathcal{D}_x$ and $\mathcal{D}$ respectively. In each case, the top horizontal map is an admissible epimorphism. As in B.3, these give short exact sequences

$$0 \to R_x \to Q_x \oplus N_x \to P_x \to 0$$
$$0 \to R_\Lambda \to Q^0(\Lambda) \oplus N^0(\Lambda) \to P^0(\Lambda) \to 0.$$

Let S_x (resp., $S(\Lambda)$) denote the above short exact sequence associated to x (resp., Λ). There is an isomorphism $F_x(S_x) \cong S(x)$ of short exact sequences in $\mathcal{D}$ induced by the isomorphisms $F_x(P_x) \cong P^0(x)$ and $F_x(Q_x \oplus N_x) \cong Q^0(x) \oplus N^0(x)$; in particular, $F_x(R_x) \cong R_{\{x\}}$. Also, for a coideal Γ of Λ with complementary ideal $\Sigma = \Lambda \setminus \Gamma$, 2.5.1 and the 9-lemma gives a short exact sequence of diagrams

$$0 \to S(\Gamma) \to S(\Lambda) \to S(\Sigma) \to 0$$

(i.e. regarding the $S(\Lambda)$ etc as exact rows of a 3×3-diagram in $\mathcal{D}$, the columns are also short exact sequences). Setting $R^0 = R(\Omega)$ in $\mathcal{D}^0$, the above defines an object R of $\mathcal{C}^0$ and a short $\mathcal{C}^0$-exact sequence $0 \to R \xrightarrow{f} Q \oplus N \xrightarrow{g} P \to 0$. Since $f = \ker g$, B.3 implies there is a pullback square

$$\begin{array}{ccc} R & \longrightarrow & Q \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array}$$

in $\mathcal{C}^0$ in which the map $R \to Q$ is an admissible epimorphism by 3.3. The verification of the remaining exact category axioms B.2 (involving monomorphisms) is dual to that given above for those involving epimorphisms.

Finally, we verify the assertion about splitting of idempotents in $\mathcal{C}^0$. Let M be in $\mathcal{C}^0$ and $e \in \operatorname{End}(M)$ be an idempotent. Corresponding to e^0 and e_x, there are split epimorphisms $\pi^0 \colon M^0 \to N^0$ in $\mathcal{D}^0$ (by 2.6) and $\pi_x \colon M_x \to N_x$ in $\mathcal{D}_x$ with $\ker \pi^0 = \ker e^0$ and $\ker \pi_x = \ker e_x$. The definitions imply there is a unique isomorphism $n_x \colon F_x(N_x) \to N^0(x)$ so $n_x \pi^0(x) = F_x(\pi_x) m_x$. This makes N an object of $\mathcal{C}^0$ and defines a split epimorphism $\pi \colon M \to N$ with $\ker \pi = \ker e$. $\square$

3.6. For any locally closed subset Λ of Ω, we continue to regard $\mathcal{C}^0_\Lambda := \mathcal{C}^0[E_\Lambda]$ as a full subcategory of $\mathcal{C}^0$. It consists of the objects M of $\mathcal{C}^0$ such that $M_x = 0$ for $x \notin \Lambda$, so is clearly closed under extensions. Thus, it coincides with the smallest extension closed subcategory of $\mathcal{C}^0$ containing $\mathcal{C}^0_x$ for all $x \in \Lambda$. The functor $\tau_\Lambda\colon \mathcal{C}^0 \to \mathcal{C}^0_\Lambda$ is clearly exact, and $\iota_\Lambda\colon \mathcal{C}^0_\Lambda \to \mathcal{C}^0$ is perfectly exact.

There is an exact functor $\tau^0\colon \mathcal{C}^0 \to \mathcal{D}$ given on objects by $M \mapsto M^0$. We now have

3.6.1. If Γ is an ideal of Ω, $i \leq 1$, and M (resp., N) is an object of $\mathcal{C}^0_\Gamma$ (resp., of $\mathcal{C}^0_{\Omega\setminus\Gamma}$) then $\mathrm{Ext}^i_{\mathcal{C}^0}(N, M) = 0$ but the natural map $\mathrm{Ext}^i_{\mathcal{C}^0}(M, N) \to \mathrm{Ext}^i_{\mathcal{D}}(M, N)$ induced by τ^0 is an isomorphism.

We leave the verification to the reader (observe that for $x \in \Omega$, $N_x = 0$ if $x \in \Gamma$ and $M_x = 0$ if $x \notin \Gamma$).

For $x \in \Omega$, there is an equivalence of exact categories $\mathcal{C}^0_x \to \mathcal{D}_x$ given by $M \mapsto M_x$. We regard this as an identification $\mathcal{C}^0_x = \mathcal{D}_x$. The above fact now implies that for $i \leq 1$ and objects M of $\mathcal{C}^0_x$ and N of $\mathcal{C}^0_y$, one has natural isomorphisms

$$\mathrm{Ext}^i_{\mathcal{C}^0}(M, N) = \begin{cases} \mathrm{Ext}^i_{\mathcal{D}_x}(M, N), & \text{if } x = y \\ \mathrm{Ext}^i_{\mathcal{D}}(\tau^0 M, \tau^0 N), & \text{if } x < y \\ 0 & \text{otherwise.} \end{cases} \tag{3.6.2}$$

It follows that $\mathcal{C}^0$ is a weakly stratified exact category with weight poset Ω and strata $\mathcal{C}^0_x$, as defined in 1.5 (1.4(i) and 1.4(ii) hold by (3.6.2) and conditions 1.5(i)–(iv) are clear from the construction and the above remarks).

Regard $F := (\mathcal{C}^0, \{\mathcal{C}^0_x\}_{x\in\Omega})$ as data satisfying the conditions 1.4. It is also clear that $\mathcal{D}^0[F] = \mathcal{C}^0$ as additive categories, and that the functors τ_Γ, ι_Γ and σ_Γ associated in Section 2 to the additive category $\mathcal{D}^0[F]$ identify naturally with the similarly denoted functors associated to the exact category $\mathcal{C}^0[E]$.

3.7. **Proof of Theorem 1.7.** Suppose that the functors $F_x\colon \mathcal{D}_x \to \mathcal{D}$ are full, exact inclusion functors as in Theorem 1.7. Define a three-term sequence in $\mathcal{C} := \mathcal{D}^0[D]$ to be short exact if it is exact in $\mathcal{D}$ and application of τ_x gives a short exact sequence in $\mathcal{D}_x$ for all $x \in \Omega$. Then it is easy to see that τ^0 gives an equivalence of additive categories between $\mathcal{C}^0[E]$ and $\mathcal{D}^0[D]$; moreover, under τ_0, short exact sequences in $\mathcal{C}$ correspond to short exact sequences in $\mathcal{D}^0$. This proves Theorem 1.7.

Remarks. It is clear that if one performs the construction of 1.7 from a weakly stratified exact category $\mathcal{D}$ with strata $\mathcal{D}_x$, the resulting category $\mathcal{C}$ is equal to $\mathcal{D}$ as weakly stratified exact category.

3.8. **Proof of Lemma 1.2.** Let $\mathcal{C}$, $\mathcal{B}$ etc be as in 1.2 Let $\mathcal{C}_{\not> x}$ be the smallest extension closed subcategory of $\mathcal{B}$ containing all strata $\mathcal{C}_y$ for $y \not> x$, regarded as a perfectly exact subcategory of $\mathcal{B}$. Since objects of $\mathcal{C}_x$ are projective in $\mathcal{B}_x$, they are certainly projective in $\mathcal{C}_{\not> x}$. The assertion 1.2(a) is then immediate from definition 1.1.

By 1.7 and 2.7, we may form a weakly stratified exact category $(\mathcal{C}', \{\mathcal{C}_x\}_{x\in\Omega})$ in which $\mathcal{C}' = \mathcal{C}$ as additive category, but such that a three-term sequence

$$0 \to M' \to M \xrightarrow{f} M'' \to 0 \tag{3.8.1}$$

in $\mathcal{C}'$ is exact iff (3.8.1) is exact in $\mathcal{B}$ and σ_x(3.8.1) is exact in $\mathcal{C}_x$ for all $x \in \Omega$, where the σ_Γ denote truncation functors in $\mathcal{C}'$.

Let (3.8.1) be a short exact sequence in $\mathcal{B}$ with M and M'' objects of $\mathcal{C}$, and assume that either M' is in $\mathcal{C}$ or idempotents split in each $\mathcal{C}_x$. To complete the proof of 1.2(b), it will be more than sufficient to show that (3.8.1) is a short exact sequence in $\mathcal{C}'$; in fact, since every stratified exact category arises by the construction 1.2, that will also show that every stratified exact category is a weakly stratified exact category with split strata (see 1.6(d)).

Without loss of generality, assume that Ω is finite and non-empty. Choose a maximal element x of Ω, so (3.8.1) may be regarded as a short exact sequence in $\mathcal{B}_x$. Let N be any object of $\mathcal{C}_x$ and $R := \mathrm{End}(N)^{\mathrm{op}}$. Then $\mathrm{Hom}_{\mathcal{B}_x}(N, (3.8.1))$ is a short exact sequence of R-modules, and in particular we have an epimorphism $\mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'')$. Equivalently, $\sigma_x(f)\colon M(x) \to M''(x)$ induces an epimorphism $\mathrm{Hom}(N, M(x)) \to \mathrm{Hom}(N, M''(x))$. Taking $N = M''(x)$, one sees that $\sigma_x(f)$ is necessarily a split epimorphism in $\mathcal{C}_x$, with kernel N', say (note that N' exists in $\mathcal{C}_x$ if idempotents split in $\mathcal{C}_x$, and $N' = M'(x)$ if M' is in $\mathcal{C}$ by taking $N = M(x)$ above). By the 9-lemma, we have a short exact sequence

$$0 \to M'/N \to M(\neq x) \to M''(\neq x) \to 0$$

in $\mathcal{B}$ (with $M'/N = N(\neq x)$ if M' is in $\mathcal{C}$). Induction on the cardinality of Ω gives that M'/N is in $\mathcal{C}'_{\neq x}$ in any case and that this last sequence is a short exact sequence in $\mathcal{C}'_{\neq x}$. This immediately implies that M' is in $\mathcal{C}$, with $M'(x) = N$, and that (3.8.1) is a short exact sequence of $\mathcal{C}'$.

3.9. **Proof of Proposition 1.6.** As every weakly stratified exact category $\mathcal{D}$ arises by the construction 1.7, 1.6(a) is a special case of (3.6.2). We saw in the proof of 1.2 that a stratified exact category is a weakly stratified exact category with split strata. Conversely, let $\mathcal{D}$ be a weakly stratified exact category with split strata $\mathcal{D}_x$. It is clear that for any locally closed $\Gamma \subseteq \Omega$, $\mathcal{D}_\Gamma$ (as defined in 1.5) is the smallest extension closed subcategory of $\mathcal{D}$ containing all $\mathcal{D}_y$ with $y \in \Gamma$. If one has an object P of $\mathcal{D}_x$ and a short exact sequence $E\colon 0 \to L \to M \to N \to 0$ in $\mathcal{D}_{\not> x}$, then $\mathrm{Hom}(P, E) = \mathrm{Hom}(P, \sigma_x(E))$ is exact since $\sigma_x(E)$ is split exact; thus P is projective in $\mathcal{D}_{\not> x}$. This proves 1.6(d).

We make some preliminary remarks for the proof of 1.6(b). If $\rho : C \to D$ is an exact functor between exact categories, the natural maps $f\colon \mathrm{Ext}^i_C(P, Q) \to \mathrm{Ext}^i_D(\rho(P), \rho(Q))$ for P, Q in C are the obvious ones for $i = 0$ and map the class of an i-fold extension

$$E\colon 0 \to Q \to R^1 \to \ldots \to R^i \to P \to 0$$

($i \geq 1$) in C to the class of the extension

$$\rho(E)\colon 0 \to \rho(Q) \to \rho(R^1) \to \ldots \to \rho(R^i) \to \rho(P) \to 0$$

in D. If now ρ has an exact left adjoint ρ' such that the counit $\rho'\rho \to \mathrm{Id}$ of the adjunction is a natural isomorphism, it follows that f is an isomorphism. This is especially well known for $i = 0$, so assume $i > 0$. Firstly, f is an injection since if the class of $\rho(E)$ is zero, so is that of $\rho'\rho(E)$ and hence that of E. Secondly, the map f is surjective since given E' representing an element of $\mathrm{Ext}^i_D(\rho(P), \rho(Q))$ with $i \geq 1$, the unit $\mathrm{Id} \to \rho\rho'$ of the adjunction gives a morphism of extensions $E' \to \rho\rho' E'$. Using the counit of the adjunction to identify $\rho(P)$ with $\rho\rho'\rho(P)$ and similarly for Q, this shows the class of E' coincides that of $\rho\rho'(E')$ and hence it is clearly in the image of f. If Γ is an ideal of Ω, the above remarks prove 1.6(b) by 2.5.4. The argument when Γ is a coideal of Ω is dual. The general case of 1.6(b) follows since every locally closed subset is a coideal of the ideal it generates.

Finally, let L, N, x, y be as in 1.6(c). Then $\operatorname{Ext}^i_{\mathcal{D}}(N,L)=\operatorname{Ext}^i_{\mathcal{D}_{\not>x}}(N,L)$ by 1.6(d) and (b), and the right hand side is zero for $i\geq 1$ since N is projective in $\mathcal{D}_{\not>x}$.

3.10. The proof of the following, using 2.6 and the remarks in B.11, is left to the reader.

Proposition. *Let $\mathcal{D}$ be a (weakly) stratified exact category with strata $\mathcal{D}_x$ for $x\in\Omega$.*

(a) *Let $\mathcal{D}'$ be the Karoubianization of $\mathcal{D}$, and regard $\mathcal{D}$ as a perfectly exact subcategory of $\mathcal{D}'$. Let $\mathcal{D}'_x$ be the full additive subcategory of $\mathcal{D}'$ consisting of direct summands of objects in $\mathcal{D}_x$. Then $\mathcal{D}'$ is a (weakly) stratified exact category with strata $\mathcal{D}'_x$.*

(b) *Idempotents split in $\mathcal{D}$ iff they split in $\mathcal{D}_x$ for all $x\in\Omega$.*

3.11. We close this section with one final trivial remark. Suppose $\mathcal{C}$ is a weakly stratified exact category. Let $\{\Omega_x\}_{x\in\Lambda}$ be a partition of Ω by locally closed subsets Ω_x indexed by an interval finite poset Λ such that $a\in\Omega_x$, $b\in\Omega_y$ and $a\leq b$ imply $x\leq y$ in Λ. Then $(\mathcal{C},\{\mathcal{C}_{\Omega_x}\}_{x\in\Lambda})$ is a weakly stratified exact category.

4. The Abelian Category $\mathcal{C}^*$

Let $\mathcal{C}$ be a svelte stratified exact category with strata $\mathcal{C}_x$ and weight poset Ω. We prove here some first general facts about the associated abelian categories $\mathcal{C}^*$ and $\mathcal{C}^\dagger$ defined as in 1.8 and Appendix B. The results of the first two subsections apply even to a svelte weakly stratified exact category.

4.1. Let Σ be a locally closed subset of Ω. Then one has abelian categories $\mathcal{C}^*_\Sigma$ and $\mathcal{C}^*=\mathcal{C}^*_\Omega$. Observe that since $\mathcal{C}^*_\Sigma$ has injective envelopes, all Ext-groups $\operatorname{Ext}^i_{\mathcal{C}^*_\Sigma}(M,N)$ exist and can be computed using an injective resolution of N. The exact functors $\mathcal{C}\xrightarrow{\tau_\Sigma}\mathcal{C}_\Sigma\xrightarrow{\iota_\Sigma}\mathcal{C}$ induce by B.15 left exact functors $\tau^*_\Sigma\colon\mathcal{C}^*_\Sigma\to\mathcal{C}^*$, $\iota^*_\Sigma\colon\mathcal{C}^*\to\mathcal{C}^*_\Sigma$, and right exact functors $\tau_{\Sigma*}\colon\mathcal{C}^*\to\mathcal{C}^*_\Sigma$, $\iota_{\Sigma*}\colon\mathcal{C}^*_\Sigma\to\mathcal{C}^*$ (we abbreviate $(X)^*=X^*$, $(X)_*=X_*$, $(X)^\dagger=X^\dagger$ etc for any X if it seems unlikely to cause confusion).

Proposition. *Let Γ be an ideal of Ω with complementary coideal $\Lambda:=\Omega\setminus\Gamma$.*

(a) *One has adjoint triples $(\tau_{\Gamma*},\tau^*_\Gamma\cong\iota_{\Gamma*},\iota^*_\Gamma)$ and $(\iota_{\Lambda*},\iota^*_\Lambda\cong\tau_{\Lambda*},\tau^*_\Lambda)$ with $\tau_{\Lambda*}\iota_{\Gamma*}=0$, $\iota^*_\Lambda\tau^*_\Gamma=0$ and $\iota^*_\Gamma\tau^*_\Lambda=0$. In particular, τ^*_Γ and ι^*_Λ are exact, τ^*_Λ and ι^*_Γ preserve injectives and $\iota_{\Lambda*}$ and $\tau_{\Gamma*}$ preserve injectives.*

(b) *The adjunction morphisms* $\mathrm{Id}\to\iota^*_\Gamma\tau^*_\Gamma$, $\mathrm{Id}\to\tau_{\Lambda*}\iota_{\Lambda*}$, $\iota^*_\Lambda\tau^*_\Lambda\to\mathrm{Id}$ *and* $\tau_{\Gamma*}\iota_{\Gamma*}\to\mathrm{Id}$ *are isomorphisms, so $\iota_{\Lambda*}$, τ^*_Λ and $\tau^*_\Gamma=\iota_{\Gamma*}$ are full embeddings. We usually identify $\mathcal{C}^*_\Gamma$ with the strict image of τ^*_Γ in $\mathcal{C}^*$.*

(c) *The functor ι^*_Λ is a quotient functor with $\mathcal{C}^*_\Gamma$ as its kernel. In particular, $\mathcal{C}^*_\Gamma$ is a Serre subcategory of $\mathcal{C}^*$.*

(d) *The components $F\to\iota_{\Gamma*}\tau_{\Gamma*}F$ (resp., $\tau^*_\Gamma\iota^*_\Gamma F\to F$) of the adjunction morphisms are epimorphisms (resp., monomorphisms) for F in $\mathcal{C}^*$, so $\iota_{\Gamma*}\tau_{\Gamma*}F$ (resp., $\tau^*_\Gamma\iota^*_\Gamma F$) is the largest quotient object (resp., subobject) of F which is in the strict image of τ^*_Γ.*

(e) *For F in $\mathcal{C}^*$, the adjunction arrows induce isomorphisms $F\cong\varinjlim_{\Gamma\in I}\tau^*_\Gamma\iota^*_\Gamma F$ and $\varprojlim_\Lambda\tau^*_\Lambda\iota^*_\Lambda F\cong F$ where the projective limit is taken over the directed family (by inclusion) of finitely generated coideals of Ω.*

(f) *The natural sequence* $0 \to \tau_\Gamma^* \iota_\Gamma^* F \to F \to \tau_\Lambda^* \iota_\Lambda^* F \to 0$ *is exact for any injective object* F *of* $\mathcal{C}^*$.

(g) *A sequence* $A\colon 0 \to F \to G \to H \to 0$ *in* $\mathcal{C}^*$ *is exact iff* $\iota_\Lambda^*(A)$ *is exact in* $\mathcal{C}_\Lambda^*$ *for all f.g. coideals* Λ *of* Ω.

Proof. First, (a) and (b) follow from 2.5.4 and obvious identities $\tau_\Gamma \iota_\Lambda = 0$, $\tau_\Lambda \iota_\Gamma = 0$ by standard properties of adjoint functors and conjugate natural transformations. Then ι_Λ^* is a quotient functor by [25, Chapter III, Prop 5], for instance. To complete the proof of (c), one must check that

4.1.1. An object F in $\mathcal{C}^*$ satisfies $\iota_\Lambda^* F = 0$ iff $F \cong \tau_\Gamma^* G$ for some G in $\mathcal{C}_\Lambda$.

We have already noted $\iota_\Lambda^* \tau_\Gamma^*(G) = 0$. Now we consider an arbitrary element F of $\mathcal{C}^*$. For any M in $\mathcal{C}$, the natural short exact sequence

$$0 \to M(\Lambda) \to M \to M(\Gamma) \to 0 \tag{4.1.2}$$

gives on application of F^* the exact sequence

$$0 \to (\tau_\Gamma^* \iota_\Gamma^* F)(M) \to F(M) \to (\tau_\Lambda^* \iota_\Lambda^* F)(M) \tag{4.1.3}$$

of abelian groups and hence an exact sequence

$$0 \to \tau_\Gamma^* \iota_\Gamma^* F \to F \to \tau_\Lambda^* \iota_\Lambda^* F \tag{4.1.4}$$

in $\mathcal{C}^*$ from which $F \cong \tau_\Gamma^* \iota_\Gamma^* F$ if $\iota_\Lambda^* F = 0$.

The above argument also shows that $\tau_\Gamma^* \iota_\Gamma^* F \to F$ is a monomorphism in C^* in general. By adjointness, $F \to \iota_{\Gamma *} \tau_{\Gamma *} F$ is an epimorphism and the rest of (d) follows. By the Yoneda lemma, (4.1.3) identifies with $\mathrm{Hom}_{C^*}(\phi(4.1.2), F)$ and since ϕ is exact, it follows that if F is injective in $\mathcal{C}^*$ then (4.1.3) is a short exact sequence for all M in C. This implies in turn that for injective F, (4.1.4) is a short exact sequence in the category of contravariant additive functors from C to abelian groups, and hence a short exact sequence in C^*, proving (f).

To prove (e), note that for M in $\mathcal{C}$ one has $M \in \mathcal{C}_\Lambda$ for some $\Lambda \in I$ and then the canonical map $(\tau_\Gamma^* \iota_\Gamma^* F)M = F(M(\Gamma)) \to F(M)$ is an isomorphism for $I \ni \Gamma \supseteq \Lambda$. This shows that $F \cong \varinjlim_{\Gamma \in I} \tau_\Gamma^* \iota_\Gamma^* F$ in the category of contravariant additive functors from $\mathcal{C}$ to abelian groups, and the first part of (e) follows. The proof of the second part of (e) is similar. Finally, (g) follows on checking that an additive functor $F\colon \mathcal{C} \to \mathbb{Z}\text{-Mod}$ is effaceable iff $F\iota_\Lambda\colon \mathcal{C}_\Lambda \to \mathbb{Z}\text{-Mod}$ is effaceable for all f.g. coideals Λ of Ω. □

Remarks. A formal setup involving abelian categories with adjoint functors satisfying several of the conditions above is studied in [44].

4.2. Here we collect some general facts about the structure of injectives of $\mathcal{C}^*$.

4.2.1. If F is in $\mathcal{C}^*$, and $F \to Q$ is an injective envelope of F in $\mathcal{C}^*$, then for any ideal Γ of Ω the induced map $\iota_\Gamma^* F \to \iota_\Gamma^* Q$ is an injective envelope in $\mathcal{C}_\Gamma^*$. In particular, if F is injective in $\mathcal{C}_\Gamma^*$, then $F \cong \iota_\Gamma^* Q$.

To see this, note first that $\iota_\Gamma^* F \to \iota_\Gamma^* Q$ is a monomorphism into an injective object since ι_Γ^* is right adjoint to the exact functor τ_Γ^*. If $0 \neq M \subseteq \iota_\Gamma^* Q$, then $0 \neq \tau_\Gamma^* M \subseteq \tau_\Gamma^* \iota_\Gamma^* Q \subseteq Q$ so $H := F \cap \tau_\Gamma^* M \neq 0$ since $F \to Q$ is an injective envelope. Now $H \subseteq \tau_\Gamma^* M$ so $H = \tau_\Gamma^* G$ for some $G \neq 0$ in $\mathcal{C}_\Gamma^*$. The inclusion $H \to F$

factors through the monomorphism $\tau_\Gamma^*\iota_\Gamma^*F \to F$ and one deduces that G is a non-zero subobject of $M \cap \iota_\Gamma^*F$. Hence $\iota_\Gamma^*F \to \iota_\Gamma^*Q$ is an essential monomorphism as required to prove the claim.

The following two assertions follow directly using 4.1(a),(c),(f) and [25, Ch III].

4.2.2. Let Γ be any ideal of Ω with complementary coideal Λ. If $M \to Q$ is an injective envelope in $\mathcal{C}^*$ of an object M of $\mathcal{C}^*$ with no non-zero subobject in $\mathcal{C}_\Gamma^*$, then $\iota_\Lambda^*M \to \iota_\Lambda^*Q$ is an injective envelope of ι_Λ^*M in $\mathcal{C}_\Lambda^*$; moreover, $\tau_\Lambda^*\iota_\Lambda^*Q \cong Q$.

4.2.3. For Λ, Γ as above, any injective object Q of $\mathcal{C}^*$ is a direct sum $Q = J \oplus \tau_\Lambda^*K$ where J is the injective envelope in $\mathcal{C}^*$ of the injective object ι_Γ^*Q of $\mathcal{C}_\Gamma^*$ and K is injective in $\mathcal{C}_\Lambda$; moreover, K is unique up to isomorphism.

The next claim relates injective objects of $\mathcal{C}^*$ to the injectives in $\mathcal{C}_{\geq x}^*$ for each $x \in \Omega$.

4.2.4. Choose for each $x \in \Omega$ a cogenerator I_x of $\mathcal{C}_x^*$ and an injective envelope J_x of I_x in $\mathcal{C}_{\geq x}^*$. Then $\prod_{x\in\Omega} \tau_{\geq x}^*J_x$ is an injective cogenerator for $\mathcal{C}^*$.

To see this, let $F \neq 0$ be in $\mathcal{C}^*$. By 4.1, $F' := \tau_\Gamma^*\iota_\Gamma^*F \neq 0$ for some $\Gamma \in I$. Choose $x \in \Lambda$ maximal so that there exists N in $\mathcal{C}_x$ with $F'(N) \neq 0$; then F' is in $\mathcal{C}_{\not> x}^*$ and $\mathrm{Hom}(F', \tau_x^*(I_x)) = \mathrm{Hom}(\iota_x^*F', I_x) \neq 0$ since $(\iota_x^*F)(N) = F(N) \neq 0$. Thus, we have a non-zero morphism $g\colon \tau_\Gamma^*\iota_\Gamma^*F \to \tau_x^*(I_x)$ in $\mathcal{C}_\Gamma^*$ for some $x \in \Gamma$. In $\mathcal{C}^*$, we have natural monomorphisms $\tau_\Gamma^*\iota_G amma^*F \to F$ and $\tau_x^*(I_x) \to \tau_{\geq x}^*(J_x)$. Hence g lifts to a non-zero map $F \to \tau_{\geq x}^*J_x$ since the latter is injective. The claim follows.

On the other hand, the following claims relates injective objects of $\mathcal{C}^*$ to injectives in $\mathcal{C}_\Gamma^*$ for $\Gamma \in I$.

4.2.5. An object F of $\mathcal{C}^*$ is injective iff ι_Γ^*F is injective in $\mathcal{C}_\Gamma^*$ for all $\Gamma \in I$.

Indeed, let F in $\mathcal{C}^*$ be such that ι_Γ^*F is injective for all $\Gamma \in I$. Let $f\colon F \to Q$ be an injective envelope of F in $\mathcal{C}^*$. By 4.2.1, ι_Γ^*f is an isomorphism for all $\Gamma \in I$; using 4.1(e) one deduces that f is an isomorphism in $\mathcal{C}^*$.

Remarks. Taken together, the last two claims reduce, to some extent, the description of injectives in $\mathcal{C}^*$ to that of injectives in $\mathcal{C}_\Sigma^*$ for finite, locally closed subsets Σ of Ω. In 4.7, we say more about the latter problem under an additional hypothesis which is implied by strong stratification of $\mathcal{C}$.

4.3. From B.9(a), one immediately deduces the following basic fact.

Lemma. $\phi(M) = \mathrm{Hom}(?, M)$ *is projective in* $\mathcal{C}_\Gamma^*$ *for any* $M \in \mathcal{C}_x$ *and any ideal* Γ *of* Ω *with* x *as a maximal element.*

4.4. If $\mathcal{C}$ has enough projective pro-objects, the existence of a right adjoint to τ_x^* for each $x \in \Omega$ is easy to see by interpreting module-theoretically (see 9.4). In this subsection, we sketch a proof of the existence of a right adjoint to τ_x^* in general.

For locally closed $\Gamma \in I$, let $\phi_\Gamma\colon \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma^*$ be the standard Gabriel-Quillen embedding. Define functors $\Lambda_x\colon \mathcal{C}_x \to \mathcal{C}^*$ and $V_x\colon \mathcal{C}_x \to \mathcal{C}^*$ by $\Lambda_x = \iota_{x*}\phi_x = \phi\iota_x$ and $V_x = \tau_x^*\phi_x$ i.e. $\Lambda_x(N) = \phi(N)$ and $V_x(N) = \mathrm{Hom}(\tau_x?, N)$ for N in $\mathcal{C}_x$ (observe that $V_x(N)$ takes short exact sequences in $\mathcal{C}$ to split exact sequences of abelian groups). Both functors V_x and Λ_x are exact since they are additive and $\mathcal{C}_x$ is a split exact category.

Now define an additive functor $j^x\colon \mathcal{C}^* \to \mathcal{C}_x^*$ by

$$j^x(F)(N) = \mathrm{Hom}_{\mathcal{C}^*}(V_x(N), F)$$

for F in $\mathcal{C}^*$ and N in $\mathcal{C}_x$. We claim that

4.4.1. The functor j^x is right adjoint to τ_x^*. Hence τ_x^* has both left and right adjoint functors, and in particular τ_x^* is exact.

To see this, one needs to construct isomorphisms

$$\theta_{F,G}\colon \operatorname{Hom}_{\mathcal{C}^*}(\tau_x^* F, G) \cong \operatorname{Hom}_{\mathcal{C}_x^*}(F, j^x(G)),$$

natural for F in $\mathcal{C}_x^*$ and G in $\mathcal{C}^*$. Define $\theta_{F,G}$ by the formula

$$\Big[\big[\theta_{F,G}(\eta)\big]_N(b)\Big]_M(f) = \big(\big[\eta\big]_M F(f)\big)(b)$$

for $\eta\colon \tau_x^* F \to G$, N in $\mathcal{C}_x$, M in $\mathcal{C}$, $f\colon \tau_x M \to N$ and $b \in F(N)$, where for a natural transformation ρ, $[\rho]_L$ denote its component at L. One may check that $\theta_{F,G}$ has the required properties (one has $[\theta_{F,G}^{-1}(\epsilon)]_M(a) = \big[[\epsilon]_{\tau_x(M)}(a)\big]_M(\mathrm{Id}_{\tau_x(M)})$ for $\epsilon\colon F \to j^x G$, M in $\mathcal{C}$ and $a \in F\tau_x(M)$).

4.5. The following result is a variant of 1.21(b); I don't know if there is a natural common generalization.

Proposition. *If Γ is an ideal of Ω with $\Lambda := \Omega \setminus \Gamma$ finite, then*

(a) *For F, G in $\mathcal{C}_\Gamma^*$, one has $\operatorname{Ext}^i_{\mathcal{C}_\Gamma^*}(F, G) \cong \operatorname{Ext}^i_{\mathcal{C}^*}(\tau_\Gamma^* F, \tau_\Gamma^* G)$ naturally.*

(b) *More generally, the right derived functor $R\tau_\Gamma^*\colon D^+(\mathcal{C}_\Gamma^*) \to D^+(\mathcal{C}^*)$ is a full embedding of derived categories.*

Proof. By induction on the cardinality of Λ, it suffices to prove the assertions in the case that $\Gamma = \Omega \setminus \{x\}$ for some maximal element x of Ω. If G is injective in $\mathcal{C}_\Gamma^*$, one has from 4.2.1 and 4.1 an exact sequence $0 \to \tau_\Gamma^* G \to Q^0 \to (\tau_x^* \iota_x^*) Q^0 \to 0$ where $\tau_\Gamma^* G \to Q^0$ is an injective envelope in $\mathcal{C}^*$. Choosing an injective resolution $0 \to \iota_x^* Q^0 \to I^\bullet$ of $\iota_x^* Q^0$ in $\mathcal{C}_x^*$ and applying τ_x^* gives an injective resolution $0 \to \tau_\Gamma^* G \to Q^\bullet$ in $\mathcal{C}^*$ where $Q^{i+1} = \tau_x^*(I^i)$. Note $\iota_\Gamma^* Q^i = 0$ for $i \geq 1$ since $\tau_x \iota_\Gamma = 0$. Now for F in $\mathcal{C}_\Gamma^*$ and $i \geq 1$, $\operatorname{Ext}^i(\tau_\Gamma^* F, \tau_\Gamma^* G)$ is a subquotient of $\operatorname{Hom}(\tau_\Gamma^* F, I^i) = \operatorname{Hom}(F, \iota_\Gamma^* I^i) = 0$ proving (a) in case G is injective. Now one can prove (a) for arbitrary G in $\mathcal{C}_\Gamma^*$ by dimension shifting, choosing an exact sequence $0 \to G \to Q \to H \to 0$ with Q injective in $\mathcal{C}_\Gamma^*$.

For (b), note that the above argument shows that $\tau_\Gamma^* G$ is right ι_Γ^*-acyclic for any injective G in $\mathcal{C}_\Gamma^*$. Computing $R\tau_\Gamma^*$ using quasi-isomorphisms to bounded-below complexes of injectives and noting that, as is well known, $R\iota_\Gamma^*$ may be computed using quasi-isomorphisms to bounded-below complexes of right ι_Γ^*-acyclic objects, we get $R\iota_\Gamma^* R\tau_\Gamma^* \cong R(\iota_\Gamma^* \tau_\Gamma^*) = \mathrm{Id}$, proving (b).

□

4.6. Here, we give some formulae which are useful for the computation of certain Ext-groups in $\mathcal{C}^\dagger$.

We regard $\tau_\Gamma^*\colon \mathcal{C}_\Gamma^* \to \mathcal{E}$ as an inclusion for $\Gamma \in I$, so $\iota_\Gamma^*(F) = \sigma_\Gamma^*(F)$ for F in $\mathcal{C}^\dagger$. Observe first that for $\Gamma \subseteq \Lambda \in I_1$, the natural inclusions $\mathcal{C}_\Gamma^* \to \mathcal{C}_\Lambda^* \to \mathcal{C}^\dagger$ induce maps

$$\operatorname{Ext}^p_{\mathcal{C}_\Gamma^*}(F, G) \to \operatorname{Ext}^p_{\mathcal{C}_\Lambda^*}(F, G) \to \operatorname{Ext}^p_{\mathcal{C}^\dagger}(F, G)$$

of Yoneda Ext-groups for F, G in $\mathcal{C}_\Gamma^*$. Since I is directed and every element of $\mathcal{C}^\dagger$ lies in $\mathcal{C}_\Gamma^*$ for some $\Gamma \in I_1$, 4.6.1. The natural map $\varinjlim_{\Gamma\in I}\mathrm{Ext}^p_{\mathcal{C}^*_\Gamma}(F,G)\to\mathrm{Ext}^p_{\mathcal{C}^\dagger}(F,G)$ is an isomorphism for any F, G in $\mathcal{C}^\dagger$ (in the colimit, we consider only the cofinal family of $\Gamma\in I$ so both F, G are in $\mathcal{C}^*_\Gamma$). In particular, $\mathrm{Ext}^p_{\mathcal{C}^\dagger}(F,G)$ is defined i.e. is a set.

4.6.2. If N is in $\mathcal{C}_x$ and F is in in both $\mathcal{C}^*_{\not>x}$ and $\mathcal{C}^\dagger$, then

$$\mathrm{Ext}^p_{\mathcal{C}^\dagger}(\phi(N),F)=\begin{cases}0, & \text{if } p>0\\ F(N), & \text{otherwise}\end{cases}$$

(note $F(N)=0$ if F is in $\mathcal{C}^*_{\not\geq x}$).

For the proof, one may assume by 4.6.1 that $\Omega\in I$ and then by 4.5 that x is maximal in Ω (since $\{\,y\in\Omega\mid y>x\,\}$ is finite). Then $\phi_x(N)$ is projective in $\mathcal{C}^*$ by 4.3 so the result is immediate.

4.6.3. For F in both $\mathcal{C}^*_{\not>x}$ and $\mathcal{C}^\dagger$ and for G in $\mathcal{C}^*_x$,

$$\mathrm{Ext}^p_{\mathcal{C}^*_x}(\iota^*_xF,G)\cong\mathrm{Ext}^p_{\mathcal{C}^\dagger}(F,\tau^*_xG)$$

(note this is zero if G is injective in $\mathcal{C}^*_x$ and $p>0$, or if F is in $\mathcal{C}^\dagger_{\not\geq x}$).

Once again the proof reduces to the case that x is maximal in Ω and $\Omega\in I$. The assertion follows by computing the first Ext-group using an injective resolution of G, noting that τ^*_x is exact by 4.4.1 and it preserves injectives since it is right adjoint to the exact functor ι^*_x (see [25, III, Cor. 6]).

We record the following easy consequences of these assertions.

4.6.4. For N in $\mathcal{C}_x$ and M in $\mathcal{C}_y$, one has $\mathrm{Ext}^p_{\mathcal{C}^\dagger}(\phi(N),\phi(M))=0$ if $p>0$ unless $x<y$ while $\mathrm{Hom}_{\mathcal{C}^\dagger}(\phi(N),\phi(M))=\mathrm{Hom}_{\mathcal{C}}(N,M)$ is zero unless $x\leq y$.

4.6.5. For G in $\mathcal{C}^*_y$ and N in $\mathcal{C}_x$, one has

$$\mathrm{Ext}^p_{\mathcal{C}^\dagger}(\phi(N),\tau^*_y(G))=\begin{cases}G(N), & \text{if } x=y \text{ and } p=0\\ 0, & \text{otherwise.}\end{cases}$$

4.6.6. For G in $\mathcal{C}^*_y$ and F in $\mathcal{C}^*_x$, one has $\mathrm{Ext}^p_{\mathcal{C}^\dagger}(\tau^*_x(F),\tau^*_y(G))=0$ unless $x\geq y$, and if $x=y$, $\mathrm{Ext}^p_{\mathcal{C}^\dagger}(\tau^*_x(F),\tau^*_x(G))=\mathrm{Ext}^p_{\mathcal{C}^*_x}(F,G)$.

Remarks. The above formulae immediately imply the standard Ext-vanishing properties listed in 1.26, cf 9.4.

4.7. Let $\mathcal{D}_x$ be the additive category of all objects of $\mathcal{C}^\dagger$ which are isomorphic to $\tau^*_x(I)$ for some injective object I of $\mathcal{C}^*_x$. Let $\mathcal{D}$ be the smallest extension closed subcategory of $\mathcal{C}^\dagger$ which contains $\mathcal{D}_x$ for all x, regarded as a perfectly exact subcategory of $\mathcal{C}^\dagger$. It follows from 4.6.6 that $(\mathcal{D},\{\mathcal{D}\}_{x\in\Omega^{\mathrm{op}}})$ is a stratified exact category (though it is svelte only when $\mathcal{C}$ contains only zero objects).

Now for any ideals $\Gamma\subseteq\Sigma$ in Ω, we have for any injective object Q of $\mathcal{C}^*$ a canonical short exact sequence $0\to\sigma^*_\Gamma Q\to\sigma^*_\Sigma Q\to\sigma^*_\Lambda Q\to 0$ where $\Lambda=\Sigma\setminus\Lambda$, as follows by applying 4.1(f) to the (injective by 4.1(a)) object $\iota^*_\Sigma Q$ of $\mathcal{C}^*_\Sigma$. We view this as asserting that Q has a possibly infinite filtration with objects σ^*_xQ as "successive subquotients."

Proposition. *Suppose that for each $x\in\Omega$, the following condition* (i) *holds:*

(i) *the functor $\iota_{x*}\colon\mathcal{C}^*_x\to\mathcal{C}^*_{\leq x}$ induced by the inclusion $\iota_x\colon\mathcal{C}_x\to\mathcal{C}_{\leq x}$ is exact.*

If in addition Ω is finite, then the injective objects of $\mathcal{C}^*$ are precisely the injective objects of the stratified exact category $\mathcal{D}$ constructed above.

Proof. We begin with some remarks on the above condition (i) for arbitrary (interval finite) Ω. It is equivalent to assume that $\iota_{x*}\colon \mathcal{C}_x^* \to \mathcal{C}_\Gamma^*$ is exact for some ideal (resp., all ideals) $\Gamma \ni x$ of Ω. Assume (i) holds. If Γ is an ideal of Ω with x as maximal element, $\iota_x^*\colon \mathcal{C}_x^* \to \mathcal{C}_\Gamma^*$ is exact and preserves injectives since it has an exact left adjoint ι_{x*} and a right adjoint τ_x^*, and therefore for any injective object Q of $\mathcal{C}^*$, $\iota_x^* Q$ is injective in $\mathcal{C}_\Gamma^*$.

Now assume that Ω is finite. Then injectives in $\mathcal{C}^*$ are objects of $\mathcal{D}$ by the above remarks, and they are clearly injective in $\mathcal{D}$. On the other hand, suppose that Q is an injective object of $\mathcal{D}$. Let $0 \to Q \to \hat{Q} \to F \to 0$ be an injective envelope of Q in $\mathcal{C}^*$. Let x be a maximal element of Ω and $\Gamma := \Omega \setminus \{x\}$. We have a short exact sequence $0 \to Q(\Gamma) \to Q \to Q(x) \to 0$ in $\mathcal{D}$. Using the facts that $\iota_\Gamma^*(Q(x)) = 0$ and $\iota_x^*(Q(\Gamma)) = 0$, one deduces that this sequence is isomorphic to the natural one $0 \to \sigma_\Gamma^* Q \to Q \to \sigma_x^* Q \to 0$ (in particular, the latter is exact). There is an exact sequence $0 \to \iota_x^* Q \to \iota_x^* \hat{Q} \to \iota_x^* F \to 0$ in which the first two terms are injective, so the sequence is split and $\iota_x^* F$ is injective in $\mathcal{C}_x^*$. also. Applying τ_x^* gives a similar split exact sequence with ι_x^* replaced by σ_x^*.

Now by induction on the cardinality of Ω, $Q(\Gamma) \cong \sigma_\Gamma^* Q$ is injective in $\mathcal{C}_\Gamma^*$ so we may identify $\sigma_\Gamma^* Q = \sigma_\Gamma^* \hat{Q}$ by 4.2.1. This gives a commutative diagram in $\mathcal{C}^*$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma_\Gamma^* Q & \longrightarrow & Q & \longrightarrow & \sigma_x^* Q & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma_\Gamma^* \hat{Q} & \longrightarrow & \hat{Q} & \longrightarrow & \sigma_x^* \hat{Q} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & \sigma_x^* F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, from which the middle vertical column is seen to be an exact sequence in $\mathcal{D}$. Injectivity of Q in $\hat{D}$ implies that Q is a direct summand of $\hat{Q}$ in $\mathcal{D}$ and hence also in $\mathcal{C}^*$. Thus Q is injective in $\mathcal{C}^*$ as required. □

4.8. Assume for the remainder of this section that $\mathcal{C}_x = \operatorname{Add} \mathbf{N}_x$ where $\mathbf{N}_x = \{N_{x,i}\}_i$ and the $N_{x,i}$ are pairwise non-isomorphic objects of $\mathcal{C}_x$ with local endomorphism rings $R_{x,i} := \operatorname{End}(N_{x,i})^{\mathrm{op}}$. We set $R_x := \operatorname{End}(\mathbf{N}_x)^{\mathrm{op}}$, and $J_{x,i} := \operatorname{Rad} R_{x,i}$, the maximal ideal of $R_{x,i}$. By C.8, there is a natural category equivalence $\mathcal{C}_x^* \cong R_x$-Mod given by $F \mapsto \oplus_i F(N_{x,i})$.

Proposition. *For any x and i, $\phi(N_{x,i})$ has a unique maximal subobject in $\mathcal{C}^*$ (or $\mathcal{C}^\dagger$). Denote the simple quotient object of $\phi(N_{x,i})$ by $L_{x,i}$. Then any simple object of $\mathcal{C}^*$ (or $\mathcal{C}^\dagger$) is isomorphic to $L_{x,i}$ for a unique x and i. Moreover,* $\operatorname{End}(L_{x,i})^{\mathrm{op}} \cong$

$R_{x,i}/J_{x,i}$. Finally, if $I_{x,i}$ denotes an injective envelope of $L_{x,i}$ in $\mathcal{C}^*$, then $\prod_{x,i} I_{x,i}$ is an injective cogenerator of $\mathcal{C}^*$.

Proof. It will be enough to prove the assertions for $\mathcal{C}^*$. For the proof of the first assertion, we may assume without loss of generality that x is maximal in Ω, since $\phi(N_{x,i}) \cong \mathrm{Hom}(\sigma_{\leq x}?, N_{x,i})$ lies in the Serre subcategory $\mathcal{C}^*_{\leq x}$ (see 4.1(c)). We have the functor $j_x \colon R_x\text{-mod} \to \mathcal{C}^*$ with $(j_x(N))(M) = \mathrm{Hom}(M(x), \mathbf{N}_x) \otimes_{R_x} N$ for M in $\mathcal{C}$ and N in R_x-Mod (note $j_x(N)$ takes short exact sequences in $\mathcal{C}$ to split exact sequences of abelian groups, so $j_x(N)$ is left exact; actually, j_x identifies with τ_x^* under our identification $\mathcal{C}_x^* = R_x$-Mod). The natural transformation $\sigma_x \to \mathrm{Id}$ induces a homomorphism $\phi(N_{x,i}) \to \mathrm{Hom}(\sigma_x?, N_{x,i}) \cong j_x(R_x e_{x,i})$. Further, $R_x e_{x,i}$ has a unique simple quotient module $L'_{x,i} = (R_x/J_x)e_{x,i}$. This gives a composite homomorphism $\nu \colon \phi(N_{x,i}) \to j_x(R_x e_{x,i}) \to j_x(L'_{x,i})$. We will show that the kernel F_0 of this homomorphism is the unique maximal subobject of $\phi(N_{x,i})$.

For M in $\mathcal{C}$, $\big(j_x(L'_{x,i})\big)(M) = \mathrm{Hom}(\sigma_x(M), \mathbf{N}_x) \otimes_{R_x} L'_{x,i}$ with $\mathrm{Hom}(\sigma_x(M), \mathbf{N}_x)$ a f.g. projective right R_x-module. Identify $F_0(M) = \ker \nu_M$. Let $j \colon M(x) \to M$ be the canonical admissible monomorphism. By the definitions, $F_0(M)$ consists of those $h \in \mathrm{Hom}(M, N_{x,i})$ such that $hj \in \mathrm{Hom}(M(x), \mathbf{N}_x)J_x e_i$. If $h \in F_0(M)$, then $hjf \in e_i J_x e_i = J_{x,i}$ for all $f \colon N_{x,i} \to M(x)$. Conversely, if $hjf \in e_i J_x e_i$ for all $f \colon N_{x,i} \to M(x)$, then $hjf \in e_k J_x e_i$ for all $f \colon N_{x,k} \to M(x)$ and all k by C.15. Since $M(x)$ is in $\mathrm{Add}\,\mathbf{N}_x$, this latter condition implies that $hj \in \mathrm{Hom}(M(x), \mathbf{N}_x)J_x e_i$. Since any homomorphism $N_{x,i} \to M$ factors through j, it follows that

(4.8.1) $$F_0(M) = \{\, h \colon M \to N_{x,i} | hf \in \mathrm{Rad}\ \mathrm{End}(N_{x,i}) \text{ for all } f \colon N_{x,i} \to M \,\}.$$

Consider any exact sequence

$$0 \to F \to \mathrm{Hom}(?, N_{x,i}) \xrightarrow{\eta} G$$

in $\mathcal{C}^*$. We regard $F(M) = \ker(\eta_M)$ in $\mathbb{Z}$-Mod. For any morphism $h \colon M \to N_{x,i}$ in $\mathcal{C}$, we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & F(N_{x,i}) & \longrightarrow & \mathrm{Hom}(N_{x,i}, N_{x,i}) & \longrightarrow & G(N_{x,i}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(M) & \longrightarrow & \mathrm{Hom}(M, N_{x,i}) & \longrightarrow & G(M)
\end{array}$$

Let $a = \eta_{N_{x,i}}(\mathrm{Id}_{N_{x,i}})$. Looking at the right hand square above shows that the right hand horizontal map in the second row is given by $h \mapsto G(h)(a)$. Suppose that F is not a subobject of F_0. This means that for some $h \colon M \to N_{x,i}$ we have $G(h)(a) = 0$ but there is some $f \colon N_{x,i} \to M$ such that hf is a unit in $\mathrm{End}(N_{x,i})$. Without loss of generality, we may assume $hf = \mathrm{Id}_{N_{x,i}}$. Then $a = G(f)G(h)(a) = 0$ and so (by the Yoneda lemma), $\eta = 0$ and $F = \mathrm{Hom}(?, N_{x,i})$. This completes the proof that $\phi(N_{x,i})$ has a unique maximal subobject.

Now we show any simple object L of $\mathcal{C}^*$ is isomorphic to $L_{x,i}$ for some x and i. Certainly, $L(N) \neq 0$ for some $N \in \mathcal{C}$. By left exactness of L, we have $L(\sigma_x(N)) \neq 0$ for some x. Since $\sigma_x(N)$ is a direct summand of a finite direct sum of objects of the form $N_{x,i}$ for various i, we have $L(N_{x,i}) \neq 0$ for some i. By the Yoneda lemma, this gives a non-zero homomorphism $\mathrm{Hom}(?, N_{x,i}) = \phi(N_{x,i}) \to L$. This map is an epimorphism since L is simple, and so $L \cong L_{x,i}$ from above.

Next, we show that if $L_{x,i} \cong L_{y,j}$ then $(x,i) = (y,j)$. Suppose without loss of generality that $x \not< y$. We may then replace Ω by the ideal generated by x and y to assume x is maximal in Ω. A non-zero homomorphism $L_{x,i} \to L_{y,j}$ then lifts to a homomorphism $f\colon \phi(N_{x,i}) \to \phi(N_{y,j})$ by projectivity of $\phi(N_{x,i})$. Now $\operatorname{Im} f$ is not contained in the maximal submodule of $\phi(N_{y,j})$, so f is an epimorphism. Since $\operatorname{Hom}(N_{x,i}, N_{y,j}) = \operatorname{Hom}(\phi(N_{x,i}), \phi(N_{y,j})) \neq 0$ we have $x \leq y$ and so $x = y$. But then $\phi(N_{y,j})$ is projective as well, so f splits to give a non-zero idempotent $\phi(N_{x,i}) \xrightarrow{f} \phi(N_{y,j}) \to \phi(N_{x,j})$ in $\operatorname{End}(\phi(N_{x,i}))$. Since $\operatorname{End}(N_{x,i}) \cong \operatorname{End}(\phi(N_{x,i})$ is local, this idempotent is the identity. Hence $\phi(N_{x,i}) \cong \phi(N_{y,j})$ and so $N_{x,i} \cong N_{y,j}$ by faithfulness of the Gabriel-Quillen embedding. Thus $x = y$ and $i = j$ as required.

Now we determine $\operatorname{End}(L_{x,i})$. Again without loss of generality, assume x is maximal in Ω. If M is any object with a unique maximal subobject in an abelian category, the endomorphisms of M preserve its unique maximal subobject. This gives a ring homomorphism

$$\alpha\colon R_{x,i} = \operatorname{End}(N_{x,i})^{\mathrm{op}} = \operatorname{End}(\phi(N_{x,i}))^{\mathrm{op}} \to \operatorname{End}(L_{x,i})^{\mathrm{op}}.$$

Since $\phi(N_{x,i})$ is projective, α is an epimorphism. From the above description of F_0, it is clear that $\ker\alpha$ contains $J_{x,i}$. Since $J_{x,i}$ is the unique maximal ideal of $R_{x,i}$, we get $\ker\alpha = J_{x,i}$.

To prove the final assertion, it is enough to show that for any object F of $\mathcal{C}^*$, there is a non-zero morphism $F \to I_{x,i}$ for some x and i. For some x and i, we have a morphism $\phi(N_{x,i}) \to F$ with non-zero image G, say, which we regard as a subobject of F. Then we get from above a non-zero composite $G \to L_{x,i} \to I_{x,i}$ which lifts to a non-zero $F \to I_{x,i}$ since $I_{x,i}$ is injective. □

4.9. We list some additional properties of $\mathcal{C}^*$, leaving proofs to the reader. Let us say that an object F of $\mathcal{C}^*$ (resp., $\mathcal{C}^\dagger$) is a highest weight object of highest weight (x,i) if every simple subquotient object of F is of the form $L_{y,j}$ for some $y \leq x$, and there is a surjection $F \to L_{x,i}$ such that for every proper subobject F' of F, the composite $F' \to F \to L_{x,i}$ is zero.

4.9.1. The object $\phi(N_{x,i})$ is a universal highest weight object of highest weight (x,i) in $\mathcal{C}^*$ (resp., $\mathcal{C}^\dagger$), in the sense that the highest weight objects of highest weight (x,i) in $\mathcal{C}^*$ (resp., $\mathcal{C}^\dagger$) are precisely the non-zero quotients of $\phi(N_{x,i})$.

For the remaining properties, we let $L^\Sigma_{x,i}$ denote the unique simple quotient of $\phi_\Sigma(N_{x,i})$ in $\mathcal{C}^*_\Sigma$, for $x \in \Sigma$.

4.9.2. Let Λ be a coideal of Ω. Then for a simple object $L_{x,i}$ of $\mathcal{C}^*$, $\iota^*_\Lambda L_{x,i}$ is zero unless $x \in \Lambda$, in which case $\iota^*_\Lambda L_{x,i} = L^\Lambda_{x,i}$. Moreover, if $x \in \Lambda$ then $\iota_{\Lambda *} L^\Lambda_{x,i}$ has a unique maximal subobject with $L_{x,i}$ as corresponding quotient object and $\tau^*_\Lambda(L^\Lambda_{x,i})$ is an essential extension of $L_{x,i}$; if also x is minimal in Λ, $\tau^*_\Lambda(L^\Lambda_{x,i}) \cong \tau^*_x(L^x_{x,i})$ and $\iota_{\Lambda *} L^\Lambda_{x,i} \cong \iota_{x*} L^x_{x,i}$ are independent of Λ.

4.9.3. For any ideal Γ of I, one has $\tau^*_\Gamma(L^\Gamma_{x,i}) \cong L_{x,i}$. Moreover, $\tau_{\Gamma *} L_{x,i} \cong \iota^*_\Gamma L_{x,i}$ is zero if $x \notin \Gamma$ and isomorphic to $L^\Gamma_{x,i}$ if $x \in \Gamma$.

5. Inverse Systems and Sheaves

The rather technical results in this section are applied to give the most general version of our construction of weakly stratified exact categories (mentioned after

1.7) and are used in the study in subsequent sections of the category $\hat{\mathcal{C}}$ of pro-objects of a (weakly) stratified exact category $\mathcal{C}$ with an infinite weight poset. The reader interested primarily in situations with finite weight posets may wish to omit this section (since $\hat{\mathcal{C}} \cong \mathcal{C}$ if Ω is finite).

5.1. Let $\mathcal{D}$ be an exact category and (to begin) I be an arbitrary poset. Let $\mathcal{D}^{I^{\mathrm{op}}}$ be the exact category of contravariant functors (i.e. inverse systems) $I \to \mathcal{D}$. We write an object of $\mathcal{D}^{I^{\mathrm{op}}}$ as $Q = \{Q_x\}_{x\in I}$ and its limit (as a functor $I^{\mathrm{op}} \to \mathcal{D}$) as $\varprojlim_{x\in I} Q_x$ when it exists.

Remarks. We assume for convenience once again that $\mathcal{D}$ is a perfectly exact subcategory of an abelian category $\mathcal{B}$. Then $\mathcal{D}^{I^{\mathrm{op}}}$ is a perfectly exact subcategory of the abelian category $\mathcal{B}^{I^{\mathrm{op}}}$.

5.2. For an object A of $\mathcal{D}$ and $x \in I$, define the ("skyscraper") object $j_x^I A$ in $\mathcal{D}^{I^{\mathrm{op}}}$ as follows. Set $(j_x^I A)_y$ equal to A if $x \le y$ and to 0 if $x \not\le y$. For $y \ge z$ in I, the canonical map $f_{y,z}^{x,A} \colon (j_x^I A)_y \to (j_x^I A)_z$ of the inverse system $j_x^I A$ is the identity map if $z \ge x$ and (necessarily) the zero map if $z \not\ge x$. Extended in the obvious way to morphisms, this defines for each $x \in \Omega$ an exact functor $j_x^I \colon \mathcal{D} \to \mathcal{D}^I$. Note that the inverse system Q in $\mathcal{D}^{I^{\mathrm{op}}}$ is isomorphic to $j_x^I A$ iff Q_y is isomorphic to A for $y \ge x$, $Q_y = 0$ if $y \not\ge x$ and $Q_y \to Q_z$ is an isomorphism for $y \ge z \ge x$.

Lemma. *Let $x, y \in I$ and A, B be objects of $\mathcal{D}$. Then there are isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{D}^{I^{\mathrm{op}}}}(j_x^I A, j_y^I B) \cong \begin{cases} \mathrm{Ext}^i_{\mathcal{D}}(A,B) & \text{if } x \le y \\ 0 & \text{if } x \not\le y, \end{cases}$$

natural in A and B, for $i = 0$ and for $i = 1$ (of course, Ext^0 *is just* Hom*).*

Proof. One has for any $y \in \Omega$ the natural exact functor $\mathcal{D}^{I^{\mathrm{op}}} \to \mathcal{D}$ given by $M = \{M_x\}_{x\in I} \mapsto M_y$. This induces natural homomorphisms of Yoneda Ext-groups $\mathrm{Ext}^p_{\mathcal{D}^{I^{\mathrm{op}}}}(M,N) \to \mathrm{Ext}^p_{\mathcal{D}}(M_y, N_y)$ for all p. In particular, there are natural homomorphisms

$$\theta_i \colon \mathrm{Ext}^i_{\mathcal{D}^{I^{\mathrm{op}}}}(j_x^I A, j_y^I B) \to \mathrm{Ext}^i_{\mathcal{D}}((j_x^I A)_y, (j_y^I B)_y). \tag{5.2.1}$$

It will suffice to show θ_i is an isomorphism for $i = 0$ or $i = 1$.

The image of an element $\{h_z\}_{z\in I} \colon j_x^I A \to j_y^I B$ of $\mathrm{Hom}(j_x^I A, j_y^I B)$ under θ_0 is by definition $h_y : (j_x^I A)_y \to (j_y^I B)_y$. On the other hand, given $h \colon (j_x^I A)_y \to (j_y^I B)_y$ in $\mathcal{D}$, define $h_z \colon (j_x^I A)_z \to (j_y^I B)_z$ by $h_z = (f_{z,y}^{y,B})^{-1} h f_{z,y}^{x,A}$ if $z \ge y$ and $h_z = 0$ otherwise. Then $\{h_z\}_{z\in I} \colon j_x^I A \to j_y^I B$ in $\mathcal{D}^{I^{\mathrm{op}}}$ and $h \mapsto \{h_z\}_{z\in I}$ defines an inverse to θ_0.

Now we consider the map θ_1. Suppose e is an element of $\mathrm{Ext}^1(j_x^I A, j_y^I B)$ represented by the class of an extension

$$0 \to j_y^I B \to M \to j_x^I A \to 0 \tag{5.2.2}$$

in $\mathcal{D}^{I^{\mathrm{op}}}$. Write $M = \{M_z\}_{z\in I}$ with canonical maps $g_{z,w} \colon M_z \to M_w$ for $z \ge w$ in I. Then $\theta_1(e)$ is by definition the class in $\mathrm{Ext}^1_{\mathcal{D}}((j_x^I A)_y, (j_y^I B)_y)$ of the extension

$$0 \to (j_y^I B)_y \to M_y \to (j_x^I A)_y \to 0. \tag{5.2.3}$$

We claim that θ_1 is injective. For suppose that $d \colon M_y \to (j_y^I B)_y$ is a splitting map for the extension (5.2.3). For $z \in \Omega$, define $c_z \colon M_z \to (j_y^I B)_z$ by

$c_z = (f^{y,B}_{z,y})^{-1} dg_{z,y}$ if $z \geq y$, and $c_z = 0$ if $z \not\geq y$. Then one can check that $\{c_z\}_{z \in I}$ is in $\mathrm{Hom}_{\mathcal{D}^{I^{\mathrm{op}}}}(M, j^I_y B)$ and defines a splitting map for (5.2.2). Now we show that θ_1 is surjective. If $x \not\leq y$, then $(j^I_x A)_y = 0$, (5.2.3) is necessarily split, $\mathrm{Ext}^1_{\mathcal{D}}((j^I_x A)_y, (j^I_y B)_y) = 0$ and θ_1 is an isomorphism of zero modules. So assume $x \leq y$. Consider an element e' of $\mathrm{Ext}^1_{\mathcal{D}}((j^I_x A)_y, (j^I_y B)_y) = \mathrm{Ext}^1_{\mathcal{D}}(A, B)$ represented by the class of an extension

$$0 \to B \to M' \xrightarrow{f} A \to 0 \tag{5.2.4}$$

in $\mathcal{D}$. Define an inverse system $M = \{M_z\}_{z \in I}$ in $\mathcal{D}^{I^{\mathrm{op}}}$ by setting M_z equal to M' if $z \geq y$, to A if $z \geq x$ but $z \not\geq y$ and to 0 if $z \not\geq x$; the maps $M_z \to M_w$ for $z \geq w$ are the obvious maps $\mathrm{Id}_{M'}$, Id_A or f whenever possible, and 0 otherwise. One obtains an extension (5.2.2) in $\mathcal{D}^{I^{\mathrm{op}}}$ in which the corresponding extension

$$0 \to (j^I_y B)_z \to M_z \to (j^I_x A)_z \to 0.$$

in $\mathcal{D}$ is (5.2.4) for $z \geq y$, is $0 \to 0 \to A \to A \to 0$ for $z \geq x$ with $z \not\geq y$, and is a short exact sequence of zero modules if $z \not\geq x$. Letting e denote the class of the extension (5.2.2), one has $\theta_1(e) = e'$ and so θ_1 is an isomorphism as required. □

Remarks. For $x \in I$, $j^I_x \colon \mathcal{D} \to \mathcal{D}^{I^{\mathrm{op}}}$ is perfectly exact.

5.3. For the remainder of Section 3, Ω denotes a fixed interval finite poset. Let I_0, I_1 be as defined in 1.15 but let I be any set of ideals of Ω which contains I_0. We order I, I_0, I_1 by inclusion. The poset I_0 is isomorphic to Ω, under the map which takes an element of Ω to the ideal it generates.

5.4. We assume till 5.6 that Ω is finite (so I_1 is the family of all open sets of Ω). Let B be any additive category. There is a natural "forgetful functor" $B^{I^{\mathrm{op}}} \to B^{I_0^{\mathrm{op}}}$ given on objects by $\{Q_U\}_{U \in I} \mapsto \{Q_U\}_{U \in I_0}$.

We say that an object ("presheaf") $Q = \{Q_U\}_{U \in I_1}$ of $B^{I_1^{\mathrm{op}}}$ satisfies the sheaf axiom if $Q_\emptyset = 0$ and for any open U, V in I_1, the restriction maps of Q give a pullback square in B

$$\begin{array}{ccc} Q_{U \cup V} & \longrightarrow & Q_U \\ \downarrow & & \downarrow \\ Q_V & \longrightarrow & Q_{U \cap V} \end{array}$$

(cf. 2.4). Let Sh_B denote the full additive category of $B^{I_1^{\mathrm{op}}}$ consisting of objects satisfying the sheaf axiom.

The defining properties of limits imply that the sheaf axiom is equivalent to the requirement that for each open set U of Ω, the canonical maps $Q_U \to Q_{\leq x}$ for $x \in U$ induce an isomorphism

$$Q_U \xrightarrow{\cong} \varprojlim_{x \in U} Q_{\leq x}$$

(in particular, these limits exist). This implies that the natural composite functor $\theta_B \colon \mathrm{Sh}_B \to B^{I_1^{\mathrm{op}}} \to B^{I_0^{\mathrm{op}}}$ is full and faithful.

If B is an exact category, we denote by Fl_B the full additive subcategory of Sh_B consisting of objects Q such that all restriction maps $Q_U \to Q_V$ are admissible epimorphisms (we call such objects Q flabby sheaves). One sees from the sheaf axiom and exact category axioms that Fl_B is a full, extension closed subcategory of the exact category $B^{I_1^{\text{op}}}$. Hence Fl_B may be naturally regarded as a perfectly exact subcategory of $B^{I_1^{\text{op}}}$.

5.5. In this subsection, we assume that B is an abelian category. For Q in $B^{I_1^{\text{op}}}$, and open sets $U_1, \dots, U_n$ and $U = \cup_{i=1}^n U_n$ of Ω, there is a standard complex

$$0 \to Q_U \xrightarrow{f} \oplus_{i=1}^n (Q_{U_i}) \xrightarrow{g} \oplus_{1 \leq i < j \leq n} Q_{U_i \cap U_j},$$

where the components of f are the restriction maps $Q_U \to Q_{U_i}$ and for $i \neq j$, the component $Q_{U_i} \to Q_{U_i \cap U_j}$ of g is the restriction map (resp., the negative of the restriction map) if $i < j$ (resp., $i > j$). Exactness of these complexes for $n = 0$ and $n = 2$ is equivalent to the sheaf axiom for Q as given above. Conversely, if Q is a sheaf, one sees by induction on n that the above complex is exact.

Hence, the notion of sheaf as defined here is the usual one for abelian B. Likewise, a sheaf Q is flabby as defined here iff the restriction map $Q_\Omega \to Q_U$ is an epimorphism for all open U in Ω i.e. iff it is flabby (flasque) in the usual sense.

Note also that $\theta_B \colon \text{Sh}_B \to B^{I_0^{\text{op}}}$ is a category equivalence for abelian B; one readily verifies that there is an inverse to θ taking an inverse system $M = \{M_U\}_{U \in I_0}$ to $\hat{M}$ in Sh_B where for $U \in I$, $\hat{M}_U := \varprojlim_{x \in U} M_{\leq x}$ (note that the finite limits exist in B and the restriction maps are determined by the universal properties of limits). Since $B^{I_0^{\text{op}}}$ is an abelian category, Sh_B is (as well-known) an abelian category: a sequence

$$0 \to M' \to M \to M'' \to 0 \tag{5.5.1}$$

in Sh_B is exact iff it is carried by θ_B to an exact sequence i.e. iff the sequence of "stalks" $0 \to M'_{\leq x} \to M_{\leq x} \to M''_{\leq x} \to 0$ is exact in B for all $x \in \Omega$. Recall the following well-known fact (e.g. [27])

5.5.2. Suppose that (5.5.1) is a short exact sequence in Sh_B and that M' is flabby. Then (5.5.1) is a short exact sequence in $B^{I_1^{\text{op}}}$; moreover, M is flabby iff M'' is flabby.

On the other hand, it follows readily from the sheaf axiom that

5.5.3. If (5.5.1) is a short exact sequence in $B^{I_1^{\text{op}}}$ and M'', M' are sheaves, then M is a sheaf and (5.5.1) is a short exact sequence of sheaves.

These facts imply that

5.5.4. For an abelian category B, Fl_B is a perfectly exact subcategory of Sh_B.

5.6. For an exact category $\mathcal{D}$, let $\alpha_{\mathcal{D}}$ denote the composite $\text{Fl}_{\mathcal{D}} \to \mathcal{D}^{I_1^{\text{op}}} \to \mathcal{D}^{I_0^{\text{op}}}$.

Lemma. *If $\mathcal{D}$ is an exact category, the functor $\alpha_{\mathcal{D}}$ is perfectly exact.*

Proof. Consider the commutative diagram of functors

$$\begin{array}{ccc} \text{Fl}_{\mathcal{D}} & \longrightarrow & \mathcal{D}^{I_0^{\text{op}}} \\ \downarrow & & \downarrow \\ \text{Fl}_{\mathcal{B}} & \longrightarrow & \mathcal{B}^{I_0^{\text{op}}} \end{array}$$

The vertical arrows are perfectly exact functors functors, and it will suffice to show the bottom arrow has the same properties. The bottom arrow factorizes as

$$\text{Fl}_{\mathcal{B}} \xrightarrow{u} \text{Sh}_{\mathcal{B}} \xrightarrow{v} \mathcal{B}^{I_1^{\text{op}}} \xrightarrow{w} \mathcal{B}^{I_0^{\text{op}}}.$$

As seen above, u is perfectly exact and wv is an equivalence of abelian categories. $\square$

Remarks. It follows that an inverse to the restriction $\alpha_{\mathcal{D}}\colon \mathrm{Fl}_{\mathcal{D}} \to \operatorname{Im}\alpha_{\mathcal{D}}$ takes an inverse system $M := \{M_\Gamma\}_{\Gamma\in I_0}$ in $\operatorname{Im}\alpha_{\mathcal{D}}$ to $\hat{M} = \{\hat{M}_U\}_{U\in I_1}$ where $\hat{M}_U = \varprojlim_{x\in U} M_{\leq x}$ (limit in $\mathcal{B}$). In particular, the limits defining the $\hat{M}_U$ exist in $\mathcal{D}$, and $M \mapsto \hat{M}_U$ is an exact functor $\operatorname{Im}\alpha_{\mathcal{D}} \to \mathcal{D}$.

5.7. Let $F = (\mathcal{D}, \{\mathcal{D}_x\}_{x\in\Omega})$ where the $\mathcal{D}_x$ are any full additive subcategories of the exact category $\mathcal{D}$ indexed by an interval finite poset Ω. For $x \in \Omega$, the ideal $\leq x$ generated by x is in I, and we abbreviate $j^I_{\leq x}\colon \mathcal{D} \to \mathcal{D}^{I^{\mathrm{op}}}$ by j^I_x or even j_x if I is fixed. We denote the strict image of the restriction $j^I_x\colon \mathcal{D}_x \to \mathcal{D}^{I^{\mathrm{op}}}$ by $j^I_x(\mathcal{D}_x)$. Then by 5.2, $F^I := (\mathcal{D}^{I^{\mathrm{op}}}, \{j^I_x(\mathcal{D}_x)\}_{x\in\Omega})$ satisfies the conditions imposed on D in Section 2, so one may define the additive category $\mathcal{D}^0[F^I]$ of objects of $\mathcal{D}^{I^{\mathrm{op}}}$ with a F^I-filtration.

The category $\mathcal{D}^0[F^I]$ is independent of the choice of I containing I_0, up to equivalence. More precisely,

Lemma. *The "forgetful" functor $\mathcal{D}^{I^{\mathrm{op}}} \to \mathcal{D}^{I_0^{\mathrm{op}}}$ given on objects by $\{M_\Gamma\}_{\Gamma\in I} \mapsto \{M_\Gamma\}_{\Gamma\in I_0}$ restricts to an equivalence $\rho_I\colon \mathcal{D}^0[F^I] \to \mathcal{D}^0[F^{I_0}]$. A functor which is an inverse to ρ_I is given by $\{M_\Gamma\}_{\Gamma\in I_0} \mapsto \{\hat{M}_\Gamma\}_{\Gamma\in I}$ where $\hat{M}_\Gamma := \varprojlim_{x\in\Gamma} M_{\leq x}$. These equivalences are compatible with the respective truncation functors σ_Γ on $\mathcal{D}^0[F^I]$ and $\mathcal{D}^0[F^{I_0}]$. Moreover, the short exact sequences in $\mathcal{D}^{I^{\mathrm{op}}}$ of objects of $\mathcal{D}^0[F^I]$ correspond under ρ_I to short exact sequence in $\mathcal{D}^{I_0^{\mathrm{op}}}$ of objects of $\mathcal{D}^0[F^{I_0}]$.*

Proof. Consider an object $M = \{M_U\}_{U\in I}$ of $\mathcal{D}^0[F^I]$. Then M is an object of $\mathcal{D}^0[(F^I)_\Sigma]$ for some finite locally closed subset Σ of Ω. We claim that

5.7.1. For any $U \in I$, the canonical map $M_U \to \hat{M}_U := \varprojlim_{x\in\Sigma\cap U} M_{\leq x}$ in $\mathcal{B}$ is an isomorphism (here, the limit exists as a finite limit in the abelian category $\mathcal{B}$ and hence it is also the limit in $\mathcal{D}$).

We prove the claim by induction on the number $n(M)$ of $x \in \Sigma$ with $\tau_x(M) \neq 0$ (alternatively, one could reduce to the case where $\Omega = \Sigma$ is finite and use Remark 5.6). If $n(M) = 1$, then $M \cong j^I_x(B)$ for some object B of $\mathcal{D}_x$ and $x \in \Sigma$, and the claim is trivial. Otherwise, one may choose a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{D}^{I^{\mathrm{op}}}$ of objects of $\mathcal{D}^0[(F^I)_\Sigma]$ with $n(M')$ and $n(M'')$ both smaller than $n(M)$. The claim follows from the commutative diagram with exact rows in $\mathcal{B}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_U & \longrightarrow & M_U & \longrightarrow & M''_U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & \hat{M}'_U & \longrightarrow & \hat{M}_U & \longrightarrow & \hat{M}''_U & &
\end{array}$$

(recall the left exactness of $\varprojlim$).

Now the claim implies that the restriction maps $M_V \to M_U$ of the inverse system M are uniquely determined by the restriction maps $M_{\leq x} \to M_{\leq y}$ for $x \geq y$ in Σ (using the universal properties of limits). It follows immediately from this that ρ_I is full and faithful. For any $\Gamma \in I$, the claim also implies

$$\varprojlim_{x\in\Gamma} M_{\leq x} \cong \varprojlim_{x\in\Gamma} \hat{M}_{\leq x} \cong \varprojlim_{x\in\Gamma\cap\Sigma} M_{\leq x} \cong \hat{M}_\Gamma \cong M_\Gamma.$$

Here, the limits are taken in $\mathcal{B}$ (and hence are also the limits in $\mathcal{D}$).

Now we show ρ_I is essentially surjective on objects. Suppose given an object $\{M_U\}_{U\in I_0}$ in $\mathcal{D}^0[(F^{I_0})_\Sigma]$. The (finite) limits defining $\hat{M}_U := \varprojlim_{x\in\Sigma\cap U} M_{\leq x}$ exist by Lemma 5.6 and Remark 5.6. They determine an object $\hat{M} = \{\hat{M}\}_{U\in I}$ of $\mathcal{D}^{I^{\mathrm{op}}}$ with restriction maps determined by the universal properties of limits. The map $M \mapsto \hat{M}$ extends naturally to a functor $\mathcal{D}^0[(F^{I_0})_\Sigma] \to \mathcal{D}^{I^{\mathrm{op}}}$, and, by Lemma 5.6 again, this functor maps short exact sequences in $\mathcal{D}^{I_0^{\mathrm{op}}}$ of objects of $\mathcal{D}^0[(F^{I_0})_\Sigma]$ to short exact sequences in $\mathcal{D}^{I^{\mathrm{op}}}$. Note that if $M \cong j_x^{I_0}(B)$ for B in $\mathcal{D}_x$ and $x\in\Sigma$, then $\hat{M}\cong j_x^I(B)$. It follows that $\hat{M}$ is in $\mathcal{D}^0[F^I]$ in general, and clearly $\rho_I\hat{M}\cong M$. The remaining claims of the lemma follow readily. □

5.8. Let F be as in the previous subsection. Suppose now that for each $U\in I$, the data $F_U = (\mathcal{D}, \{\mathcal{D}_y\}_{y\in U})$ satisfies the conditions 1.4(i) and 1.4(ii). Then we may define the exact category $\mathcal{D}^0[F_U]$ for $U\in I$ and the associated truncation functors σ_Λ for Λ locally closed in U. If $U\subseteq V$ are in I, then $\mathcal{D}^0[F_U]$ is a full additive subcategory of $\mathcal{D}^0[F_V]$.

Let $\hat{\mathcal{D}}^0 = \hat{\mathcal{D}}^0[F^I]$ denote the full additive subcategory of $\mathcal{D}^{I^{\mathrm{op}}}$ consisting of inverse systems $\{Q_U\}_{U\in I}$ satisfying the following conditions:

(i) Q_U is in $\mathcal{D}^0[F_U]$ for any $U\in I$

(ii) for $U\subseteq V$ in I, the restriction map $Q_V\to Q_U$ of Q is an admissible epimorphism in $\mathcal{D}$ with kernel $\sigma_{V\setminus U}Q_V$.

For any locally closed subset Γ of Ω, define the "truncation" functor $\hat{\sigma}_\Gamma = \hat{\sigma}_{\Gamma,I}\colon \hat{\mathcal{D}}^0[F^I]\to\hat{\mathcal{D}}^0[F^I]$ with $(\hat{\sigma}_\Gamma Q)_U = \sigma_{\Gamma\cap U}Q_U$ and restriction maps

$$\sigma_{\Gamma\cap V}Q_V \to \sigma_{\Gamma\cap U}Q_V \xrightarrow{\cong} \sigma_{\Gamma\cap U}Q_U$$

for $V\supseteq U$ in I.

Lemma. (a) *In general, $\hat{\mathcal{D}}^0[F^I]$ contains $\mathcal{D}^0[F^I]$ as its full additive subcategory of objects Q with $\hat{\tau}_x(Q)$ non-zero for only finitely many x; these two categories coincide if $\Omega\in I_1$.*

(b) *The forgetful functor $\{Q_U\}_{U\in I}\mapsto\{Q_U\}_{U\in I_0}$ induces a category equivalence $\hat{\mathcal{D}}^0[F^I]\to\hat{\mathcal{D}}^0[F^{I_0}]$ provided $I_0\subseteq I\subseteq I_1$.*

(c) *Suppose that the data $F=(\mathcal{D},\{\mathcal{D}_y\}_{y\in\Omega})$ itself satisfies the conditions* 1.4(i) *and* 1.4(ii)*. Then there is a natural equivalence of additive categories $\theta = \theta_I\colon \mathcal{D}^0[F]\to\mathcal{D}^0[F^I]$ mapping an object Q of $\mathcal{D}^0[F]$ to the inverse system $\{Q(U)\}_{U\in I}$ (with restriction maps $Q(U)\to Q(V)$ compatible with all the canonical admissible epimorphisms $Q\to Q(W)$).*

Proof. For this proof, we denote the truncation functors in $\mathcal{D}^0[F^I]$ by σ'_Γ and those on $\mathcal{D}^0[F_U]$ by σ_Γ.

We first prove (a). Suppose $Q=\{Q_U\}_{U\in I}$ is an object of $\mathcal{D}^0[F^I]$. Then for $U\in I$, the definitions immediately show that Q_U is an object of $\mathcal{D}^0[F_U]$, with $\sigma_\Gamma(Q_U) = (\sigma'_\Gamma Q)_U$ for any locally closed subset Γ of U; for Γ a coideal of Γ' (resp., Γ' an ideal of Γ), the natural map $\sigma_\Gamma(Q_U)\to\sigma_{\Gamma'}(Q_U)$ identifies with the natural map $(\sigma'_\Gamma Q)_U\to(\sigma'_{\Gamma'}Q)_U$. For $U\subseteq V$ in I and $z\in V$, the map $\sigma_z(Q_V)\to\sigma_z(Q_U)$ induced by $Q_V\to Q_U$ identifies with $(\sigma'_z Q)_V\to(\sigma'_z Q)_U$, which by the definitions is an isomorphism of objects of $\mathcal{D}_z$ if $z\in U$ and is the zero map otherwise. It follows readily that $Q_V\to Q_U$ is an admissible epimorphism with kernel $\sigma_{V\setminus U}Q_V$, so Q is in $\hat{\mathcal{D}}^0[F^I]$.

Now suppose Q is in $\hat{\mathcal{D}}^0[F^I]$. If Λ is a coideal of Γ and $\Sigma = \Gamma \setminus \Lambda$, there is a natural short exact sequence

$$0 \to \hat{\sigma}_\Lambda Q \to \hat{\sigma}_\Gamma Q \to \hat{\sigma}_\Sigma Q \to 0$$

in $\mathcal{D}^{I^{\text{op}}}$ with $\hat{\sigma}_\Lambda Q \to \hat{\sigma}_\Gamma Q$ given by $\{\sigma_{\Lambda\cap U} Q_U \to \sigma_{\Gamma\cap U} Q_U\}_{U\in I}$ (and similarly for the other map). It is easily seen that $\hat{\sigma}_z(Q) \cong j_z^I(\sigma_z(Q_{\leq z}))$ for $z \in \Omega$. From these facts, it follows that Q is in $\mathcal{D}^0[F^I]$ provided $\hat{\sigma}_z(Q) \neq 0$ for only finitely many $z \in \Omega$.

Consider arbitrary $x_1, \dots, x_n$ in Ω, and let V' be the ideal of Ω they generate. Choose any finite coideal Σ of V' which contains all (finitely many) $z \in \Omega$ with $\sigma_z(Q_{\leq x_i}) \neq 0$ for some i. For any $U \in I$ with $U \subseteq V'$ and any $z \in U \setminus \Sigma$, one has $\sigma_z(Q_U) = 0$ (if, say, $z \leq x_i$, consider the restriction maps $Q_U \to Q_{\leq z} \leftarrow Q_{\leq x_i}$ and use the definition of $\hat{\mathcal{D}}^0[F^I]$). If $\Omega \in I_1$, choose $x_1, \dots, x_n$ as generators of Ω as an ideal; the above shows that $\hat{\sigma}_z(Q) \neq 0$ implies $z \in \Sigma$, and since Σ is finite, it follows that Q is in $\mathcal{D}^0[F^I]$ with $\sigma'_\Gamma Q \cong \hat{\sigma}_\Gamma Q$ for all locally closed $\Gamma \subseteq \Omega$. This completes the proof of (a).

Now let Q be in $\hat{\mathcal{D}}^0[F^I]$ and $\Omega' \in I_1$. Define $J := \{\, V \in I | V \subseteq \Omega' \,\}$. Observe that $\{Q_U\}_{U\in J}$ is in $\hat{\mathcal{D}}^0[(F_{\Omega'})^J]$. Taking $V' \supseteq \Omega'$ above, we see that $\tau_z(\{Q_U\}_{U\in J})$ is nonzero for only finitely many $z \in \Omega'$, so $\{Q_U\}_{U\in J}$ is even in $\mathcal{D}^0[(F_{\Omega'})^J]$. If $I \subseteq I_1$, it follows immediately from this using 5.7 that an inverse equivalence to the forgetful functor in (b) is given by $\{Q_U\}_{U\in I_0} \mapsto \{\hat{Q}_U\}_{U\in I}$ where $\hat{Q}_U := \varprojlim_{x\in U} Q_{\leq x}$. This completes the proof of (b).

In the situation (c), it is readily checked using 5.7 that the functor taking an object $Q = \{Q_U\}_{U\in I}$ of $\mathcal{D}^0[F^I]$ to $\varprojlim_{U\in I} Q_U$ gives an inverse equivalence; in fact, one may take $I = I_1$ and then the limit in question is canonically isomorphic to Q_U for any $U \in I_1$ which contains the finitely many $z \in \Omega$ with $\hat{\sigma}_z(Q) \neq 0$. □

Remarks. The embedding $\hat{\mathcal{D}}^0[F^I] \to \mathcal{D}^0[F^I]$ of (a) and equivalences as in (b), (c) of the lemma are obviously compatible with the respective truncation functors on these categories. Moreover, they are easily seen to preserve exact sequences in the following sense. For a "base" additive category $\hat{\mathcal{D}}^0[F^I]$ (resp., $\mathcal{D}^0[F^I]$ or $\mathcal{D}^0[F]$), define the corresponding "ambient" exact category as $\mathcal{D}^{I^{\text{op}}}$ (resp., $\mathcal{D}^{I^{\text{op}}}$ or $\mathcal{D}$). Let $F \colon X \to Y$ denote the embedding of (a) or the equivalence as in (b) or (c); then a three term exact sequence of objects of the base category X is exact in the ambient exact category of X iff application of F gives a short exact sequence of objects of the ambient exact category of Y (this is trivial for (a) and follows using 5.6 for (b) and (c)).

5.9. We now give the more general construction of weakly stratified exact categories mentioned after Theorem 1.7.

Let $E := \{F_x : \mathcal{D}_x \to \mathcal{D}\}_{x\in\Omega}$ denote an arbitrary family of exact functors from exact categories $\mathcal{D}_x$ into a fixed exact category $\mathcal{D}$, indexed by the interval finite poset Ω. Then for any I with $I_0 \subseteq I \subseteq I_1$, the family $E^I := \{j_x^I F_x \colon \mathcal{D}_x \to \mathcal{D}^{I^{\text{op}}}\}_{x\in\Omega}$ satisfies the condition 3.1, by 5.2, so the weakly stratified exact category $\mathcal{C}^0[E^I]$ may be defined as in Section 3.

Theorem. *$\mathcal{C}^0[E^I]$ is independent of I (with $I_0 \subseteq I \subseteq I_1$) up to equivalence of weakly stratified exact categories. Moreover, if E itself satisfies the conditions of* 3.1, *then $\mathcal{C}^0[E]$ is equivalent as weakly stratified exact category to $\mathcal{C}^0[E^I]$.*

Proof. The first claim follows from Lemma 5.7, and the second claim follows from the Lemma and Remark in 5.8. □

5.10. Maintain the notation of the previous subsection. There is a natural exact functor $H\colon \mathcal{C}^0[E^{I_1}] \to \mathcal{D}$ determined on objects by

$$Q = \{Q_U\}_{U\in I_1} \to \varprojlim_{U\in I_1} Q_U$$

(the limit is canonically isomorphic to Q_U for any $U \in I_1$ which contains all the finitely many $z \in \Omega$ with $\hat{\sigma}_z(Q) \neq 0$). One can check that H is faithful if all the F_x are faithful. Further, using the second assertion of the theorem, one sees that H is full and faithful if E satisfies the conditions of 3.1 and all the F_x are full and faithful. Finally, if in addition the F_x are (strict) inclusion functors, then H obviously induces an equivalence $\mathcal{C}^0[E^I] \to \mathcal{C}$ where $\mathcal{C}$ is as defined in Theorem 1.7.

6. Pro-objects

In this section, we define the category $\hat{\mathcal{C}}$ of pro-objects of a weakly stratified exact category and discuss its functoriality in $\mathcal{C}$.

6.1. Throughout this section, we fix a weakly stratified exact category $\mathcal{C}$ with strata $\mathcal{C}_x$ and weight poset Ω. Define $F := (\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})$ (note that F satisfies the assumptions of 2.1). Now for any I with $I_0 \subseteq I \subseteq I_1$, we form F^I as in Section 5 and then define the additive subcategory $\hat{\mathcal{D}}^0[F^I]$ as in Section 2.

Thus, $\hat{\mathcal{D}}^0[F^I]$ is the full additive subcategory of the exact category $\mathcal{C}^{I^{\mathrm{op}}}$ consisting of inverse systems $\{Q_U\}_{U\in I}$ on I such that Q_U is in $\mathcal{C}_U$ and the restriction map $Q_V \to Q_U$ for $V \supseteq U$ is an admissible epimorphism in $\mathcal{C}$ with kernel $\sigma_{V\setminus U} Q_V$. It is easily checked that $\hat{\mathcal{D}}^0[F^I]$ is closed under extensions in $\mathcal{C}^{I^{\mathrm{op}}}$. We may therefore regard $\hat{\mathcal{D}}^0[F^I]$ as a perfectly exact subcategory of $\mathcal{C}^{I^{\mathrm{op}}}$. We denote this exact category as $\hat{\mathcal{C}}_I := \hat{\mathcal{D}}^0[F^I]$; it coincides with $\hat{\mathcal{C}}_I$ defined in 1.15. Moreover, the truncation functors $\hat{\sigma}_\Gamma\colon \hat{\mathcal{D}}^0[F^I] \to \hat{\mathcal{D}}^0[F^I]$ for Γ locally closed in Ω, with $(\hat{\sigma}_\Gamma Q)_U \cong \sigma_{\Gamma\cap U} Q_U$, coincide with the functors $\hat{\sigma}_{\Gamma,I}$ defined for $\hat{\mathcal{C}}_I$ in 1.15 and are readily seen to be exact. In fact, all the assertions of Lemma 1.15 are immediate consequences of the definitions together with Lemma 5.7, Lemma 5.8 and Remark 5.8. As in 1.15, we henceforward denote $\hat{\mathcal{C}}_I$ for $I_0 \subseteq I \subseteq I_1$ just as $\hat{\mathcal{C}}$, θ_I as θ and $\hat{\sigma}_{\Gamma,I}$ as $\hat{\sigma}_\Gamma$.

It is easy to check that idempotents split in $\hat{\mathcal{C}}$ if they split in $\mathcal{C}$.

6.2. For a locally closed subset Γ of Ω, we define $\hat{\mathcal{C}}_\Gamma$ to be the perfectly exact subcategory of $\hat{\mathcal{C}}$ consisting of objects Q with Q_U in $\mathcal{C}_\Gamma$ for all $U \in I_1$ (equivalently, for all $U \in I_0$).

We let $\hat{\tau}_\Gamma\colon \hat{\mathcal{C}} \to \hat{\mathcal{C}}_\Gamma$ denote the restriction of σ_Γ, and $\hat{\iota}_\Gamma\colon \hat{\mathcal{C}}_\Gamma \to \hat{\mathcal{C}}$ denote the inclusion. Observe that for $\{Q_U\}_{U\in I_1}$ in $\mathcal{C}_\Gamma$, $\Gamma \in I_1$, one has $\hat{\sigma}_x(Q) \neq 0$ iff $\sigma_x(Q_\Gamma) \neq 0$. One therefore has from 5.8(a),(c) that

6.2.1. If $\Gamma \in I_1$, then θ restricts to an equivalence of exact categories $\mathcal{C}_\Gamma \to \hat{\mathcal{C}}_\Gamma$. Moreover, $\hat{\sigma}_\Gamma(M) \cong \theta(M_\Gamma)$ for any M in $\hat{\mathcal{C}}$.

The following facts follow readily from their analogues in 2.5.

6.2.2. For any locally closed subsets Γ and Σ of Ω such that Σ is a coideal of Γ there are short exact sequences

$$0 \to \hat{\sigma}_\Sigma Q \to \hat{\sigma}_\Gamma Q \to \hat{\sigma}_{\Gamma\setminus\Sigma} Q \to 0 \tag{6.2.3}$$

functorial in Q for Q in $\hat{\mathcal{C}}$

6.2.4. If Λ is an ideal of Ω, then $\operatorname{Hom}(\hat{\sigma}_{\Omega\setminus\Lambda} M, \hat{\sigma}_\Lambda N) = 0$ for all M, N in $\hat{\mathcal{C}}$.

6.2.5. If Γ is an ideal (resp., coideal) of Ω then $\hat{\tau}_\Gamma$ is left (resp., right) adjoint to $\hat{\iota}_\Gamma$ and the counit $\hat{\tau}_\Gamma \hat{\iota}_\Gamma \to 1$ (resp., unit $1 \to \hat{\tau}_\Gamma \hat{\iota}_\Gamma$) is a natural isomorphism.

6.2.6. For two locally closed subsets Γ, Σ of Ω there is a natural isomorphism $\hat{\sigma}_\Sigma \hat{\sigma}_\Gamma \cong \hat{\sigma}_{\Sigma\cap\Gamma}$.

Consider two objects M, N of $\hat{\mathcal{C}}$. There is a natural inverse system of abelian groups $\{\operatorname{Hom}_{\mathcal{D}}(M_U, N_U)\}_{U\in I}$. The maps $\operatorname{Hom}_{\hat{\mathcal{C}}}(M,N) \to \operatorname{Hom}_{\mathcal{C}}(M_U, N_U)$ for $U \in I$ (from the definition of Hom in $\hat{\mathcal{C}}$) induce isomorphisms of abelian groups

$$\operatorname{Hom}_{\hat{\mathcal{C}}}(M,N) \cong \varprojlim_{U\in I} \operatorname{Hom}_{\mathcal{C}}(M_U, N_U). \tag{6.2.7}$$

Also, for in M in $\hat{\mathcal{C}}$, and in Q in $\mathcal{C}_\Gamma$ with $\Gamma \in I_1$, the natural map gives an isomorphism

$$\operatorname{Hom}_{\hat{\mathcal{C}}}(M, \theta(Q)) \cong \operatorname{Hom}_{\mathcal{C}}(M_\Gamma, Q). \tag{6.2.8}$$

Applying this equation and the definition of projective object to the admissible epimorphism $N \to \hat{\sigma}_U(N) \cong \hat{\theta}(N_U)$ for N in $\hat{\mathcal{C}}$ shows that

6.2.9. If M is a projective object of $\hat{\mathcal{C}}$, then for each $U \in I$, the natural map $\operatorname{Hom}_{\hat{\mathcal{C}}}(M,N) \cong \operatorname{Hom}_{\mathcal{C}}(M_U, N_U)$ is an epimorphism.

6.3. Consider weakly stratified exact categories $(\mathcal{C}^{(i)}, \{\mathcal{C}^{(i)}_x\}_{i\in\Omega^{(i)}})$ for $i \in \mathbb{Z}$. For notational simplicity, we henceforward abbreviate $X^{(i)}$ as X^i for any X. There are various standard objects X^i associated to $\mathcal{C}^i$ in the same way as X is associated to $\mathcal{C}$; for instance, we have posets $I^i_0 \subseteq I^i \subseteq I^i_1$ of ideals of Ω^i, categories $\hat{\mathcal{C}}^i$ of pro-objects of $\mathcal{C}^i$, truncation functors σ^i_Γ, $\hat{\sigma}^i_\Gamma$, perfectly exact functors $\theta^i \colon \mathcal{C}^i \to \hat{\mathcal{C}}^i$ etc.

Let $F \colon \mathcal{C}^1 \to \mathcal{C}^2$ be a fixed covariant functor. For $\Lambda \subseteq \Omega^2$, we define

$$F^{-1}(\Lambda) := \{\, y \in \Omega^1 | \sigma_\Lambda F(A) \neq 0 \text{ for some } A \text{ in } \mathcal{C}^1_y \,\}.$$

Similarly, for $\Gamma \subseteq \Omega^1$, we define

$$F(\Gamma) := \{\, z \in \Omega^2 | \sigma_z F(A) \neq 0 \text{ for some } A \text{ in } \mathcal{C}^1_y,\ y \in \Gamma \,\}.$$

If Λ is an ideal of Ω^2, Γ is an ideal of Ω^1 containing $F^{-1}(\Lambda)$ and M is in $\mathcal{C}^1$ with $\sigma_\Lambda F M \neq 0$, then it follows if F is left or right exact that $\sigma_\Gamma M \neq 0$.

We say that F is stable backwards for f.g. ideals if for any f.g. ideal Λ of Ω^2, there exists a f.g. ideal Γ of Ω^1 with $\Gamma \supseteq F^{-1}(\Lambda)$. We say that F is stable forwards for f.g. ideals if for any f.g. ideal Γ of Ω^1, there is a f.g. ideal Λ of Ω^2 with $\Lambda \supseteq F(\Gamma)$. We say F is stable for f.g. ideals if it is stable forwards and stable backwards for f.g. ideals. Similarly, we define what is meant by saying F is stable, or stable forwards, or stable backwards, for f.g. coideals (recalling $(\mathcal{C}^i)^{\mathrm{op}}$ is weakly stratified with weight poset $(\Omega^i)^{\mathrm{op}}$, each stability condition for f.g. coideals is dual to the corresponding stability condition for f.g.ideals). Finally, we say that

F is bistable if F is stable for both f.g. ideals and for f.g. coideals (i.e. F and $F^{\mathrm{op}}\colon (\mathcal{C}^1)^{\mathrm{op}} \to (\mathcal{C}^2)^{\mathrm{op}}$ are both stable for f.g. ideals). Observe that each of the above stability conditions (stability forwards or backwards for ideals or coideals) is preserved under the operations of taking direct sums or direct summands of functors, or composites of right (resp., left) exact functors. There are relationships between stability properties of adjoint functors; we shall not need them, so they are left to the interested reader.

6.4. **Proof of Theorem 1.16.** We regard $\mathcal{C}^i$ as a full subcategory of $\hat{\mathcal{C}}^i$ via the completely exact embedding $\theta^i\colon \mathcal{C}^i \to \hat{\mathcal{C}}^i$. Consider an object $M = \{M_\Lambda\}_{\Lambda\in I^1}$ of $\hat{\mathcal{C}}^1$. For $\Gamma \in I^2$ and $\Lambda, \Sigma \in I^1$ with $\Lambda \supseteq \Sigma$, there is an exact sequence

$$\sigma_\Gamma F \sigma_{\Lambda\setminus\Sigma} M_\Lambda \to \sigma_\Gamma F M_\Lambda \to \sigma_\Gamma F M_\Sigma \to 0$$

in $\mathcal{C}^2$ obtained by applying $\sigma_\Gamma F$ to the evident canonical short exact sequence in $\mathcal{C}^1$. Now if $\Sigma \supseteq F^{-1}(\Gamma)$, then $\sigma_\Gamma F \sigma_{\Lambda\setminus\Sigma} M_\Lambda = 0$. Hence $\varprojlim_{\Lambda\in I^1} \sigma_\Gamma F(M_\Lambda)$ exists and is canonically isomorphic to $\sigma_\Gamma F(M_\Lambda)$ for any $\Lambda \supseteq F^{-1}(\Gamma)$. Now for $I^2 \ni \Gamma \supseteq \Gamma'$ and $\Lambda \in I^1$, the natural maps $\sigma_\Gamma F(M_\Lambda) \to \sigma_{\Gamma'} F(M_\Lambda)$ induce maps $\varprojlim_{\Lambda\in I^1} \sigma_\Gamma F(M_\Lambda) \to \varprojlim_{\Lambda\in I^1} \sigma_{\Gamma'} F(M_\Lambda)$, defining an object $\hat{F}(M)$ of $\hat{\mathcal{C}}^2$ with $(\hat{F}M)_\Gamma := \varprojlim_{\Lambda\in I^1} \sigma_\Gamma F(M_\Lambda)$ for $\Gamma \in I^2$. This construction extends in an obvious way to define a functor $\hat{F}\colon \hat{\mathcal{C}}^1 \to \hat{\mathcal{C}}^2$ as in (a). (Right) exactness of $\hat{F}$ follows immediately from the (right) exactness of F and exactness of the truncation functors. It is clear that $\hat{F}$ extends F in the sense that there is a natural isomorphism $\hat{F}\hat{\theta}_1 \cong \hat{\theta}_2 F$.

Now we define $\hat{\epsilon}$. The components $\hat{\epsilon}_M$ for M in $\hat{\mathcal{C}}^1$ are induced by the morphisms

$$\sigma_\Gamma(\epsilon_{M_\Lambda})\colon \sigma_\Gamma F(M_\Lambda) \to \sigma_\Gamma F'(M_\Lambda).$$

It is clear that $\widehat{\mathrm{Id}_F} = \mathrm{Id}_{\hat{F}}$ and that if $\epsilon'\colon F' \mapsto F''$ is another natural transformation with F'' right exact and stable backwards for f.g. ideals, then $\widehat{\epsilon'\epsilon} = \hat{\epsilon}'\hat{\epsilon}$, where $(\epsilon'\epsilon)_M = \epsilon'_M \epsilon_M$ for M in $\mathcal{C}^1$ etc. Thus, $\epsilon \mapsto \hat{\epsilon}$ is functorial.

On the other hand, we claim that $F \mapsto \hat{F}$ is functorial up to coherent natural isomorphisms. More precisely, suppose that $G\colon \mathcal{C}^2 \to \mathcal{C}^3$ is another right exact functor which is stable backwards for f.g. ideals. There is a natural isomorphism $\widehat{GF} \cong \hat{G}\hat{F}$ defined as follows. For M in $\hat{\mathcal{C}}^1$ and $\Gamma \in I^3$, one has

$$(\hat{G}\hat{F}M)_\Gamma = \varprojlim_{\Lambda\in I^2} \sigma_\Gamma G \varprojlim_{\Sigma\in I^1} \sigma_\Lambda F M_\Sigma \cong \sigma_\Gamma G \sigma_\Lambda F M_\Sigma$$

provided $\Lambda \supseteq G^{-1}\Gamma$ and $\Sigma \supseteq F^{-1}\Lambda$. But then application of $\sigma_\Gamma G$ to the admissible epimorphism $FM_\Sigma \to \sigma_\Lambda F M_\Sigma$ gives an isomorphism

$$(\widehat{GF}M)_\Gamma \cong \sigma_\Gamma G F M_\Sigma \cong \sigma_\Gamma G \sigma_\Lambda F M_\Sigma \cong (\hat{G}\hat{F}M)_\Gamma$$

since $\Sigma \supseteq (GF)^{-1}(\Gamma)$. One can check this isomorphism is well-defined; over all $\Gamma \in I^3$, these isomorphisms yield isomorphisms $\widehat{GF}M \cong \hat{G}\hat{F}M$ which are the components of the desired natural isomorphism $\widehat{GF} \cong \hat{G}\hat{F}$.

These natural isomorphisms are coherent in the following sense: given another stable exact functor $H\colon \mathcal{C}^0 \to \mathcal{C}^1$, the diagram

$$\begin{array}{ccc} \widehat{GFH} & \xrightarrow{\cong} & \widehat{GF}\hat{H} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \hat{G}\widehat{FH} & \xrightarrow{\cong} & \hat{G}\hat{F}\hat{H} \end{array}$$

commutes. This follows since for N in $\hat{\mathcal{C}}^0$, $\Gamma \in I^3$, $\Lambda \supseteq G^{-1}\Gamma$ in I^2, $\Sigma \supseteq F^{-1}\Lambda$ in I^1 and $\Psi \supseteq H^{-1}\Sigma$ in I^0, the diagram

$$\begin{array}{ccc} \sigma_\Gamma GFHN_\Psi & \xrightarrow{\cong} & \sigma_\Gamma GF\sigma_\Sigma HN_\Psi \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \sigma_\Gamma G\sigma_\Lambda FHN_\Psi & \xrightarrow{\cong} & \sigma_\Gamma G\sigma_\Lambda F\sigma_\Sigma HN_\Psi \end{array}$$

commutes. For F, F', ϵ, G, H as above, there are also commutative diagrams

$$\begin{array}{ccc} \widehat{GF} & \xrightarrow{\widehat{G\epsilon}} & \widehat{GF'} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \hat{G}\hat{F} & \xrightarrow{\hat{G}\hat{\epsilon}} & \hat{G}\hat{F}' \end{array} \qquad\qquad \begin{array}{ccc} \widehat{FH} & \xrightarrow{\widehat{\epsilon H}} & \widehat{F'H} \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \hat{F}\hat{H} & \xrightarrow{\hat{\epsilon}\hat{H}} & \hat{F}'\hat{H} \end{array}$$

the straightforward verification of which is left to the reader.

The proof of the proposition can be completed as follows. Suppose that $G\colon \mathcal{C}^2 \to \mathcal{C}^1$ is a right exact right adjoint to F. We let $\nu\colon \mathrm{Id}_{\mathcal{C}^1} \to GF$ and $\epsilon\colon FG \to \mathrm{Id}_{\mathcal{C}^2}$ be the unit and counit of the adjunction, respectively. Define the composite natural transformations

$$\mathrm{Id}_{\hat{\mathcal{C}}^1} \xrightarrow{\hat{\nu}} \widehat{GF} \xrightarrow{\cong} \hat{G}\hat{F}$$
$$\hat{F}\hat{G} \xrightarrow{\cong} \widehat{FG} \xrightarrow{\hat{\epsilon}} \mathrm{Id}_{\hat{\mathcal{C}}^2}.$$

We claim that the triangular identities $(\epsilon F)(F\nu) = \mathrm{Id}_F$ and $(G\epsilon)(\nu G) = \mathrm{Id}_G$ for the adjoint pair (F, G) imply the corresponding identities for an adjoint pair $(\hat{F}, \hat{G})$ with the above natural transformations as unit and counit respectively. One triangular identity for $(\hat{F}, \hat{G})$ follows by inspection of the commutative diagram

$$\begin{array}{ccccc} & & & & \hat{F} \\ & & & \nearrow{\scriptstyle\widehat{\epsilon F}} & \uparrow{\scriptstyle\hat{\epsilon}\hat{F}} \\ & & \widehat{FGF} & \xrightarrow{\cong} & \widehat{FG}\hat{F} \\ & \nearrow{\scriptstyle\widehat{F\nu}} & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \hat{F} & \xrightarrow{\hat{F}\hat{\nu}} & \hat{F}\widehat{GF} & \xrightarrow{\cong} & \hat{F}\hat{G}\hat{F} \end{array}$$

using the coherence properties proved above, and the other follows similarly.

6.5. Let $\{M_j\}_{j\in J}$ be a family of objects of $\hat{\mathcal{C}}$; we write $M_j = \{M_{j,\Gamma}\}_{\Gamma\in I}$. Suppose that for each $\Gamma \in I$, there are only finitely many $j \in J$ with $M_{j,\Gamma} \neq 0$. Then the direct sum and direct product of $\{M_j\}_{j\in J}$ exist in $\hat{\mathcal{C}}$ and are canonically isomorphic; denoting them by M, we have $M_\Gamma = \oplus_j M_{j,\Gamma} = \prod_{j\in J} M_{j,\Gamma}$. Denoting the projections and inclusions as $p_j \colon M \to M_j$ and $i_j \colon M_j \to M$, we have $p_j i_j = \mathrm{Id}_{M_j}$ for all j. Moreover, $\sum_j i_j p_j = \mathrm{Id}_M$ in the sense that for each $\Gamma \in I$, $\sum_j i_{j,\Gamma} p_{j,\Gamma} = \mathrm{Id}_{M_\Gamma}$, the latter sum involving only finitely many non-zero terms. Henceforward, all direct sum and products considered in $\hat{\mathcal{C}}$ are assumed to be of this form; we call them convergent direct sums (products). Finite direct sums are obviously convergent.

7. Projective objects of $\mathcal{C}$ and $\hat{\mathcal{C}}$

In this subsection, we consider projective objects in a weakly stratified exact category $\mathcal{C}$ with strata $\mathcal{C}_x$ and weight poset Ω. Many of the later results require that $\mathcal{C}$ be a stratified exact category.

7.1. Let $\mathcal{P}$ denote the full additive subcategory of projective objects in $\mathcal{C}$.

Lemma. *For an object P of $\mathcal{C}$, the following conditions are equivalent:*

(i) *P is in $\mathcal{P}$*
(ii) $\mathrm{Ext}^1_{\mathcal{C}}(P, ?) = 0$
(iii) *for all ideals Γ of Ω and all M in $\mathcal{C}$, the natural map*

$$f \mapsto \sigma_\Gamma(f) \colon \hom(P, M) \to \hom(P(\Gamma), M(\Gamma))$$

is an epimorphism, and for all $x \in \Omega$ and for any admissible epimorphism $f : N' \to N''$ in $\mathcal{C}_x$, any homomorphism $P \to N''$ can be factored through f.

Proof. Clearly, P is projective in $\mathcal{C}$ iff it is projective in $\mathcal{C}_\Sigma$ for all finite, locally closed subsets Σ with P in $\mathcal{C}_\Sigma$. Hence in the proof we assume without loss of generality that Ω is finite.

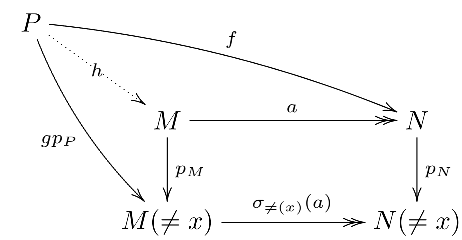
In general, an object P of an exact category is projective iff $\mathrm{Ext}^1(P, ?) = 0$, establishing the equivalence of (i) and (ii). Clearly, (i) implies (iii) as well. The proof that (iii) implies (i) will be given after the following lemma.

7.2. Suppose that x is a maximal element of Ω. For any M in $\mathcal{C}$, we denote by i_M and p_M the indicated maps in the canonical exact sequence

$$0 \to M(x) \xrightarrow{i_M} M \xrightarrow{p_M} M(\neq x) \to 0$$

in $\mathcal{C}$.

Lemma. *Suppose that P in $\mathcal{C}$ satisfies* 7.1(iii)*. Then for any admissible epimorphism $a \colon M \to N$ and maps $f \colon P \to N$ and $g \colon P(\neq x) \to M(\neq x)$ in $\mathcal{C}$ satisfying $p_N f = \sigma_{\neq x}(a) g p_P \colon P \to N(\neq x)$, there is a map $h \colon P \to M$ making the following* *diagram commute:*



Proof. By the first condition in 7.1(iii), there exists a map $h'\colon P \to M$ with $gp_P = p_M h'$. Then

$$p_N(f - ah') = p_N f - p_N a h' = \sigma_{\neq x}(a)(gp_P - p_M h') = 0.$$

Since $i_N = \ker p_N$, we may write $f - ah' = i_N m$ for some map $m'\colon P \to N(x)$. By the second condition in 7.1(iii), $m' = \sigma_x(a) m$ for some $m\colon P \to M(x)$. Then $h := h' + i_M m$ is as required. □

7.3. **Completion of proof of Lemma 7.1.** To show (iii) implies (i), we proceed by induction on $\sharp(\Omega)$. Let $x \in \Omega$ be a maximal element. Suppose that $P \in \mathcal{C}$ satisfies 7.1(iii) and that we have an admissible epimorphism $a\colon M \to N$ and map $f\colon P \to N$. Then $\sigma_{\neq x} a\colon \sigma_{\neq x} M \to \sigma_{\neq x} N$ is an admissible epimorphism and $\sigma_{\neq x}\colon P(\neq x) \to N(\neq x)$ is a map, so by induction there is a map $g\colon P(\neq x) \to M(\neq x)$ with $\sigma_{\neq x}(a) g = f$. In the notation of the previous lemma, one has $\sigma_{\neq x}(a) g p_P = f p_P = p_N f$ and the existence of a map $h\colon P \to M$ with $f = ah$ follows by the lemma. □

7.4. The second condition in 7.1(iii) holds automatically if $\mathcal{C}$ is a stratified exact category, since admissible epimorphisms in $\mathcal{C}_x$ are split. Hence

Corollary. *If $\mathcal{C}$ is a stratified exact category, then P in $\mathcal{C}$ is projective iff for all M in $\mathcal{C}$ and all ideals $\Gamma \in I$, the natural map* $\operatorname{Hom}(P, M) \to \operatorname{Hom}(P(\Gamma), M(\Gamma))$ *is an epimorphism.*

7.5. Let Γ be an ideal of Ω with a maximal element x. Let P (resp., N) be an object of $\mathcal{C}$ (resp., $\mathcal{C}_x$). Set $\Gamma' := \Gamma \setminus \{x\} \in I$. The short exact sequence

$$0 \to P(x) \to P(\Gamma) \to P(\Gamma') \to 0 \tag{7.5.1}$$

in $\mathcal{C}$ gives an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(P(\Gamma'), N) \to \operatorname{Hom}(P(\Gamma), N) \to \operatorname{Hom}(P(x), N) \to \\ \operatorname{Ext}^1(P(\Gamma'), N) \to \operatorname{Ext}^1(P(\Gamma), N) \to \operatorname{Ext}^1(P(x), N). \end{aligned} \tag{7.5.2}$$

Lemma. *The exact sequence* (7.5.2) *is independent of the ideal* Γ *with* x *as maximal element, up to isomorphism.*

Proof. Let $\Sigma := \{\, z \in \Omega | z \not> x \,\} \supseteq \Gamma$ and set $\Sigma' := \Sigma \setminus \{x\}$. Then Σ is another ideal of Ω with maximal element x. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P(\Sigma\setminus\Gamma) & = & P(\Sigma\setminus\Gamma) & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & P(x) & \to & P(\Sigma) & \to & P(\Sigma') & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & P(x) & \to & P(\Gamma) & \to & P(\Gamma') & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that $P(\Sigma \setminus \Gamma)$ has a finite filtration with successive subquotients in $\mathcal{C}_y$ for various y in Ω not comparable to x, so $\mathrm{Hom}(P(\Sigma\setminus\Gamma), N) = \mathrm{Ext}^1(P(\Sigma\setminus\Gamma), N) = 0$. Taking the start of the long exact $\mathrm{Ext}(?, N)$ sequences from the rows and columns of this diagram gives an isomorphism of the sequences (7.5.2) associated to Γ and Σ, as required. □

7.6. The construction of projective objects in many stratified categories of interest hinges on the following observation.

Proposition. *Consider an object P of $\mathcal{C}$ with $P(x)$ projective in $\mathcal{C}_x$ for all $x \in \Omega$. Then P is projective in $\mathcal{C}$ iff for all $x \in \Omega$ and N in $\mathcal{C}_x$, the map*

$$\mathrm{Hom}(P(x), N) \to \mathrm{Ext}^1(P(< x), N)$$

induced by the short exact sequence $0 \to P(x) \to P(\leq x) \to P(< x) \to 0$ is an epimorphism.

Proof. Clearly, $\mathrm{Ext}^1_{\mathcal{C}}(P, ?) = 0$ iff $\mathrm{Ext}^1(P, N) = 0$ for all x and all N in $\mathcal{C}_x$ i.e. iff $\mathrm{Ext}^1(P(\leq x), N) = 0$ for all such x and N, since

$$\mathrm{Hom}(P(\not\leq x), N) = 0 = \mathrm{Ext}^1(P(\not\leq x), N).$$

Since $\mathcal{C}_x$ is closed under extensions in $\mathcal{C}$ and $P(x)$ is projective in $\mathcal{C}_x$, we have $\mathrm{Ext}^1_{\mathcal{C}}(P(x), N) = 0$. The desired conclusion follows from the exact sequence (7.5.2) taking Γ equal to the ideal generated by x. □

7.7. Let $\hat{\mathcal{P}}$ denote the full subcategory of projective objects of $\hat{\mathcal{C}}$.

Lemma. *For $P = \{P_\Gamma\}_{\Gamma\in I}$ in $\hat{\mathcal{C}}$, the following conditions* (i)–(iii) *are equivalent:*

(i) *P is projective in $\hat{\mathcal{C}}$*
(ii) *P_Γ is projective in $\mathcal{C}_\Gamma$ for all $\Gamma \in I$*
(iii) *$P_{\leq x}$ is projective in $\mathcal{C}_{\leq x}$ for all $x \in \Omega$.*

Proof. Assume (i) holds. By 6.2.5, $\hat{\tau}_\Gamma$ is left adjoint to the exact inclusion $\hat{\iota}_\Gamma$, so $\hat{\tau}_\Gamma P$ is projective in $\hat{\mathcal{C}}_\Gamma$. Then P_Γ is projective in $\mathcal{C}_\Gamma$ by 6.2.1, proving (i) implies (ii). It is trivial that (ii) implies (iii), since $I \supseteq I_0$. Now we prove that (iii) implies (i). For $M = \{M_\Gamma\}_{\Gamma\in I_1}$ in $\hat{\mathcal{C}} = \hat{\mathcal{C}}_{I_1}$, write M' for the inverse system $M' := \{M'_x\}_{x\in\Omega}$ where $M'_x := M_{\leq x}$. Identifying Ω with I_0 via the order isomorphism sending x

to the ideal it generates, we regard M' as the object of $\hat{\mathcal{C}}_{I_0}$ corresponding to M under the standard equivalence $\hat{\mathcal{C}}_{I_1} \cong \hat{\mathcal{C}}_{I_0}$; we use similar notation for morphisms. Suppose given an admissible epimorphism $f\colon M \to A$ and a map $g\colon P \to A$ in $\hat{\mathcal{C}}$. We consider families $h' = \{h_x\colon P'_x \to M'_x\}_{x\in\Gamma}$ of morphisms in $\mathcal{C}$ satisfying the following conditions: Γ is an ideal of Ω containing the ideal $\Omega_0 := \{\, y \in \Omega | P'_y = 0 \,\}$, $g_x = f_x h_x$ for all $x \in \Gamma$, and the diagrams

(7.7.1)
$$\begin{CD} P'_x @>{h_x}>> M'_x \\ @VVV @VVV \\ P'_y @>{h_y}>> M'_y \end{CD}$$

commute for all $y \le x$ in Γ. Given another such family $g = \{g_x\colon P'_x \to M'_x\}_{x\in\Sigma}$, we write $h \preceq g$ if $\Gamma \subseteq \Sigma$ and $h_x = g_x$ for all $x \in \Gamma$. By Zorn's lemma, there is maximal element h in the set of such families in the order $\preceq$. To complete the proof, it is sufficient to show $\Gamma = \Omega$, since then h' defines a morphism $P' \to M'$ in $\hat{\mathcal{C}}_{I_0}$ with $g' = f'h'$ and the corresponding morphism $h : P \to M$ in $\hat{\mathcal{C}}$ satisfies $g = fh$.

If $z \in \Omega \setminus \Gamma$, there are only finitely many $x \le z$ with $x \notin \Omega_0$, so we may choose a minimal element x of $\Omega \setminus \Gamma$. Note that $\Gamma' := \Gamma \cup \{x\}$ is an ideal of Ω. Define the poset $\Sigma := \{\, y \in \Omega | y < x \,\} \subseteq \Gamma$. We obtain a contradiction to maximality of h by constructing a map h_x such that the following diagram commutes:

$$\begin{array}{ccccc} P'_x & & \xrightarrow{\quad g_x \quad} & & A'_x \\ \downarrow & \overset{h_x}{\searrow} & & \overset{f_x}{\nearrow} & \downarrow \\ & & M'_x & & \\ \downarrow & & \downarrow & & \downarrow \\ \varprojlim_{y\in\Sigma} P'_y & \longrightarrow & \varprojlim_{y\in\Sigma} M'_y & \longrightarrow & \varprojlim_{y\in\Sigma} A'_y. \end{array}$$

In this diagram, the vertical maps and maps in the bottom row are the canonical ones from the universal properties of inverse limits. The bottom row identifies with $P_\Sigma \to M_\Sigma \to A_\Sigma$ in $\mathcal{C}_\Sigma$ in which the last arrow is induced by $f\colon M \to A$. The vertical maps identify with the canonical admissible epimorphisms $B_{\le x} \to B_{<x}$ for B one of P, M or A. The rightmost trapezoid and outer rectangle are commutative since f (resp. g) is a morphism in $\hat{\mathcal{C}}$. The existence of h_x with the required properties follows from 7.2. □

7.8. We record the following simple relations between projectives in $\mathcal{C}$, in $\hat{\mathcal{C}}$ and in $\mathcal{C}_\Sigma$ for locally closed $\Sigma \in I$.

Suppose that P is projective in $\mathcal{C}$. For any $\Gamma \in I$, σ_Γ has an exact right adjoint ι_Γ, so $P(\Gamma)$ is projective in $\mathcal{C}_\Gamma$. This implies by 7.7 that $\theta(P)$ is projective in $\hat{\mathcal{C}}$.

If Λ is any coideal of Ω, then ι_Λ has the exact right adjoint σ_Λ, so any projective object of $\mathcal{C}_\Lambda$ is projective in $\mathcal{C}$ and hence projective in $\hat{\mathcal{C}}$. Further, any projective object of $\hat{\mathcal{C}}_\Lambda$ is projective in $\hat{\mathcal{C}}$.

Finally, suppose that $\hat{\mathcal{C}}$ has enough projective objects. Then if $\Gamma \in I$, $\mathcal{C}_\Gamma$ has enough projective objects; for given M in $\mathcal{C}_\Gamma$, one may choose choose an admissible epimorphism $P \to \theta(M)$ with P projective in $\hat{\mathcal{C}}_\Gamma$, and then one has the admissible epimorphism $P_\Gamma \to \theta(M)_\Gamma \cong M$ in $\mathcal{C}_\Gamma$.

7.9. **Start of proof of 1.18.** For the remainder of this section, we assume that $\mathcal{C}$ is a stratified exact category over G such that each stratum $\mathcal{C}_x$ is svelte. Fix a set $\mathbf{N}_x = \{N_{x,i}\}_i$ of objects of $\mathcal{C}_x$ satisfying $\mathcal{C}_x = \operatorname{add}\mathbf{N}_x$. Define $R_x := \operatorname{end}(\mathbf{N}_x)^{\mathrm{op}} = \oplus_{i,i'} e_{x,i} R_x e_{x,i'}$ and $R_{x,i} = \operatorname{end}(\mathbf{N}_{x,i})^{\mathrm{op}} \cong e_{x,i} R e_{x,i}$.

In this subsection, we prove 1.18(a). For any P in $\hat{\mathcal{C}}$ and $x \in \Omega$, choose $\Gamma \in I_1$ with maximal element x. The short exact sequence

$$0 \to P(x) \to P(\Gamma) \to P(\Gamma') \to 0 \tag{7.9.1}$$

in $\mathcal{C}$ gives an exact sequence

$$(*_{P,x}) \quad 0 \to \hom(P(\Gamma'), \mathbf{N}_x) \to \hom(P(\Gamma), \mathbf{N}_x) \xrightarrow{f_{P,x}} \hom(P(x), \mathbf{N}_x) \xrightarrow{g_{P,x}} \operatorname{ext}^1(P(\Gamma'), \mathbf{N}_x) \to \operatorname{ext}^1(P(\Gamma), \mathbf{N}_x) \to \operatorname{ext}^1(P(x), \mathbf{N}_x).$$

of R_x-modules which is independent of Γ by 7.5. It follows immediately from 7.7 and 7.6 that P is projective in $\hat{\mathcal{C}}$ iff $g_{P,x}$ is an epimorphism for all $x \in \Omega$, proving 1.18(a).

7.10. The following lemma proves 1.18(c) and most of 1.18(b).

Lemma. *A standard family of projectives in $\hat{\mathcal{C}}$ exists if either* $\operatorname{ext}^1(N_{x,i}, \mathbf{N}_y)$ *is a graded Noetherian right R_y-module for all $x < y \in \Omega$ and all i, or $\mathcal{C}_\Gamma$ has enough projective objects for all $\Gamma \in I$.*

Proof. Assume first that $\Omega \in I_1$, and identify $\hat{\mathcal{C}} = \mathcal{C}$. If Σ is a finite coideal of Ω, then an object P of $\mathcal{C}_\Sigma$ is clearly projective in $\mathcal{C}_\Sigma$ iff it is projective in $\mathcal{C}$. Hence we may assume for now that Ω is finite.

Suppose that Γ is an ideal in Ω with maximal element x and Q is an object of $\mathcal{C}_{\Gamma \setminus \{x\}}$. Then by C.9, if $\operatorname{ext}^1(Q, \mathbf{N}_x)$ is a f.g. R_x-module, there is an object P of $\mathcal{C}_\Gamma$ such that $P(\Gamma \setminus \{x\}) \cong Q$ and

$$g_{P,x} \colon \hom(P(x), \mathbf{N}_x) \to \operatorname{ext}^1(P(\neq x), \mathbf{N}_x)$$

is an epimorphism. If such an object P exists, we denote an arbitrary choice of one such object by $P = x * Q$ (this definition of $x * Q$ depends only on x and Q, and not on the choice of the ideal Γ as above). Note that by 1.18(a),

7.10.1. If Q is projective in $\mathcal{C}_{\Gamma \setminus \{x\}}$ then $x * Q$ is projective in $\mathcal{C}_\Gamma$ (if it exists).

Now construct some objects of $\mathcal{P}$ as follows. Let $x \in \Omega$; choose a compatible ordering $x = x_0, x_1, \ldots, x_n$ of $\{\, y \in \Omega | y \geq x \,\}$. For any object N of $\mathcal{C}_x$, define if possible (i.e. if the required ext^1 groups are f.g.) $\operatorname{Proj}(N) := (x_n * \ldots * (x_1 * N))$. Since N is projective in $\mathcal{C}_{\not> x}$, 7.10.1 implies the following.

7.10.2. If N is in $\mathcal{C}_x$ and $\operatorname{Proj}(N)$ is defined, then $\operatorname{Proj}(N)$ is a projective object in $\mathcal{C}$ with $\operatorname{Proj}(N) \cong N$ and $\sigma_y \operatorname{Proj}(N) = 0$ unless $y \geq x$.

To complete the proof of the lemma in case $\Omega \in I_1$, it is sufficient to check that $\operatorname{Proj}(N)$ is always defined under the hypotheses of the lemma. Under the Noetherian hypothesis of the lemma, $x * Q$ is defined since $\operatorname{ext}^1(Q, \mathbf{N}_z)$ is a finitely-generated right R_z-module for any $z \in \Omega$ and $Q = Q(\not\geq z)$ in $\mathcal{C}$; this follows from the ext^1 terms of the long exact $\operatorname{ext}(?, \mathbf{N}_z)$ sequences and the definition of $\mathcal{C}$ on recalling C.2.1. On the other hand, suppose that $\mathcal{C}$ has enough projectives. Let Γ be an ideal of Ω with maximal element z, and Q be a projective object of $\mathcal{C}_{\Gamma \setminus \{z\}}$. Choose an admissible epimorphism $f\colon P \to Q$ with P projective in $\mathcal{C}$. By 7.6, the right R_z-module $\operatorname{ext}^1(P(\Gamma\setminus\{z\}, \mathbf{N}_z)$ is f.g., since in $(*_{P,z})$, $\hom(P(z), \mathbf{N}_z)$ is a f.g. projective right R_z-module. But f gives an admissible epimorphism $\sigma_{\Gamma\setminus\{z\}}f\colon \sigma_{\Gamma\setminus\{z\}}P \to Q$, so Q is a direct summand of $\sigma_{\Gamma\setminus\{z\}}P$. Therefore, $\operatorname{ext}^1(Q, \mathbf{N}_z)$ is a f.g. R_z-module and so $z * Q$ is defined. This implies that all $z * Q$ occurring in the definition of $\operatorname{Proj}(N_x)$ are defined.

Now we consider the general case i.e. we no longer assume $\Omega \in I_1$. It is clear from the proof of the special case above that if $\Lambda \subseteq \Sigma \in I$, any projective object Q' of $\mathcal{C}_\Lambda$ is isomorphic to the truncation $\sigma_\Lambda(Q)$ of some projective object Q of $\mathcal{C}_\Sigma$ such that $Q'(x) \neq 0$ only if $Q(y) \neq 0$ for some $y \leq x$. An Zorn's lemma argument similar to (but simpler than) that in the proof of 7.7 now shows that

7.10.3. If $\Gamma \in I_1$ and P' is a projective object of $\mathcal{C}_\Gamma$, then there is a projective object $P = \{P_U\}_{U\in I_1}$ of $\hat{\mathcal{C}}$ with $P_\Gamma \cong P'$ and $P(x) \neq 0$ only if $P'(y) \neq 0$ for some $y \leq x$.

Applying this to the objects $P = N_{x,i}$ in $\mathcal{C}_{\leq x}$ proves the lemma in general. □

7.11. The following lemma isolates the main part of the proof of 1.18(d).

Lemma. *Let* $\mathbf{P}$ *be a standard family of projective in* $\hat{\mathcal{C}}$. *If* M *is any object in* $\hat{\mathcal{C}}$, *there is a short exact sequence* $0 \to N \to P' \to M \to 0$ *in* $\hat{\mathcal{C}}$ *in which* P' *is a direct summand of a convergent direct sum of translates of objects from* $\mathbf{P}$. *Moreover, one may assume that the supports of* P' *and* N *are contained in the support* Γ *of* M *and that the support of* N *contains none of the minimal elements of* Γ. *If* M *is in the strict image of* $\theta\colon \mathcal{C} \to \hat{\mathcal{C}}$, *one may require in addition that* P' *is in* $\operatorname{add}\mathbf{P}$.

Proof. For $x \in \Gamma$, one may choose a projective object P'_x of $\hat{\mathcal{C}}$ in $\operatorname{add}\{P_{x,i}\}_i$ so there is an admissible epimorphism $P'_x(x) \to M(x)$. We take $P'_x = 0$ if $M(x) = 0$. Using projectivity of P'_x, choose a morphism $h_x\colon P'_x \to \hat{\sigma}_{\geq x}M$ in $\hat{\mathcal{C}}$ in which $h_x(x)\colon P'_x(x) \cong (\hat{\sigma}_{\geq x}M)(x)$ identifies with $P'_x(x) \to M(x)$. Define the convergent direct sum $P' = \oplus_{x\in\Gamma} P'_x$. Composing the h_x with the natural inclusion $\hat{\sigma}_{\geq x}M \to M$ gives maps $P'_x \to M$ which induce by the universal property of the direct sum a map $h\colon P' \to M$. Each map $h(y)\colon P'(y) \to M(y)$ with $y \in \Omega$ is a split epimorphism since $P'_y(y) \to M(y)$ is a split epimorphism, so h is an admissible epimorphism in $\hat{\mathcal{C}}$. This shows that $\hat{\mathcal{C}}$ has enough projectives, and implies that the projectives in $\hat{\mathcal{C}}$ are precisely the direct summands of the convergent direct sum of translates of objects of $\mathbf{P}$. Now repeat the above argument choosing P'_x so in addition $P'_x(x) \to M(x)$ is an isomorphism, which is possible by 7.10.3. Then by construction $h(y)$ is an isomorphism if y is a minimal element of the support of M. Taking $N = \ker h$, the evident short exact sequence has the required properties. □

7.12. **Completion of proof of Theorem 1.18.** We have remarked that if $\hat{\mathcal{C}}$ has enough projectives, then $\mathcal{C}_\Gamma$ has enough projectives for all $\Gamma \in I_1$ and seen that this latter condition implies existence of a standard family of projective objects of $\hat{\mathcal{C}}$. By the preceding lemma, existence of a standard family of projectives implies $\hat{\mathcal{C}}$ has enough projectives; hence 1.18(b) holds, and we have already proved 1.18(c). Finally, 1.18(d) follows immediately from 7.11 (the assertions concerning the sets Γ_i are easily checked).

8. $\mathcal{C}^\dagger$ AS A MODULE CATEGORY

In this section, we assume that the fixed, weakly stratified exact category $\mathcal{C}$ is svelte. We study the abelian categories $\mathcal{C}^*$, $\mathcal{C}^\dagger$ associated to $\mathcal{C}$ as in 1.8 and Appendix B, and show how they can be described module-theoretically if $\hat{\mathcal{C}}$ has enough projective objects.

8.1. Consider again the situation of 6.3. Let Λ be an ideal of Ω^2 and Γ be an ideal of Ω^1. Identify $\mathcal{C}^{1*}_\Gamma$ (resp., $\mathcal{C}^{2*}_\Lambda$) with its strict image under τ^{1*}_Γ (resp., τ^{2*}_Γ). It is easy to check from the definition of F^* that if $\Gamma \supseteq F^{-1}(\Lambda)$ then F^* maps $\mathcal{C}^{2*}_\Lambda$ into $\mathcal{C}^{1*}_\Gamma$. Hence if F is stable backwards for f.g. ideals, then F^* restricts to a functor $F^\dagger\colon \mathcal{C}^{2\dagger} \to \mathcal{C}^{1\dagger}$.

Also, if $\Lambda \supseteq F(\Lambda)$ then F_* maps $\mathcal{C}^{1*}_\Gamma$ into $\mathcal{C}^{2*}_\Lambda$ (since F_* maps the generators $\phi^1(M)$, M in $\mathcal{C}^1_\Gamma$, of $\mathcal{C}^{1*}_\Gamma$ to elements $\phi^2(FM)$ in $\mathcal{C}^{2*}_\Lambda$ where $\phi^i\colon \mathcal{C}^i \to \mathcal{C}^{i*}$ is the Gabriel-Quillen embedding). Hence if F is stable forwards for f.g. ideals, then F_* restricts to a functor $F_\dagger\colon \mathcal{C}^{1\dagger} \to \mathcal{C}^{2\dagger}$.

8.2. For the remainder of this section, we assume that the condition below holds.

Assumption. There is a set $\mathbf{P} := \{P_j\}_{j\in J}$ of projective objects of $\hat{\mathcal{C}}$ such that for any coideal Λ of Ω and any object M of $\hat{\mathcal{C}}_\Lambda$, there is an admissible epimorphism $Q \to M$ such that Q is a convergent direct sum of translates of objects P_j in $\mathbf{P}$ with P_j in $\hat{\mathcal{C}}_\Lambda$.

This implies that $\hat{\mathcal{C}}$ and hence $\mathcal{C}$ has only a set of isomorphism classes of objects, so $\hat{\mathcal{C}}^*$ and $\mathcal{C}^*$ are defined. In fact, we shall construct an equivalence of abelian categories between $\mathcal{C}^*$ and a full abelian subcategory $\hat{\mathcal{E}}$ of $\mathrm{end}(\mathbf{P})^{\mathrm{op}}$-Mod.

8.3. We set $I = I_1$ and write $P_j = \{P_{j,\Gamma}\}_{\Gamma\in I}$. For any ideal Γ of Ω, we have the family of objects $\mathbf{P}_\Gamma := \hat{\sigma}_\Gamma(\mathbf{P}) := \{\hat{\sigma}_\Gamma(P_j)\}_{j\in J}$ of $\hat{\mathcal{C}}$. If $\Gamma \in I_1$, $\mathbf{P}_\Gamma$ identifies via θ with the family $\mathbf{P}_\Gamma = \{P_{j,\Gamma}\}_{j\in J}$ of objects of $\mathcal{C}_\Gamma$.

Notice that for any coideal Λ of Ω, any projective object in $\hat{\mathcal{C}}_\Lambda$ is a direct summand of a convergent direct sum of translates of objects P_j in $\hat{\mathcal{C}}_\Lambda$, and there are sufficiently many projective objects in $\hat{\mathcal{C}}_\Lambda$. Note also that for $\Gamma \in I$, $P_{j,\Gamma}$ is projective in $\mathcal{C}_\Gamma$ by 7.7. It follows readily that

8.3.1. If Σ is any locally closed subset of Ω which generates an ideal Γ contained in I, then $\mathcal{C}_\Sigma$ has sufficiently many projective objects, and those projectives are precisely the objects of $\mathrm{add}\,\{\, P_{j,\Gamma} \mid P_{j,\Gamma} \text{ in } \mathcal{C}_\Sigma \,\}$.

8.4. Define for any ideal Γ of Ω the J-diagonalizable G-graded rings $\mathcal{A} := \mathrm{end}_{\hat{\mathcal{C}}}(\mathbf{P})$ and $\mathcal{A}_\Gamma := \mathrm{end}_{\hat{\mathcal{C}}}(\mathbf{P}_\Gamma)$. There is a natural epimorphism $\mathcal{A} \to \mathcal{A}_\Gamma$ of J-diagonalizable G-graded rings. If $\Gamma \supseteq \Lambda$, the admissible epimorphisms $\hat{\sigma}_\Gamma P_i \to \hat{\sigma}_\Lambda P_i$ determine similarly epimorphisms $\mathcal{A}_\Gamma \to \mathcal{A}_\Lambda$; these define an inverse system $(\mathcal{A}_\Gamma)_\Gamma$ of J-diagonalizable G-graded rings. By (6.2.7), we have $\mathcal{A} \cong \varprojlim_{\Gamma\in I} \mathcal{A}_\Gamma$ as G-graded J-diagonalizable ring.

8.5. For any ideal Γ of Ω, the inclusion functor $j_\Gamma\colon \mathcal{A}_\Gamma\text{-mod} \to \mathcal{A}\text{-mod}$ has on general ring-theoretic grounds a right adjoint k_Γ given on objects M of $\mathcal{A}$-mod by $k_\Gamma\colon M \mapsto \oplus_j \mathrm{hom}_{\mathcal{A}}(\mathcal{A}_\Gamma e_j, M)$ and a left adjoint i_Γ given by $i_\Gamma(M) = \mathcal{A}_\Gamma \otimes_{\mathcal{A}} M$. The adjunction morphisms $\mathrm{Id} \xrightarrow{\cong} k_\Gamma j_\Gamma$ and $i_\Gamma j_\Gamma \to \mathrm{Id}$ are natural isomorphisms, while the components of $j_\Gamma k_\Gamma \to \mathrm{Id}$ (resp., $\mathrm{Id} \to j_\Gamma i_\Gamma$) are monomorphisms (resp., epimorphisms).

We regard $\mathcal{A}_\Gamma$-mod as a full subcategory of $\mathcal{A}$-mod via j_Γ; then for M in $\mathcal{A}$-mod, $j_\Gamma k_\Gamma M$ (resp., $j_\Gamma i_\Gamma M$) is the largest subobject (resp., quotient object) of M in $\mathcal{A}_\Gamma$-mod. We have the direct system $\{\oplus_j \hom_{\mathcal{A}}(\mathcal{A}_\Gamma e_j, M)\}_{\Gamma\in I}$ of graded $\mathcal{A}$-submodules of M and a canonical monomorphism

$$\psi_M \colon \varinjlim_{\Gamma\in I} \oplus_j \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_\Gamma e_j, M) \to M.$$

Definition. Let $\mathcal{E}$ denote the full abelian subcategory of $\mathcal{A}$-mod consisting of graded modules which are $\mathcal{A}_\Gamma$ modules for some $\Gamma \in I$, and $\hat{\mathcal{E}}$ be the full additive subcategory of graded $\mathcal{A}$-modules M such that ψ_M is an isomorphism.

The preceding remarks imply that $\hat{\mathcal{E}}$ consists of the graded $\mathcal{A}$-modules M which are the directed union of the family of their graded submodules which are in $\mathcal{E}$. Clearly, $\hat{\mathcal{E}}$ is closed under formation of subobjects and quotients in $\mathcal{A}$-mod, and hence it is a (perfectly exact) abelian subcategory of $\mathcal{A}$-mod. Furthermore, $\hat{\mathcal{E}}$ is a complete abelian category with exact filtered colimits. In fact, colimits in $\hat{\mathcal{E}}$ are induced by colimits in $\mathcal{A}$-mod and filtered colimits in $\mathcal{A}$-mod are exact. To explicitly describe limits in $\hat{\mathcal{E}}$, note the exact inclusion functor from $j\colon \hat{\mathcal{E}} \to \mathcal{A}$-mod has a right adjoint k given on objects by $k\colon M \mapsto \varinjlim_{\Gamma\in I} \oplus_j \hom_{\mathcal{A}}(\mathcal{A}_\Gamma e_j, M)$. To take the limit of a functor to $\hat{\mathcal{E}}$, one takes its limit in $\mathcal{A}$-mod and then applies k.

8.6. For $\Gamma \in I$ and $M \in \mathcal{C}_\Gamma$, (6.2.8) gives that $\hom_{\hat{\mathcal{C}}}(\mathbf{P}, \theta(M)) \cong \hom_{\mathcal{C}}(\mathbf{P}_\Gamma, M)$ is in $\mathcal{A}_\Gamma$-mod. Define the exact functor $\varphi = \hom(\mathbf{P}, \theta ?)\colon \mathcal{C} \to \hat{\mathcal{E}}$ (with its strict image in $\mathcal{E}$).

Theorem. *All statements of* Theorem 1.19 *hold for the weakly stratified exact category* $\mathcal{C}$ *over* G *under the assumption* 8.2.

Proof. By C.8, (C.8.1) and (C.8.2), we have inverse equivalences $\alpha_\Gamma\colon \mathcal{C}^*_\Gamma \to \mathcal{A}_\Gamma$-mod and $\beta_\Gamma\colon \mathcal{A}_\Gamma\text{-mod} \to \mathcal{C}^*_\Gamma$ given by $\alpha_\Gamma(G) = G(\mathbf{P}_\Gamma)$ and

$$\beta_\Gamma(N)(M) = \mathrm{Hom}_{\mathcal{A}_\Gamma}(\varphi_\Gamma(M), N) \tag{8.6.1}$$

where $\varphi_\Gamma(M) := \hom_{\mathcal{C}}(\mathbf{P}_\Gamma, M)$. These equivalences are compatible with the natural inclusions $\mathcal{C}^*_\Gamma \to \mathcal{C}^*_\Lambda$ and $\mathcal{A}_\Gamma$-mod $\to \mathcal{A}_\Lambda$-mod for $\Gamma \subseteq \Lambda$ in I_1. We denote by $j_\Gamma\colon \mathcal{A}_\Gamma\text{-mod} \to \hat{\mathcal{E}}$ and $k_\Gamma\colon \mathcal{A}\text{-mod} \to \hat{\mathcal{E}}$ the restrictions of the functors j_Γ, k_Γ defined in 8.5.

Define functors $\alpha\colon \mathcal{C}^*_\Gamma \to \mathcal{A}_\Gamma$-mod and *beta*: $\mathcal{A}_\Gamma\text{-mod} \to \mathcal{C}^*_\Gamma$ by

$$\alpha(F) = \varinjlim_{\Gamma\in I}\big((j_\Gamma \alpha_\Gamma \iota^*_\Gamma)(F)\big)$$

and

$$\beta(M) = \varinjlim_{\Gamma\in I}\big(\tau^*_\Gamma \beta_\Gamma k_\Gamma)(M)\big).$$

It will be shown that α and β define inverse category equivalences satisfying the conditions (a)–(c) of Theorem 1.19. This will in particular prove Theorem 1.19, since a standard family $\mathbf{P}$ of projectives in $\hat{\mathcal{C}}$ (for a stratified exact category $\mathcal{C}$) satisfies the assumption 8.2 by Lemma 7.11.

Firstly, for M in $\hat{\mathcal{E}}$ and N in $\mathcal{C}$, the definitions give

$$\big((\tau^*_\Gamma \beta_\Gamma k_\Gamma)(M)\big)(N) = \mathrm{Hom}_{\mathcal{A}_\Gamma}(\varphi_\Gamma(N_\Gamma), k_\Gamma(M) \cong \mathrm{Hom}_{\mathcal{A}}(\varphi_\Gamma(N(\Gamma)), M).$$

The right hand side is independent of Γ up to canonical isomorphism provided N is in $\mathcal{C}_\Gamma$. It follows that the colimit defining $\beta(M)$ can be calculated pointwise (i.e. the pointwise colimit is left exact), that

$$\beta(M)(N) = \operatorname{Hom}_{\hat{\mathcal{E}}}(\varinjlim_{\Gamma\in I} \varphi_\Gamma(N), M) \tag{8.6.2}$$

in general and that $\beta(M)(N) \cong \operatorname{Hom}_{\hat{\mathcal{E}}}(\varphi_\Gamma(N), M)$ for N in $\mathcal{C}_\Gamma$. The functor α is explicitly described as follow: for $G \in \mathcal{C}^*$, there is a natural direct system $\{G(\mathbf{P}_\Gamma)\}_{\Gamma\in I}$ in $\hat{\mathcal{E}}$ and

$$\alpha(G) = \varinjlim_{\Gamma\in I} G(\mathbf{P}_\Gamma). \tag{8.6.3}$$

We now assert that

8.6.4. For $\Gamma \in I$, $\iota_\Gamma^*\beta \cong \beta_\Gamma k_\Gamma$ and $k_\Gamma\alpha \cong \alpha_\Gamma\iota_\Gamma^*$ as well.

The first follows easily using 8.6.2. For F in $\mathcal{C}^*$, one has

$$k_\Gamma\alpha(F) = \oplus_j \operatorname{Hom}_{\hat{\mathcal{E}}}(\mathcal{A}_\Gamma e_j, \varinjlim_{\Sigma\in I} F(P_\Sigma)).$$

The natural map $F(\mathbf{P}_\Gamma) \to \varinjlim_{\Sigma\in I} F(\mathbf{P}_\Sigma)$ is a monomorphism. To show $k_\Gamma\alpha(F) \cong \alpha_\Gamma\iota_\Gamma^*(F)$, it will suffice to show that under the monomorphism

$$\epsilon_{j,\Gamma} \colon \operatorname{Hom}_{\hat{\mathcal{E}}}(\mathcal{A}_\Gamma e_j, \varinjlim_{\Sigma\in I} F(P_\Sigma)) \to e_j \varinjlim_{\Sigma\in I} F(\mathbf{P}_\Sigma) = \varinjlim_{\Sigma\in I} F(P_{j,\Sigma})$$

given by $h \mapsto h(e_j)$ for $h\colon \mathcal{A}_\Gamma e_j \to \varinjlim_{\Sigma\in I} F(P_\Sigma)$, $\operatorname{Hom}_{\hat{\mathcal{E}}}(\mathcal{A}_\Gamma e_j, \varinjlim_{\Sigma\in I} F(P_\Sigma))$ identifies with the $\mathbb{Z}$-submodule $F(P_{j,\Gamma})$ of $\varinjlim_{\Sigma\in I} F(P_{j,\Sigma})$.

Since $\mathcal{A}_\Gamma e_j$ is a f.g. $\mathcal{A}$-module, the above claim will follow once it is checked that for any $\Sigma \supseteq \Gamma$ in I, the image L of $\operatorname{Hom}_{\mathcal{A}_\Sigma}(\mathcal{A}_\Gamma e_j, F(\mathbf{P}_\Sigma))$ under $h \mapsto h(e_j)$ coincides with the image of the monomorphism $F(g)\colon F(P_{j,\Gamma}) \to F(P_{j,\Sigma})$ induced by the canonical admissible epimorphism $g\colon P_{j,\Sigma} \to P_{j,\Gamma}$. Now from the definitions and 2.1.1, L consists of all $m \in F(P_{j,\Sigma})$ such that $F(f)(m) = 0$ for all $f\colon P_{i,\Sigma} \to P_{j,\Sigma}$ which factor through the inclusion $P_{j,\Sigma}(\Sigma\setminus\Gamma) \to P_{j,\Sigma}$. If $m = F(g)n$ where $n \in F(P_{j,\Gamma})$ then $F(f)(m) = F(gf)(n) = 0$ since $\operatorname{Hom}(P_{j,\Sigma}(\Sigma\setminus\Gamma), P_{j,\Gamma}) = 0$. On the other hand, by 8.3.1 there is an exact sequence $Q \to P_{j,\Sigma} \xrightarrow{g} P_{j,\Gamma} \to 0$ with $Q = Q(\Sigma\setminus\Gamma)$ in $\operatorname{Add}\mathbf{P}_\Sigma$. Applying F gives an exact sequence

$$0 \to F(P_{j,\Gamma}) \xrightarrow{F(g)} F(P_{j,\Sigma}) \to F(Q)$$

from which one sees that L is contained in $\operatorname{Im} F(g)$. This completes the proof of 8.6.4; observe that these natural isomorphisms for $\Gamma \in I$ are coherent in a natural sense.

Now for F, G in C, one has $F \cong \varinjlim_{\Gamma\in I} \tau_\Gamma^*\iota_\Gamma^*$ and

$$\operatorname{Hom}_{\mathcal{C}^*}(F,G) \cong \varprojlim_{\Gamma\in I} \operatorname{Hom}_{\mathcal{C}_\Gamma^*}(\iota_\Gamma^* F, \iota_\Lambda^* F).$$

Analogous results hold in $\hat{\mathcal{E}}$. Using 8.6.4, it follows immediately that α and β are inverse category equivalences. The assertions 1.19(a)–(b) now follow from the definitions of α and β using 8.6.4 and C.8 (note that any equivalence of categories between abelian categories is exact). Finally, 1.19(c) follows noting each $\mathcal{A}_\Gamma e_j \cong \varphi_\Gamma(P_{j,\Gamma})$ is in the strict image of φ_Γ and using B.8, B.9 and B.11. $\square$

Remarks. The standard automorphisms $T_g\colon M \mapsto M\langle g^{-1}\rangle$ for $g \in G$ of $\mathcal{A}$-mod by grading shift preserve $\mathcal{E}$ and $\hat{\mathcal{E}}$. One can check that the automorphism on $\mathcal{C}^*$ corresponding via β to T_g is given by $F \mapsto T^*_{g^{-1}}F = FT_{g^{-1}}$ where $T_g\colon \mathcal{C} \to \mathcal{C}$ also denote the standard automorphisms of $\mathcal{C}$. Note also that $\varphi T_g = T_g\varphi\colon \mathcal{C} \to \hat{\mathcal{E}}$.

8.7. The equivalence constructed above reduces to C.8 if Ω is a singleton set. Note C.8 also makes it possible to give a direct description of C^* for $C = \hat{\mathcal{C}}$, taking $\mathbf{Q}$ to be a set of representatives of isomorphism classes of convergent direct sums of translates of objects of $\mathbf{P}$. It is then possible to describe $\operatorname{end}(\mathbf{Q})^{\mathrm{op}}$ in terms of $\operatorname{end}(\mathbf{P})^{\mathrm{op}}$, since for any two convergent direct sums $Q_1 = \oplus_p P_{l_p}\langle g_p\rangle$ and $Q_2 = \prod_q P_{m_q}\langle h_q\rangle$ in $\mathbf{Q}$ with the l_p and m_q in J, we have $\hom(Q_1, Q_2) \cong \prod_{p,q}\hom(P_{l_p}\langle g_p\rangle, P_{m_q}\langle g_q\rangle)$.

8.8. We can now prove Proposition 1.20 under the weaker conditions of this section.

Proposition. *Suppose that $\mathcal{C}$ is a perfectly exact subcategory over G of an abelian category $\mathcal{B}$ over G, and that $\mathcal{B}$ has Serre subcategories $\mathcal{B}_\Gamma$ over G for $\Gamma \in I$ satisfying the conditions* 1.20(i)-(v). *Then the statements in Proposition* 1.20 *hold.*

Proof. We use C.7 for the proof. We claim first that

8.8.1. for any $\Gamma \subseteq \Lambda \in I$, any $j \in J$ and any object M of $\mathcal{B}_\Gamma$, the canonical admissible epimorphism $P_{j,\Lambda} \to P_{j,\Gamma}$ induces an isomorphism

$$\hom(P_{j,\Gamma}, M) \xrightarrow{\cong} \hom(P_{j,\Lambda}, M).$$

It is enough for this to show that $\hom(P_{j,\Lambda}(\Lambda \setminus \Gamma), M) = 0$. By 8.3.1, we may choose an admissible epimorphism $Q \to P_{j,\Lambda}(\Lambda \setminus \Gamma)$ with $Q \cong Q(\Lambda \setminus \Gamma)$ in add $\mathbf{P}_\Lambda$ (so Q is projective in $\mathcal{B}_\Lambda$) and it will suffice to show that $\hom(Q, M) = 0$. But M is a quotient of a (possibly infinite) direct sum of translates of objects $P_{j,\Gamma}$ which are in $\mathcal{B}_\Gamma$ and hence in $\mathcal{B}_\Lambda$, and by (v) $\hom(Q, ?)$ preserves infinite direct sums from $\mathcal{B}_\Lambda$. Hence we are reduced to showing that $\hom(Q, P_{j,\Gamma}) = 0$ for all j, which is immediate from 2.1.1 since Q is in $\mathcal{C}_{\Lambda\setminus\Gamma}$ and P is in $\mathcal{C}_\Lambda$.

Now we can define a functor $F\colon \mathcal{B} \to \mathcal{E}$ by

$$F = \varinjlim_{\Lambda \in I} \hom(\mathbf{P}_\Lambda, ?).$$

By the last fact above, the restriction of F to $\mathcal{B}_\Gamma$ for $\Gamma \in I$ is equivalent to $\hom(\mathbf{P}_\Gamma, ?)$, which by the first fact above gives an equivalence $\mathcal{B}_\Gamma \xrightarrow{\cong} \mathcal{A}_\Gamma$-mod. It is clear that F gives an equivalence $\mathcal{B} \to \mathcal{E}$ with $F\iota \cong \varphi$, as desired.

Conversely, suppose given the weakly stratified exact category $\mathcal{C}$ satisfying the assumption 8.2. Then $\mathcal{B}_\Gamma := \mathcal{A}_\Gamma$-mod is obviously closed under subquotients in $\mathcal{B} := \mathcal{E}$, and it is closed under extensions by 4.1(c) and 8.6, so $\mathcal{B}_\Gamma$ is a Serre subcategory of $\mathcal{B}$. It is clear that the conditions (i)–(v) of 1.20 hold. □

8.9. Suppose given two exact categories $\mathcal{C}^i$ for $i = 1, 2$ over G and a right exact functor $F\colon \mathcal{C}^1 \to \mathcal{C}^2$ over G (i.e. $T^2_g F = FT^1_g$ for $g \in G$ (cf 6.3). One has the corresponding left (resp., right) exact functor $F^*\colon \mathcal{C}^{2*} \to \mathcal{C}^{1*}$ (resp., $F_*\colon \mathcal{C}^{1*} \to \mathcal{C}^{2*}$).

Suppose $\mathbf{P}^i = \{P^i_j\}_{j\in J^i}$ is a set of projective objects in $\hat{\mathcal{P}}^i$ satisfying the assumption 8.2. Define $\mathcal{A}^i = \operatorname{End}(\mathbf{P}^i)$, $\hat{\mathcal{E}}^i$, α^i, β^i, φ^i etc for $\mathcal{C}^i$ as for $\mathcal{C}$. Regarding α^i as an identification, we may regard F^* (resp., F_*) as a left (resp., right) exact functor

$'F^*\colon \hat{\mathcal{E}}^2 \to \hat{\mathcal{E}}^1$ (resp., $'F_*\colon \hat{\mathcal{E}}^1 \to \hat{\mathcal{E}}^2$). It is convenient to have explicit formulae available for these functors in terms of standard functors on module categories.

Define the family $\{F_{\Lambda,\Gamma}\}_{(\Lambda,\Gamma)\in I^2\times(I^1)^{\mathrm{op}}}$ of $(\mathcal{A}^2,\mathcal{A}^1)$-bimodules with

$$F_{\Lambda,\Gamma} := \hom(\mathbf{P}^2_\Lambda, F(\mathbf{P}^1_\Gamma))$$

where $F(\mathbf{P}^1_\Gamma) := \{F(P^1_{j,\Gamma})\}_j$. For $\Lambda \supseteq \Lambda'$ in I^2 and $\Gamma \supseteq \Gamma'$ in I^1, there are natural maps $F_{\Lambda',\Gamma} \to F_{\Lambda,\Gamma'}$ given by

$$\big(P^2_{i,\Lambda'}\langle g\rangle \xrightarrow{f} F(P^1_{j,\Gamma})\big) \mapsto \big(P^2_{i,\Lambda}\langle g\rangle \to P^2_{i,\Lambda'}\langle g\rangle \xrightarrow{f} F(P^1_{j,\Gamma}) \to F(P^1_{j,\Gamma'})\big)$$

which make $\{F_{\Lambda,\Gamma}\}_{(\Lambda,\Gamma)\in I^2\times(I^1)^{\mathrm{op}}}$ into a direct system.

For fixed Γ, the direct system $\{F_{\Lambda,\Gamma}e_j\}_{\Lambda\in I^2}$ is stable for Λ sufficiently large that $F(P_{j,\Gamma})$ is in $\mathcal{C}^2_\Lambda$ and so we may define the $(\mathcal{A}^2,\mathcal{A}^1)$-bimodule $F_\Gamma = \varinjlim_{\Lambda\in I^2} F_{\Lambda,\Gamma}$. There is a natural inverse system $\{F_\Gamma\}_{\Gamma\in I^1}$.

Corollary. *The following formulae for $'F^*$ and $'F_*$ hold:*

$$'F^* = \varinjlim_{\Gamma\in I^1} \big(\oplus_{j\in J^1} \hom_{\mathcal{A}^2}(F_\Gamma e_j, ?)\big)$$

$$'F_* = \varinjlim_{\Gamma\in I^1} \Big(F_\Gamma \otimes_{\mathcal{A}^1} \big(\oplus_{j\in J^1} \hom_{\mathcal{A}^1}(\mathcal{A}^1_\Gamma e_j, ?)\big)\Big)$$

with the colimits calculated pointwise.

Proof. Directly from the definitions, $'F^* = \alpha^1 F^* \beta^2$ is given by

$$\oplus_{j\in J^1} \varinjlim_{\Gamma\in I^1} \hom_{\mathcal{A}^2}\big(\varinjlim_{\Lambda\in I^2} \oplus_{i\in J^2} \hom_{\mathcal{C}^2}(P^2_{i,\Lambda}, F(P^1_{j,\Gamma})), ?\big)$$

which gives the first formula. Now in the special case $\Omega \in I^1$, the formula for $'F^*$ simplifies to

$$'F^* = \oplus_{j\in J^1} \hom_{\mathcal{A}^2}(F_\Omega e_j, ?)$$

and in that case the left adjoint $'F_*$ of $'F^*$ is clearly given by

$$'F_* = F_\Omega \otimes_{\mathcal{A}^1} ?.$$

Now consider the general case. For any functor G in $\mathcal{C}^{1*}$, one has a direct system $\{G\sigma_\Gamma\}_{\Gamma\in I^1}$ of functors in $\mathcal{C}^{1*}$ and a canonical isomorphism $G \cong \varinjlim_{\Gamma\in I^1} G\sigma_\Gamma$ in $\mathcal{C}^{1*}$. To see this, note the canonical isomorphism $G(M) \cong \varinjlim_{\Gamma\in I^1} G\sigma_\Gamma(M)$ for M in $\mathcal{C}$, by stability of the direct system.

Since F_* is a left adjoint to F^*, it preserves colimits and hence $F_* \cong \varinjlim_{\Gamma\in I^1} F_*\sigma^*_\Gamma$. But $\sigma_\Gamma = \iota_\Gamma\tau_\Gamma$ and $\tau^*_\Gamma = \iota_{\Gamma*}$ so $F_* \cong \varinjlim_{\Gamma\in I^1}(F\iota_\Gamma)_*\iota^*_\Gamma$; there is of course a corresponding formula for $'F_*$. Now the given formula for $'F_*$ follows from the explicit formulae (from the special case examined above) for $'(F\iota_\Gamma)_*$ and $'\iota^*_\Gamma$. For completeness, we should indicate how the canonical maps of the inverse system whose colimit appears in the formula for $'F_*$ may be described module-theoretically. Consider $\Gamma \subseteq \Sigma$ in I^1. Let $j\colon \mathcal{C}_\Gamma \to \mathcal{C}_\Sigma$ be the inclusion, and $k\colon \mathcal{C}_\Sigma \to \mathcal{C}_\Gamma$ be the restriction of the truncation functor σ_Γ. Since $F_*\iota_{\Gamma*} = F_*\iota_{\Sigma*}j_*$, we obtain an equivalence of functors

$$F_\Gamma \otimes_{\mathcal{A}^1_\Gamma} ? \cong F_\Sigma \otimes_{\mathcal{A}^1_\Sigma} \mathcal{A}^1_\Gamma \otimes_{\mathcal{A}^1_\Gamma} ?$$

from $\mathcal{A}^1_\Gamma$-mod to $\mathcal{A}^2$-mod, or equivalently, an isomorphism $F_\Gamma \cong F_\Sigma \otimes_{\mathcal{A}^1_\Sigma} \mathcal{A}^1_\Gamma$ of $(\mathcal{A}^2, \mathcal{A}^1_\Gamma)$-bimodules.

Now the canonical map $(F\iota_\Gamma)_*\iota_\Gamma^* \to (F\iota_\Sigma)_*\iota_\Sigma^*$ considered above is

$$F_*\iota_{\Gamma *}\iota_\Gamma^* \cong F_*\iota_{\Sigma *}j_*j^*\iota_\Sigma^* = F_*\iota_{\Sigma *}(k^*j^*)\iota_\Sigma^* \to F_*\iota_{\Sigma *}\iota_\Sigma^*$$

(using the natural isomorphism $\mathrm{Id} \cong jk$). In terms of modules, this is the map

$$F_\Gamma \otimes_{\mathcal{A}^1} \big(\oplus_j \hom_{\mathcal{A}^1}(\mathcal{A}^1_\Gamma e_j, ?)\big) \xleftarrow{\cong} F_\Sigma \otimes_{\mathcal{A}^1} \big(\oplus_j \hom_{\mathcal{A}^1}(\mathcal{A}^1_\Gamma e_j, ?)\big) \longrightarrow F_\Sigma \otimes_{\mathcal{A}^1} \big(\oplus_j \hom_{\mathcal{A}^1}(\mathcal{A}^1_\Sigma e_j, ?)\big).$$

□

9. $\boldsymbol{\Delta}$-modules and $\boldsymbol{\nabla}$-modules

For the rest of the paper, we fix a stratified exact category $\mathcal{C}$ such that $\hat{\mathcal{C}}$ has enough projectives, and a standard family $\mathbf{P} = \{P_{x,i}\}$ of projectives in $\hat{\mathcal{C}}$. Unexplained notation is as in Section 1 (see especially 1.17 and 1.19) unless otherwise indicated.

9.1. For any locally closed subset Σ of Ω, define the family $\hat\sigma_\Sigma(\mathbf{P}) = \{\hat\sigma_\Sigma(P_{x,i})\}_{x,i}$ in $\hat{\mathcal{C}}$. Then $\mathcal{A}_\Sigma := \mathrm{end}(\hat\sigma_\Sigma(\mathbf{P}))^{\mathrm{op}}$ is a ring and there is a natural (diagonalizable, graded) ring homomorphism $\mathcal{A} \to \mathcal{A}_\Sigma$ induced by $f \mapsto \hat\sigma_\Sigma f$. By pullback along this homomorphism, we obtain an exact functor $\mathcal{A}_\Sigma\text{-mod} \to \mathcal{A}\text{-mod}$. In particular, $\mathrm{ext}^i(\hat\sigma_\Sigma(\mathbf{P}), ?)$ may be regarded as a functor to $\mathcal{A}$-mod, and the long exact $\mathrm{ext}^i(\hat\sigma_\Sigma(\mathbf{P}), ?)$-sequences are then exact sequences of $\mathcal{A}$-modules; similar remarks apply to right modules and $\mathrm{ext}^i(?, \hat\sigma_\Sigma(\mathbf{P}))$.

Define the $(\mathcal{A}, \mathcal{A}_\Sigma)$-bimodule $\mathcal{A}(\Sigma) := \hom(\mathbf{P}, \hat\sigma_\Sigma(\mathbf{P}))$ and regard it as a (graded) $(\mathcal{A}, \mathcal{A})$-bimodule as above. As $(\mathcal{A}, \mathcal{A})$-bimodule, $\mathcal{A}(\Sigma)$ is a subquotient bimodule of $\mathcal{A}$, by projectivity of the objects of $\mathbf{P}$. If Λ is a coideal of Σ, we have again by projectivity an exact sequence

$$0 \to \mathcal{A}(\Lambda) \to \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma \setminus \Lambda) \to 0$$

of $(\mathcal{A}, \mathcal{A})$-bimodules. Observe also that if Σ is an ideal of Ω, then projectivity implies that $\mathcal{A}_\Sigma$ is a quotient ring of $\mathcal{A}$ and that $\mathcal{A}(\Sigma) \cong \mathcal{A}_\Sigma$ as $(\mathcal{A}, \mathcal{A})$-bimodule.

If $\Gamma \in I_1$, then $\mathcal{A}_\Gamma \cong \mathrm{End}(\mathbf{P}_\Gamma)^{\mathrm{op}}$ where $\mathbf{P}_\Gamma := \{P_{x,i,\Gamma}\}_{x,i}$. There is a natural inverse system of diagonalizable rings $\mathcal{A}_\Gamma \cong \mathrm{End}(\mathbf{P}_\Gamma)^{\mathrm{op}}$ for $\Gamma \in I_1$ with $\mathcal{A}$ as its projective limit. Using Theorem 1.19, we identify $\mathcal{C}^*$ with $\hat{\mathcal{E}}$, $\mathcal{C}^\dagger$ with $\mathcal{E}$ and $\mathcal{C}^*_\Gamma = \mathcal{A}_\Gamma\text{-mod}$ for $\Gamma \in I_1$.

If Σ above is contained in an ideal $\Gamma \in I_1$, then $\mathcal{A}_\Sigma$-modules regarded as $\mathcal{A}$-modules are actually $\mathcal{A}_\Gamma$-modules and hence lie in $\mathcal{E}$.

9.2. **Proof of Lemma 1.21.** We first prove 1.21(b). Observe that there is a natural map

$$\varinjlim_{\Lambda \in I_1, \Lambda \supseteq \Gamma} \mathrm{Ext}^i_{\mathcal{C}^*_\Lambda}(M, N) \to \mathrm{Ext}^i_{\mathcal{E}}(M, N).$$

It is clear from the point of view of Yoneda Ext-groups that this map is an isomorphism; for I_1 is directed, and given an i-fold extension $0 \to N \to N_1 \to \cdots \to N_i \to M \to 0$ in $\mathcal{E}$, there is some $\Lambda \in I$, $\Lambda \supseteq \Gamma$ with all terms of the sequence in $\mathcal{C}^*_\Lambda$ (incidentally, this also shows that Yoneda $\mathrm{Ext}^i_{\mathcal{E}}$-groups are defined i.e. the classes of extensions are sets).

To prove (b), it will therefore suffice to show that all the natural maps

$$\mathrm{Ext}^i_{\mathcal{A}_\Gamma}(M, N) \to \mathrm{Ext}^i_{\mathcal{A}_\Lambda}(M, N)$$

are isomorphisms. If $i \leq 1$, this is true since $\mathcal{C}_\Gamma^*$ is a Serre subcategory of $\mathcal{E}$ (by Proposition 1.20). The remainder of the proof of (b) follows that of [15, Statement 3]. We first prove $\mathrm{Ext}^i_{\mathcal{A}_\Lambda}(\mathcal{A}_\Gamma e_{x,j}, N) = 0$ for $\mathcal{A}_\Gamma$-modules N and $i \geq 1$. To do this, use 7.11 to choose a projective resolution $P^\bullet \to P_{x,j,\Gamma} \to 0$ in $\mathcal{C}_\Lambda$ such that the map $P^0 \to P_{x,j,\Gamma}$ is the canonical admissible epimorphism $P_{x,j,\Lambda} \to P_{x,j,\Gamma}$ and P^k is a direct sum of translates of objects $P_{y,l,\Lambda}$ with $y \in \Lambda \setminus \Gamma$ for all $k \geq 1$. Compute the Ext^i-group in question using the projective resolution $\varphi(P^\bullet)$ of $\mathcal{A}_\Gamma$ in $\mathcal{A}_\Lambda$-Mod; it vanishes since $\mathrm{Hom}(\varphi(P^i), N) = 0$ for $i \geq 1$ (observe $\varphi(P_{y,l,\Lambda}) = \mathcal{A}_\Lambda e_{y,l}$ and $e_{y,l}N = 0$ if $y \notin \Gamma$).

It follows from the preceding paragraph that $\mathrm{Ext}^i_{\mathcal{A}_\Lambda}(F, N) = 0$ for $i \geq 1$ and any projective $\mathcal{A}_\Gamma$-module F. The proof of (b) can now be completed using dimension shifting, taking a short exact sequence $0 \to L \to F \to M \to 0$ with F projective in $\mathcal{A}_\Gamma$-Mod.

Now to prove (a), we may by (b) and 1.6(b) assume that $\Omega \in I_1$. Choose a projective resolution $P^\bullet \to M \to 0$ in $\mathcal{C}$. Then

$$\mathrm{Ext}^i_{\mathcal{C}}(M, N) \cong \mathrm{H}^i(\mathrm{Hom}_{\mathcal{C}}(P^\bullet, N) \cong \mathrm{H}^i(\mathrm{Hom}(\varphi(P^\bullet), \varphi(N)) \cong \mathrm{Ext}^i_{\mathcal{E}}(\varphi(M), \varphi(N))$$

since $\varphi(P^\bullet) \to \varphi(M) \to 0$ is a projective resolution of M in $\mathcal{A}$-mod (i.e. in $\mathcal{E}$).

Remarks. Combining similar arguments to those above with those in the proof of 4.5 shows that if $\hat{\mathcal{C}}$ has enough projectives and $\Gamma \subseteq \Lambda \in I_1$, then the left derived functor $L\tau_\Gamma^* \colon \mathrm{D}^-(\mathcal{C}_\Gamma^*) \to \mathrm{D}^-(\mathcal{C}_\Lambda^*)$ is a full embedding, with $L\tau_{\Gamma *} L\tau_\Gamma^* \cong \mathrm{Id}$ (one has to note additionally that each object of $\mathrm{D}^-(\mathcal{C}_\Gamma^*)$ can be represented by a bounded above complex of projectives which are (possibly infinite) direct sums of projectives of the form $\phi_{\mathcal{C}_\Gamma}(P)$ with P projective in $\mathcal{C}_\Gamma$). Together with 4.5(b) and 4.1, this has implications for "recollement" of these derived categories, particularly if Ω is finite; see [44].

9.3. We describe in module-theoretic terms some of the standard functors defined on certain subcategories of $\mathcal{E}$ (cf 4.1). The formulae follow easily from those in 8.9.

Fix $\Gamma \subseteq \Lambda$ in I_1; let $i \colon \mathcal{C}_\Gamma \to \mathcal{C}_\Lambda$ be the inclusion and $t \colon \mathcal{C}_\Lambda \to \mathcal{C}_\Gamma$ denote truncation at Γ. Then we have an adjoint triple $(t_*, t^* = i_*, i^*)$ of functors between $\mathcal{A}_\Lambda$-mod and $\mathcal{A}_\Gamma$-mod such that $t_* = \mathcal{A}_\Gamma \otimes_{\mathcal{A}_\Lambda} ?$, $t^* = i_*$ is the natural inclusion $\mathcal{A}_\Gamma$-mod $\to \mathcal{A}_\Lambda$-mod and $i^* = \oplus_{x,j} \mathrm{hom}_{\mathcal{A}_\Lambda}(\mathcal{A}_\Gamma e_{x,j}, ?)$.

Suppose now instead that $\Gamma \in I_1$ and Σ is a coideal of Γ. Then $\mathcal{C}_\Sigma^*$ identifies with $\mathcal{A}'_\Sigma$-Mod where $\mathcal{A}'_\Sigma := \oplus_{x \in \Sigma, i} \oplus_{x' \in \Sigma, i'} e_{x,i} \mathcal{A}_\Gamma e_{x',i'}$. Let $j \colon \mathcal{C}_\Sigma \to \mathcal{C}_\Gamma$ be the inclusion and $s \colon \mathcal{C}_\Gamma \to \mathcal{C}_\Sigma$ be truncation at Σ. Then the localization functor $j^* = s_* \colon \mathcal{A}_\Gamma$-mod $\to \mathcal{A}'_\Sigma$-mod is just $M \mapsto \oplus_{x \in \Sigma, i} e_{x,i} M$; the left adjoint of j^* is

$$j_* = (\oplus_{x \in \Sigma, i} \mathcal{A}_\Gamma e_{x,i}) \otimes_{\mathcal{A}'_\Sigma} ?$$

and the right adjoint of s_* is

$$s^* = \oplus_{x',i'} \mathrm{hom}_{\mathcal{A}'_\Sigma}(\oplus_{x \in \Sigma, i} e_{x,i} \mathcal{A}_\Gamma e_{x',i'}, ?).$$

9.4. We record some basic properties of the families $\Delta = \Delta^{\mathcal{A}}$, $\nabla = \nabla^{\mathcal{A}}$, $\Delta^{\mathcal{A}^{\mathrm{op}}}$, $\nabla^{\mathcal{A}^{\mathrm{op}}}$ of modules (recall their definitions in 1.25 and 1.27).

Note first that under the identification $\mathcal{E} = \mathcal{C}^\dagger$, $\Delta_{x,i}$ identifies with $\Lambda(N_{x,i})$ and $\nabla_{x,i}$ identifies with $V_x(N_{x,i})$, in the notation of Section 4. Proposition 1.26 therefore follows immediately from the statements in 4.6.

If we also identify identify $\mathcal{C}_x^* = R_x$-mod for $x \in \Omega$, the functor $j_x \cong \tau_x^*$ of Section 4 identifies with $\nabla_x \otimes_{R_x} ?\colon R_x\text{-mod} \to \hat{\mathcal{E}}$. For using 9.3 and 9.4, $\tau_x^*\colon R_x\text{-mod} \to \hat{\mathcal{E}}$ is given by

$$\tau_x^* \cong \oplus_{y,i} \operatorname{hom}_{R_x}(\oplus_j e_{x,j}\mathcal{A}_{\leq x} e_{y,i}, ?) \cong \oplus_{y,i} \operatorname{hom}_{R_x}(\Delta_x^{\mathcal{A}^{\mathrm{op}}} e_{y,i}, ?) \cong \nabla_x \otimes_{R_x} ?.$$

In module-theoretic terms, the right adjoint j^x of j_x (see 4.4) is therefore simply given by $j^x \cong \operatorname{hom}_{\hat{\mathcal{E}}}(\boldsymbol{\nabla}_x, ?)$

Fix $x \in \Omega$ and an ideal $\Gamma \in I$ of Ω with x as maximal element. Observe that $\Delta_{x,i} \cong \mathcal{A}_\Gamma e_{x,i}$ (resp., $\Delta_{x,i}^{\mathcal{A}^{\mathrm{op}}} \cong e_{x,i}\mathcal{A}_\Gamma$) is a f.g. projective left (resp., right) $\mathcal{A}_\Gamma$-module. One has

$$\operatorname{end}_{\mathcal{A}}(\boldsymbol{\Delta}_x)^{\mathrm{op}} \cong \oplus_{i,j} e_{x,i}\mathcal{A}_\Gamma e_{x,j} \cong \operatorname{end}(\mathbf{N}_x)^{\mathrm{op}} \cong R_x$$

and

$$\operatorname{end}_{\mathcal{A}^{\mathrm{op}}}(\boldsymbol{\Delta}_x^{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} \cong (\oplus_{i,j} e_{x,i}\mathcal{A}_\Gamma e_{x,j})^{\mathrm{op}} \cong R_x^{\mathrm{op}}.$$

Note that $e_{y,i}\nabla_x = \operatorname{hom}(P_{y,i}(x), \mathbf{N}_x)$ (resp., $\Delta_x^{\mathcal{A}^{\mathrm{op}}} e_{y,i} \cong \operatorname{hom}(\mathbf{N}_x, P_{y,i}(x))$) is a f.g. projective left (resp., right) R_x-module since $P_{y,i}(x)$ is in $\operatorname{add}\mathbf{N}_x$. It follows using C.9 that for each x, one has

$$\oplus_{y,i,j} \operatorname{hom}_{R_x}(\Delta_x^{\mathcal{A}^{\mathrm{op}}} e_{y,i}, R_x e_j) \cong \nabla_x$$

as $(\mathcal{A}, R_x)$-bimodule and

$$\oplus_{y,i,j} \operatorname{hom}_{R_x^{\mathrm{op}}}(e_{y,i}\Delta_x, e_j R_x) \cong \Delta_x^{\mathcal{A}^{\mathrm{op}}}$$

as $(R_x, \mathcal{A})$-bimodule.

9.5. The following observation is also useful. By 1.19, for any M in $\mathcal{C}$, $\varphi(M)$ has a finite filtration with successive subquotients equal to the $(\mathcal{A}, \operatorname{end}(M))$-bimodules,

$$\varphi(M(x)) \cong \operatorname{hom}(\mathbf{P}_{\leq x}, \mathbf{N}_x) \otimes_{R_x} \operatorname{hom}(\mathbf{N}_x, M(x)) \cong \Delta_x \otimes_{R_x} \operatorname{hom}(\mathbf{N}_x, M(x)),$$

where $\operatorname{hom}(\mathbf{N}_x, M(x))$ is a f.g. projective R_x-module.

9.6. Proposition 1.26 allows the construction of additional stratified exact categories as follows.

Proposition. *Define $\mathcal{C}^\Delta$ (resp., $\mathcal{C}^\nabla$) to be the full additive subcategory of $\hat{\mathcal{E}}$ consisting of modules with a finite filtration with successive subquotients in* $\operatorname{add}\boldsymbol{\Delta}_x$ *(resp.,* $\operatorname{add}\boldsymbol{\nabla}_x$*) for various $x \in \Omega$. Regard them as perfectly exact subcategories of $\mathcal{E}$. Then $(\mathcal{C}^\Delta, \{\operatorname{add}\boldsymbol{\Delta}_x\}_{x\in\Omega})$ (resp., $(\mathcal{C}^\nabla, \{\operatorname{add}\boldsymbol{\nabla}_x\}_{x\in\Omega^{\mathrm{op}}})$) are stratified exact categories.*

Proof. This follows readily from the definition in 1.1 and the ext-vanishing properties 1.26. □

Remarks. It would be more natural to define instead $\mathcal{C}^\Delta$ (resp., $\mathcal{C}^\nabla$) as subcategories of $\mathcal{C}^*$ using as x-stratum the strict image of V_x (resp., of Λ_x) in $\mathcal{C}^*$; the categories as we have defined them identify with the Karoubianizations of these (cf B.11) so the difference is immaterial for most purposes.

Observe also that by C.9, $\operatorname{add}\boldsymbol{\Delta}_x$ and $\operatorname{add}\boldsymbol{\nabla}_x$ are both equivalent as additive categories to the category of f.g. projective R_x-modules.

10. Stratified rings

10.1. **Proof of Proposition 1.22 and 1.32.** It is clear that a family $(\mathcal{A}, \{\mathcal{A}_\Gamma\}_{\Gamma\in I})$ obtained as in 1.19 from a split stratified exact category such that $\mathcal{C}$ has enough projectives satisfies conditions 1.22(i)–(iii); for the first part of (ii), observe that the kernel of $\mathrm{end}(\mathbf{P}_\Lambda)^{\mathrm{op}} \to \mathrm{end}(\mathbf{P}_\Gamma)^{\mathrm{op}}$ consists of those endomorphisms which factor through an object of $\mathcal{C}_{\Lambda\setminus\Gamma}$, or equivalently those which factor through an object of $\mathrm{add}\,\{P_{x,i,\Lambda}\}_{x\in\Lambda\setminus\Gamma}$.

Conversely, let A be any left stratified ring. We consider the full abelian category B consisting of all A-modules which are $A(\Gamma)$-modules (i.e. are annihilated by the kernel of $A \mapsto A(\Gamma)$) for some $\Gamma \in I_1$. We claim that $A(\Gamma)$-mod is a Serre subcategory of B. It is enough to check it is a Serre subcategory of $A(\Lambda)$-mod for $\Gamma \subseteq \Lambda$ in I_1. Closure of $A(\Gamma)$-mod under subquotients in $A(\Lambda)$-mod is clear. On the other hand, consider an exact sequence $0 \to L \to M \to N \to 0$ of $A(\Lambda)$-modules in which L and N are $A(\Gamma)$-modules. By 1.22(ii), $A(\Gamma) = A(\Lambda)/J$ for some idempotent ideal J of $A(\Lambda)$. Then $JN = 0$ so $JM = J^2M \subseteq JL = 0$ and M is a $A(\Gamma)$-module as required.

Note that by 1.22(ii), if x is maximal in $\Gamma \in I_1$, then $A(\Gamma)e_{x,i} = A(\leq x)e_{x,i}$ is a projective $A(\Gamma)$-module and so $\mathrm{Hom}_B(A(\leq x)e_{x,i}, M) \cong e_{x,i}M$ for any $A(\Gamma)$-module M; in particular, this vanishes if $e_{x,i}M = 0$. By 1.22(i), it follows that

$$\mathrm{hom}_B(A(\leq x)e_{x,i}, A(\leq y)e_{y,j}) = 0 \qquad \text{unless } x \leq y.$$

Let $C_x := \mathrm{add}\,\{A(\leq x)e_{x,i}\}_i$ and C be the smallest extension-closed full additive subcategory of B containing all C_x. It follows from 1.2 (taking B_x to be the Serre subcategory of all objects of B which are $A(\Gamma)$-modules for some $\Gamma \in I_1$ with maximal element x) that $(C, \{C_x\}_{x\in\Omega})$ is a split stratified exact category if endowed with the class of short B-exact sequences of objects of C. The condition 1.22(ii) implies that $A(\Gamma)e_{x,i}$ is in C_Γ for $\Gamma \in I$, and it is obviously a projective object of C_Γ, so $\hat{C}$ has enough projectives using 1.18. There is an obvious projective object $P'_{x,i} = \{A(\Gamma)e_{x,i}\}_{\Gamma\in I}$ in $\hat{C}$. The family $\mathbf{P}' := \{P'_{x,i}\}_{x,i}$ is a standard family of projectives in $\hat{\mathcal{C}}$ (with respect to the families of standard objects $\mathbf{N}'_x := \{\mathcal{A}_{\leq x}e_{x,i}\}_i$ for $x \in \Omega$ in $\mathcal{C}$). Moreover, one has isomorphisms $\mathrm{end}(\mathbf{P}')^{\mathrm{op}} = A$ of rings and $\{\mathrm{end}(\hat{\sigma}_\Gamma\mathbf{P}')^{\mathrm{op}}\}_{\Gamma\in I} \cong \{A(\Gamma)\}_{\Gamma\in I}$ of inverse systems of rings, canonically. This completes the proof of the first assertion of Proposition 1.22.

Assuming that $(A, \{A(\Gamma)\})$ is left stratified, it follows further that the abelian category C^* identifies with the full subcategory of A-Mod consisting of objects which are the directed unions of their subobjects in B. The family $\{A(\leq x)e_{x,i}\}_i$ in C^* constitutes a full set of $\boldsymbol{\Delta}$-modules of highest weight x, and the opposite ring of the endomorphism ring of this family of $\boldsymbol{\Delta}$-modules is $k_x := \oplus_{i,j}e_{x,i}A(\leq x)e_{x,j}$. We define the subquotient (A, A)-bimodule $A(\Sigma) := \mathrm{hom}(\mathbf{P}', \hat{\sigma}_\Sigma(\mathbf{P}'))$ of A for any locally closed subset Σ of Ω; this is compatible with the existing notation for the quotient ring $A(\Gamma)$ of A for $\Gamma \in I$. By 9.5, we have for $x \in \Omega$ that

$$A(x) = A(\leq x)/A(< x) \cong (\oplus_i A(\leq x)e_{x,i}) \otimes_{k_x} (\oplus_i e_{x,i}A(\leq x))$$

where for each y and j, $\oplus_i e_{x,i}A(\leq x)e_{y,j}$ is a f.g. projective k_x-module. Now if $(A, \{A(\Gamma)\})$ is stratified, it follows by symmetry that $\oplus_i e_{y,j}A(\leq x)e_{x,i}$ is a f.g. right k_x-module.

Conversely, suppose that $(A, \{A(\Gamma)\})$ is left stratified and each $\oplus_i e_{y,j}A(\leq x)e_{x,i}$ is f.g. projective as right k_x-module. To show that $(A, \{A(\Gamma)\})$ is stratified, it will be enough to show that for $\Gamma \subseteq \Lambda$ in I, the kernel V of the epimorphism $A(\Lambda) \to A(\Gamma)$ is such that for each y and j, $e_{y,j}V$ has a finite filtration with successive subquotients in add$\{e_{z,k}A(\leq z)\}_k$ for various k and z with $z \geq y$ and $z \in \Lambda \setminus \Gamma$. Set $\Lambda' = \Lambda \cap (\Gamma \cup \{ z | z \not\geq y \})$. Then $\Gamma \subseteq \Lambda' \subseteq \Lambda$ are all in I_1. By 1.22(i)–(ii), $e_{y,j}A(\Lambda' \setminus \Gamma) = 0$ since $e_{y,j}A(\leq z)e_{z,k} = 0$ for any $z \in \Lambda' \setminus \Gamma$. From the short exact sequence

$$0 \to A(\Lambda \setminus \Lambda') \to A(\Lambda \setminus \Gamma) \to A(\Lambda' \setminus \Gamma) \to 0,$$

it follows that $e_{y,j}A(\Lambda \setminus \Gamma) = e_{y,j}A(\Lambda \setminus \Lambda')$. Now $A(\Lambda \setminus \Lambda')$ has a finite filtration as (A, A)-bimodule with successive subquotients $A(x)$ for x in the finite set $\Lambda \setminus \Lambda'$. The above formula for $A(x)$ and our assumption that $\oplus_i e_{y,j}A(\leq x)e_{x,i}$ is f.g. projective as right k_x-module imply that, as required, $e_{y,j}A(x)$ is in add$\{e_{x,i}A(\leq x)\}_i$. This completes the proof.

Remarks. Let $C'_x := \{\operatorname{add} e_{x,i}A(\leq x)\}_i$ and let C' be the smallest extension closed additive category of the abelian category of A^{op}-mod containing all C'_x with $x \in \Omega$. It follows by the same argument as in the proof of the proposition that $(C', \{C'_x\}_{x\in\Omega})$ is a split stratified exact category, if C' is regarded as a perfectly exact subcategory of $\mathcal{A}^{\mathrm{op}}$-mod. However, one will not in general have $e_{x,i}A(\Gamma)$ in C' for all $\Gamma \in I_1$ and x, i unless A is stratified.

10.2. We now restate some consequences of 1.22 and the above remark in the case of a left-stratified ring $(\mathcal{A}, \{\mathcal{A}(\Gamma)\}_{\Gamma\in I})$ obtained from a stratified exact category $(\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})$ by the construction 1.19.

10.2.1. Let B be the smallest extension closed subcategory of right $\mathcal{A}$-modules containing all $\{\operatorname{add} \mathbf{\Delta}_x^{\mathcal{A}^{\mathrm{op}}}\}_x$, endowed with the short exact sequences of right $\mathcal{A}$-modules in B. Then $\mathcal{C}^{\mathbf{\Delta}^{\mathcal{A}^{\mathrm{op}}}} := (B, \{\operatorname{add} \mathbf{\Delta}_x^{\mathcal{A}^{\mathrm{op}}}\}_{x\in\Omega})$ is a stratified exact category.

10.2.2. The pair $(\mathcal{A}, \{\mathcal{A}(\Gamma)\}_{\Gamma\in I})$ is stratified iff each "weightspace" $e_{y,j}\Delta_x^{\mathcal{A}^{\mathrm{op}}} \cong \hom(P_{y,j,\leq x}, \mathbf{N}_x)$ of $\Delta_x^{\mathcal{A}^{\mathrm{op}}}$ is f.g. projective as right R_x-module. In that case, the functor $\iota_{x*}\colon \mathcal{C}_x^* \to \mathcal{C}_{\leq x}^*$ is exact for all $x \in \Omega$, so the condition (i) of Proposition 4.7 holds.

To establish the last claim, note that ι_{x*} identifies with $\mathbf{\Delta}_x \otimes_{R_x} ?\colon R_x\text{-mod} \to \mathcal{A}_{\leq x}\text{-mod}$ by 9.4.

Assume for the remainder of this subsection that $\mathcal{A}$ is stratified. Then B^* identifies with the full subcategory of diagonalizable right $\mathcal{A}$-modules which are directed unions of their right $\mathcal{A}_\Gamma$-submodules as Γ ranges over I_1. One can check that the family $\mathbf{\Delta}_x^{\mathcal{A}^{\mathrm{op}}}$ (resp., $\mathbf{\nabla}_x^{\mathcal{A}^{\mathrm{op}}}$) identifies with a full family of $\mathbf{\Delta}$-modules (resp., $\mathbf{\nabla}$-modules) with highest weight x in B^*. Moreover, it follows by the symmetry between $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ that

$$\Delta_x \cong \oplus_{y,i,j} \hom_{R_x^{\mathrm{op}}}(\nabla_x^{\mathcal{A}^{\mathrm{op}}} e_{y,i}, R_x e_j)$$

$$\operatorname{end}_{\mathcal{A}^{\mathrm{op}}}(\mathbf{\nabla}_x^{\mathcal{A}^{\mathrm{op}}}) \cong R_x^{\mathrm{op}}.$$

11. Relationship with quasi-hereditary and cellular algebras

In this section, we indicate some conditions under which algebras $\mathcal{A}$ arising from our constructions are integral quasi-hereditary algebras or cellular algebras. The results are included only for their possible interest to readers familiar with those classes of rings, so we just consider the case of ungraded unital rings and finite weight posets.

11.1. Suppose for the remainder of this section that we are given a fixed stratified exact category $(C, \{C_x\})_{x\in\Omega}$ where Ω is a finite poset and $C_x = \operatorname{Add} N_x$ for some N_x. We assume also that C is a k-category over a given commutative ring k i.e. for each M, N in C, $\operatorname{Hom}(M, N)$ has a given k-module structure compatible with its natural abelian group structure, such that composition of maps is k-bilinear. We suppose C contains a projective object P such that for any M in C, there is an admissible epimorphism $Q \to M$ with Q in $\operatorname{Add} P$.

11.2. See [11] for the definition of split integral quasihereditary k-algebras and split heredity ideals.

Proposition. *Suppose for all $x \in \Omega$ that the natural map $k \to \operatorname{End}(N_x)$ is an isomorphism and that $\operatorname{Hom}(P, N_x)$ is a f.g. projective k-module. Then if k is Noetherian, $B := \operatorname{End}(P)^{\mathrm{op}}$ is a split integral quasihereditary k-algebra.*

Proof. In view of the recursive definition of integral quasi-hereditary k-algebras, it is enough to check that for a maximal element x of Ω, $B(x) = \operatorname{Hom}(P, P(x))$ is f.g. projective as k-module and is a split heredity ideal of B. But

$$B(x) = \operatorname{Hom}(P, P(x)) \cong \operatorname{Hom}(P, N_x) \otimes_k \operatorname{Hom}(N_x, P(x))$$

with $\operatorname{Hom}(P, N_x)$ (resp., $\operatorname{Hom}(N_x, P(x)) = \operatorname{Hom}(N_x, P)$) f.g. projective as left (resp., right) B-module and as k-module. So $B(x)$ is certainly f.g. projective as left or right A-module, and as k-module. We have $B(x)^2 = B(x)$ since $\operatorname{Add} P(x) = \operatorname{Add} N_x$ is in $\operatorname{Add} P$. Finally, $\operatorname{End}(B(x)) \cong \operatorname{End}(P(x))$ is isomorphic to the endomorphism ring over k of the f.g. projective faithful module $\operatorname{Hom}(N_x, P(x))$ over $k = \operatorname{End}(N_x)$. □

11.3. For the definition of a cellular basis and cellular algebra, see [28].

We make the following assumptions in addition to our standing assumptions on $\mathcal{C}$. First, assume that $P = \oplus_{x\in\Omega} P_x$ where $P_x = P_x(\geq x)$ and $P_x(x) = N_x$. Second, assume that there is a contravariant self-equivalence δ of the additive k-category $\operatorname{Add} P$ satisfying $\delta(P_x) = P_x$ for all x and $\delta^2 = \mathrm{Id}$ (for simplicity we take strict equalities rather than isomorphisms satisfying suitable extra conditions).

Proposition. *If each module $\operatorname{Hom}(P, N_x)$ is f.g. free as k-module and $\operatorname{End}(N_x) = k$ for all x, then $B := \operatorname{End}(P)^{\mathrm{op}}$ is a cellular k-algebra.*

Proof. We indicate how a cellular basis may be constructed, leaving the proof to the reader. Note first that the map $f \mapsto \delta(f)$ defines a k-algebra anti-involution ω of B, and $\omega(e_x) = e_x$ where $e_x \colon P \to P_x \to P$ is the natural idempotent. If Γ is a coideal of Ω and $f \in B(\Gamma) = \operatorname{Hom}(P, P(\Gamma)) \subseteq B$, then $\omega(f) \in B(\Gamma)$ also; for f factors through an object of $\operatorname{Add}\{P_x\}_{x\in\Gamma}$, and hence so does $\delta(f)$.

Choose for each $x \geq y$ in Ω a k-basis $\{b'_{x,y,i}\}_i$ of $\operatorname{Hom}(N_x, P_y(\leq x))$ and lift these basis elements to elements $b_{x,y,i} \colon P_x \to P_y$. Assume without loss of generality that the (unique for each x) element $b_{x,x,i}$ is e_x. For y, z in Ω and any coideal Γ of Ω, $e_y B(\Gamma) e_z$ consists of homomorphisms $P_y \to P_z$ which factor through an object of $\operatorname{Add}\{P_x\}_{x\in\Gamma}$. One can verify by induction on the cardinality of Γ that $e_y B(\Gamma) e_z$ has the elements $\omega(b_{x,y,i}) b_{x,z,j}$ with $x \in \Gamma$ as k-basis.

For $x \in \Omega$, let $T(x) = \{b_{x,y,i}\}_{x\geq y,i}$. The elements $c^x_{s,t} := \omega(s)t$ for $y \in \Omega$ and $s, t \in T(x)$ are readily seen to form a cellular basis for B over k. □

Remarks. For $x \leq y$ the images in $\mathrm{Hom}(P_x, N_y)$ of the elements $\{\omega(b_{y,x,i})\}_i$ in $\mathrm{Hom}(P_x, P_y)$ form a k-basis of $\mathrm{Hom}(P_x, N_y)$. Thus, $\mathrm{Hom}(P_x, N_y)$ is a f.g. free k-module, so B is integral quasi-hereditary as well if k is Noetherian.

12. Tilting modules

We return to the study of a general stratified exact category $\mathcal{C}$ with weight poset Ω such that $\hat{\mathcal{C}}$ has enough projective objects. Until the last subsection of this section, we assume that Ω is finite, so that $\hat{\mathcal{E}} = \mathcal{E} = \mathcal{A}$-mod.

12.1. Recall the stratified exact categories $\mathcal{C}^\Delta$ and $\mathcal{C}^\nabla$ defined as subcategories of $\mathcal{A}$-mod in 9.6. We define $\mathcal{F} = \mathcal{F}[\mathcal{C}]$ to be the full additive subcategory of $\mathcal{A}$-modules which are objects of both $\mathcal{C}^\Delta$ and $\mathcal{C}^\nabla$.

Lemma. *Objects of $\mathcal{F}$ are injective as objects of the exact category $\mathcal{C}^\Delta$ and projective as objects of $\mathcal{C}^\nabla$.*

Proof. Observe that by 1.26,

12.1.1. $\mathrm{ext}^i_{\mathcal{A}}(M, N) = 0$ if $i > 0$, M is in $\mathcal{C}^\Delta$ and N is in $\mathcal{C}^\nabla$ (for instance, if M and N are both in $\mathcal{F}$).

We now show that an object M of $\mathcal{F}$ is injective as an object of $\mathcal{C}^\Delta$. Choose a coideal Γ of Ω with a minimal element x. By (the dual of) Proposition 7.6, it is sufficient to show that the short exact sequence

$$0 \to M(\Gamma \setminus \{x\}) \to M(\Gamma) \to M(x) \to 0$$

induces an epimorphism f in the corresponding long exact $\mathrm{ext}(\mathbf{\Delta}_x, ?)$-sequence

$$\mathrm{hom}(\mathbf{\Delta}_x, M(x)) \xrightarrow{f} \mathrm{ext}^1(\mathbf{\Delta}_x, M(\Gamma \setminus \{x\}) \to \mathrm{ext}^1(\mathbf{\Delta}_x, M(\Gamma).$$

We prove that in fact $\mathrm{ext}^1(\mathbf{\Delta}_x, M(\Gamma)) = 0$. As part of the long exact $\mathrm{ext}(\mathbf{\Delta}_x, ?)$-sequence corresponding to

$$0 \to M(\Gamma) \to M \to M(\Omega \setminus \Gamma) \to 0,$$

we find the exact sequence

$$\mathrm{hom}(\mathbf{\Delta}_x, M(\Omega \setminus \Gamma)) \to \mathrm{ext}^1(\mathbf{\Delta}_x, M(\Gamma)) \to \mathrm{ext}^1(\mathbf{\Delta}_x, M).$$

Here, the first term is zero by 1.26 since $x \notin \Omega \setminus \Gamma$ and the last term is zero by 12.1.1 above, so the middle term is zero as required. An argument dual to the above one shows that an object of $\mathcal{F}$ is projective in $\mathcal{C}^\nabla$. □

12.2. For the remainder of this section, we assume in addition that the split stratified exact category $(\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})^{\mathrm{op}}$ has enough projective objects i.e. that $\mathcal{C}$ has enough injective objects. We fix a family $\mathbf{Q} = \{Q_{x,i}\}_{x\in\Omega}$ of injective objects of $\mathcal{C}$ with $Q_{x,i} = Q_{x,i}(\leq x)$ and $Q_{x,i}(x) = N_{x,i}$ (it is not essential to choose the family $\mathbf{Q}$ of injectives as done here, so it is indexed in the same way as $\mathbf{P}$, but we do so for notational simplicity). We call a family $\mathbf{Q}$ arising as above a standard family of injective objects of $\mathcal{C}$.

Set $\mathcal{B} = \mathrm{end}(\mathbf{Q})^{\mathrm{op}}$. By abuse of notation, we write $e_{x,i} \in \mathcal{B}$ for the projection on $Q_{x,i}$ (recall also $e_{x,i} \in \mathcal{A}$ denotes the projection on $P_{x,i}$). Define the (perfectly exact by 1.19) contravariant functor $\psi := \mathrm{hom}(?, \mathbf{Q})\colon \mathcal{C} \to \mathcal{B}^{\mathrm{op}}$-mod. Define the $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{T} = \mathrm{hom}(\mathbf{P}, \mathbf{Q})$. We write $\mathbf{T}$ for the family of $\mathcal{A}$-modules $\{\mathcal{T}e_{x,i}\}_{x,i}$, and $\mathbf{T}'$ for the family of right $\mathcal{B}$-modules $\{e_{x,i}\mathcal{T}\}_{x,i}$.

The bimodule structure on $\mathcal{T}$ gives ring homomorphisms

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\cong} \mathrm{end}_{\mathcal{A}}(\mathbf{T}), \qquad \mathcal{A} \xrightarrow{\cong} \mathrm{end}_{\mathcal{B}^{\mathrm{op}}}(\mathbf{T}') \tag{12.2.1}$$

which are isomorphisms by Theorem 1.19.

12.3. **Proof of 1.28.** Theorem 1.28(a) follows immediately by the general fact C.14, and Theorem 1.28(b) is restated as part (a) of the following Proposition.

Proposition. *Assume that $\mathcal{B}^{\mathrm{op}}$ is a stratified ring.*

(a) $\mathcal{F} = \mathrm{add}\,\mathbf{T}$

(b) *Given M in $\mathcal{C}^{\Delta}$ (resp., in $\mathcal{C}^{\nabla}$), there is an admissible monomorphism $M \to T$ in $\mathcal{C}^{\Delta}$ (resp., an admissible epimorphism $T \to M$ in $\mathcal{C}^{\nabla}$) with T in $\mathrm{add}\,\mathbf{T}$. In particular, $\mathcal{C}^{\Delta}$ has enough injectives (resp., $\mathcal{C}^{\nabla}$ has enough projectives).*

(c) *The functor $\mathrm{hom}_{\mathcal{A}}(\mathbf{T}, ?)$ induces an equivalence $\mathcal{C}^{\nabla^{\mathcal{A}}} \to \mathcal{C}^{\Delta^{\mathcal{B}}}$ of stratified exact categories, with inverse $\mathcal{T}\otimes_{\mathcal{B}}?$.*

Proof. Objects of $\mathrm{add}\,\mathbf{T}$ are injective in $\mathcal{C}^{\Delta}$ since objects of $\mathbf{Q}$ are injective in $\mathcal{C}$, using for instance Lemma 1.21 and Remark 9.6. The part of 1.28(b) concerning $\mathcal{C}^{\Delta}$ follows using Remark 9.6 again. Thus, $\mathrm{add}\,\mathbf{T}$ consists of the injective objects of $\mathcal{C}^{\Delta}$ and hence contains $\mathcal{F}$ by Lemma 12.1.

Now since $Q_{x,i}$ is injective in $\mathcal{C}$, $\mathcal{T}e_{x,i} = \mathrm{hom}(\mathbf{P}, Q_{x,i})$ has a finite filtration as $\mathcal{A}$-module with successive subquotients

$$\begin{aligned}\mathrm{hom}(\mathbf{P}(y), Q_{x,i}) &\cong \mathrm{hom}(\mathbf{P}(y), \mathbf{N}_y) \otimes_{\mathrm{end}(\mathbf{N}_y)^{\mathrm{op}}} \mathrm{hom}(\mathbf{N}_y, Q_{x,i}) \\ &= \nabla_y \otimes_{R_y} \mathrm{hom}(\mathbf{N}_y, Q_{x,i}).\end{aligned} \tag{12.3.1}$$

Our assumption that $\mathcal{B}^{\mathrm{op}}$ is stratified is equivalent to the statement that each $\mathrm{hom}(\mathbf{N}_y, Q_{x,i})$ is a f.g. projective R_y-module. It follows that $\mathrm{hom}(\mathbf{P}(y), Q_{x,i})$ is in $\mathrm{add}\,\boldsymbol{\nabla}_y$. So $\mathcal{T}e_{x,i}$ is in $\mathcal{C}^{\nabla}$ and thus is in $\mathcal{F}$, proving 12.3(a).

By 12.1, objects of $\mathrm{add}\,\mathbf{T}$ are projective in $\mathcal{C}^{\nabla}$. Using the above formula for $\mathrm{hom}(\mathbf{P}(y), Q_{x,i})$ and noting $Q_{x,i} = Q_{x,i}(\leq x)$ with $Q_{x,i}(x) = N_{x,i}$, one readily sees that $\mathbf{T}$ is a standard family of projective objects of the stratified exact category $\mathcal{C}^{\nabla}$ (corresponding to the standard families of objects $\boldsymbol{\nabla}_x$ for $x \in \Omega$). The part of (b) concerning $\mathcal{C}^{\nabla}$ follows.

Now we prove (c). For M in $\mathcal{C}^{\nabla^{\mathcal{A}}}$, we have by 1.26 that $\mathrm{R}\left(\mathrm{hom}_{\mathcal{A}}(\mathbf{T}, ?)\right)M \cong \mathrm{hom}_{\mathcal{A}}(\mathbf{T}, M)$, with the right hand side regarded as a complex concentrated in degree 0. Now using 1.26, 9.5 and C.9, we have

$$\begin{aligned}\mathrm{hom}_{\mathcal{A}}(\mathbf{T}, \nabla_{x,i}) &= \mathrm{hom}_{\mathcal{A}}(\mathbf{T}(x), \nabla_{x,i}) = \mathrm{hom}_{\mathcal{A}}(\Delta_x \otimes_{R_x} \mathrm{Hom}(\mathbf{N}_x, \mathbf{Q}(x)), \nabla_{x,i}) \\ &= \mathrm{hom}_{R_x}(\mathrm{hom}(\mathbf{N}_x, \mathbf{Q}(x)), R_x e_i) = \mathrm{hom}(\mathbf{Q}(x), N_{x,i}) = \Delta^{\mathcal{B}}_{x,i}.\end{aligned}$$

This implies that $\mathrm{hom}_{\mathcal{A}}(\mathbf{T}, ?)$ restricts to an exact functor $\mathcal{C}^{\nabla^{\mathcal{A}}} \to \mathcal{C}^{\Delta^{\mathcal{B}}}$. It also follows now from C.13 that $\mathcal{T} \otimes^{L}_{\mathcal{B}} \Delta^{\mathcal{B}}_{x,i} = \nabla_{x,i}$ i.e. $\mathrm{tor}^{\mathcal{B}}_i(T, \Delta^{\mathcal{B}}_x) = 0$ for $i > 0$ and $\mathcal{T} \otimes_{\mathcal{B}} \Delta^{\mathcal{B}}_{x,i} = \nabla_{x,i}$. Thus, $\mathcal{T}\otimes^{L}_{\mathcal{B}}?$ induces an exact functor $\mathcal{C}^{\Delta^{\mathcal{B}}} \to \mathcal{C}^{\nabla^{\mathcal{A}}}$. The desired equivalence of exact categories is the restriction of the derived category equivalence Theorem 1.28(a). □

Remarks. Replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$, one obtains by symmetry equivalences

$$\mathrm{D}^b(\mathcal{B}^{\mathrm{op}}\text{-mod}) \to \mathrm{D}^b(\mathcal{A}^{\mathrm{op}}\text{-mod}), \qquad \mathcal{C}^{\nabla^{\mathcal{B}^{\mathrm{op}}}} \to \mathcal{C}^{\Delta^{\mathcal{A}^{\mathrm{op}}}}$$

provided for the second that $\mathcal{A}$ is stratified.

12.4. Assume that $\mathcal{B}^{\text{op}}$ is stratified. The ring $\text{end}_{\mathcal{C}^{\text{op}}}(\mathbf{Q}^{\text{op}})^{\text{op}} \cong \mathcal{B}^{\text{op}}$ is associated to $\mathcal{C}^{\text{op}}$ in exactly the same way as $\mathcal{A}$ is associated to $\mathcal{C}$. Identifying left $\mathcal{B}^{\text{op}}$-modules with right $\mathcal{B}$-modules gives identifications of (choices of) the families of $\boldsymbol{\Delta}$-and $\boldsymbol{\nabla}$-objects as $\Delta_x^{\mathcal{B}^{\text{op}}} = \text{hom}(\mathbf{N}_x, \mathbf{Q})$ and $\nabla_x^{\mathcal{B}^{\text{op}}} = \text{hom}(\mathbf{N}_x, \mathbf{Q}(x))$. It follows from the above proof and 9.5 that

Corollary. *If $\mathcal{B}^{\text{op}}$ is stratified, then $\mathbf{T} = \text{Hom}(\mathbf{P}, \mathbf{Q})$ has two natural filtrations as $(\mathcal{A}, \mathcal{B})$-bimodule: one filtration has successive subquotients*

$$\text{Hom}(\mathbf{P}, \mathbf{Q}(x)) \cong \Delta_x^{\mathcal{A}} \otimes_{R_x} \nabla_x^{\mathcal{B}^{\text{op}}},$$

while the other has successive subquotients

$$\text{Hom}(\mathbf{P}(x), \mathbf{Q}) \cong \nabla_x^{\mathcal{A}} \otimes_{R_x} \Delta_x^{\mathcal{B}^{\text{op}}}.$$

12.5. Suppose that Ω is infinite and that $\hat{\mathcal{C}}$ and $\widehat{\mathcal{C}^{\text{op}}}$ both have enough projectives. Choose standard families family $\mathbf{P} = \{P_{x,i}\}_{x,i}$ of projectives in $\hat{\mathcal{C}}$ and $\mathbf{Q} = \{Q_{x,i}^{\text{op}}\}_{x,i}$ of projectives in $\widehat{(\mathcal{C}^{\text{op}})}$. Here, $Q_{x,i}$ is a direct system $\{Q_{x,i,\Lambda}\}_{\Lambda \in I'_1}$ of objects of $\mathcal{C}$, where I'_1 is the family of coideals of Ω which are contained in a f.g. coideal, ordered by inclusion and $Q_{x,i,\Lambda} = Q_{x,i,\Lambda}(\Lambda \cap (\leq x))$ is injective in $\mathcal{C}_\Lambda$. Define $\mathcal{A} = \text{end}(\mathbf{P})^{\text{op}}$ and $\mathcal{B} = \text{end}(\mathbf{Q})^{\text{op}}$. Now for any x, i, y, j, one has the direct system $\text{hom}(P_{x,i,\Gamma}, Q_{y,j,\Lambda})_{\Gamma \in I_1, \Lambda \in I'_1}$. This direct system stabilizes once $\Lambda \cap \Gamma \ni x, y$; in particular, the direct limit $e_{x,i}\mathcal{T}e_{y,j} := \varinjlim_{\Gamma,\Lambda} \text{hom}(P_{x,i,\Gamma}, Q_{y,j,\Lambda})$ exists. There is a natural graded $(\mathcal{A}, \mathcal{B})$-bimodule structure on $\mathcal{T} := \oplus_{x,i,y,j} e_{x,i}\mathcal{T}e_{y,j}$. It is natural to expect that many of the preceding results for finite weight posets may have extensions to infinite Ω, using $\mathbf{T} := \{\mathcal{T}e_{x,i}\}_{x,i}$ as a substitute for the family of tilting modules and suitable abelian subcategories (perhaps $\mathcal{E}$ or $\hat{\mathcal{E}}$) in place of $\mathcal{A}$-mod, etc.

13. k-STRUCTURE AND BASE CHANGE

In this section, we suppose our given stratified category $(\mathcal{C}, \{\mathcal{C}_x\}_{x \in \Omega})$ over G is a k-category, where k is a fixed Z-graded unital ring for some subgroup Z of the center of G (see 1.31). Note that $\hat{\mathcal{C}}$, $\hat{\mathcal{E}}$, $\mathcal{E}$ etc are naturally k-categories and $\mathcal{A}$, $\mathcal{A}_\Gamma$, R_x etc are graded k-algebras. Let k' be a Z-graded commutative unital k-algebra. For a G-graded, J-diagonalizable k-algebra A and a module M in A-mod, we let A' and M' denote the G-graded, J-diagonalizable k'-algebra $A' := k' \otimes_k A$ and module $M' := k' \otimes M$ in A'-mod obtained by base change $k' \otimes_k ?$, unless otherwise indicated.

13.1. **Proof of Lemma 1.31.** We use the well-known fact that any left Artinian graded unital ring is also left Noetherian as graded ring.

First, it follows from the assumptions and C.2.1 that if M and N are in $\mathcal{C}$, then $\text{hom}_{\mathcal{C}}(M, N)$ is f.g. as k-module. Now choose $\Gamma \in I$ so M, N are in $\mathcal{C}_\Gamma$, and a projective resolution $P^\bullet \to M \to 0$ in $\mathcal{C}_\Gamma$. Then using 1.6, $\text{ext}^i_{\mathcal{C}}(M, N)$ is a subquotient of $\text{hom}(P^i, N)$ and hence it is f.g. as k-module. This proves 1.31(a).

Clearly, any object M of $\mathcal{E}$ with all weightspaces $e_{x,i}M$ f.g. over k is in $\mathcal{E}_{\text{fg}}$. Now $e_{y,j}\nabla_{x,i} = \text{hom}(P_{y,j}(x), N_{x,i})$ is k-f.g. If M is in $\mathcal{C}$, then M is in $\mathcal{C}_\Gamma$ for some $\Gamma \in I_1$ and $e_{x,i}\varphi(M) \cong \text{Hom}(P_{x,i,\Gamma}, M)$ is f.g. as k-module. This shows in particular that all $\mathcal{A}_\Gamma e_{x,i}$, $\Delta_{x,i}$ and tilting modules (for Ω finite) have f.g. weightspaces over k, proving (b).

If M is in $\mathcal{E}_{\mathrm{fg}}$, then M is a $\mathcal{A}_\Gamma$-module for some $\Gamma \in I_1$. The weightspace $e_{x,i}M$ is a f.g. $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module by definition, and we have just seen that $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$ is f.g. as k-module, so $e_{x,i}M$ is f.g. as k-module also. This proves 1.31(c). Clearly, $\mathcal{E}_{\mathrm{fin}}$ is a full subcategory of $\mathcal{E}_{\mathrm{fg}}$. If k is Artinian and M is in $\mathcal{E}_{\mathrm{fg}}$, then each weight space $e_{x,i}M$ is Noetherian and Artinian as k-module, hence also as $e_{x,i}\mathcal{A}e_{x,i}$-module, showing M is in $\mathcal{E}_{\mathrm{fin}}$. This proves 1.31(d).

If Ω and each I_x is finite, then $\mathcal{A} = \oplus_{x,i,y,j} e_{x,i}\mathcal{A}_\Omega e_{x,j}$ and $M = \oplus_{x,i} e_{x,i}M$ are f.g. k-modules, if M is in $\mathcal{E}_{\mathrm{fg}}$. This proves 1.31(d), and the final assertion is well known (and easily proved by an argument like part of that for 1.31(a)).

13.2. We say that $\mathbf{\Delta}$-modules (for $\mathcal{C}$) are k-projective (resp., k-flat) if each $\Delta_{x,i}$ is a graded projective k-module (resp., flat k-module).

Proposition. *Suppose that $\mathbf{\Delta}$-modules are k-projective. Then for all M in $\mathcal{C}$, there is an isomorphism of k-modules $\varphi(M) \cong \oplus_y \left(\Delta_y \otimes_{R_y} \hom(\mathbf{N}_y, M(y))\right)$ and in particular, $\varphi(M)$ is a projective k-module. Moreover, short exact sequences in $\mathcal{C}^\Delta$ are necessarily k-split.*

If each $e_{x,j}\Delta_{y,i}$ is f.g. k-projective and each set Ω and I_x for $x \in \Omega$ is finite, then $\mathcal{A}$ and each $\varphi(M)$ for M in $\mathcal{C}$ is a f.g. graded projective k-module.

Proof. Let M be in $\mathcal{C}$. By exactness of φ, $\varphi(M)$ has a finite filtration with successive subquotients $\varphi(M(y))$ in $\operatorname{add}\varphi(\mathbf{N}_y) = \operatorname{add}\mathbf{\Delta}_y$ for $y \in \Omega$. So $\varphi(M)$ is isomorphic in k-mod to $\oplus_y \varphi(M(y))$, hence it is in $\operatorname{add} k$. Thus the first assertion holds by 9.5, and the other statements follow readily. □

13.3. **Flat base change.** Assume in this subsection that we are given a commutative k-algebra k' which is flat as k-module (i.e. $k' \otimes_k ?$ is an exact functor k-mod $\to k'$-mod).

Let A be a G-graded J-diagonalizable k-algebra for some J. Suppose given for each $x \in \Omega$ a full, additive subcategory $\mathcal{D}_x$ of the abelian category $\mathcal{D} = A$-mod, regarded as a split exact category. Assume the conditions 1.4(i) and 1.4(ii) hold, so one may construct the stratified exact category $\mathcal{C}$ as in 1.7 from $\mathcal{D}$ and the $\mathcal{D}_x$.

Note that if if A is left Noetherian, M is a f.g. A-module and N is any A-module, the natural maps

$$k' \otimes_k \operatorname{ext}^n_A(M, N) \cong \operatorname{ext}^n_{A'}(M', N') \tag{13.3.1}$$

are isomorphisms for all $i \geq 0$.

Throughout this subsection, we assume that the above maps are isomorphisms for all M, N in $\mathcal{C}$ and $i \leq 1$. Define the full additive subcategories $\mathcal{D}'_x = \operatorname{add}\{N'_{x,i}\}_i$ of $\mathcal{D}' := A'$ mod , and regard them as split exact categories. These satisfy the assumptions 1.4(i) and 1.4(ii), so one may form from $\mathcal{D}'$ and its subcategories $\mathcal{D}'_x$ the split stratified exact category $\mathcal{C}'$ as in 1.7. The functor $k' \otimes_k ?\colon M \mapsto M'$ restricts to a (trivially bistable) exact functor $F\colon \mathcal{C} \to \mathcal{C}'$, satisfying $\hom_{\mathcal{C}'}(M', N') \cong \hom_{\mathcal{C}}(M, N)'$ for M, N in $\mathcal{C}$. In turn, F induces the exact functor $\hat{F}\colon \hat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ of the associated categories of pro-objects. Note that by 1.18, $\hat{F}$ maps projectives in $\hat{\mathcal{C}}$ to projectives in $\widehat{\mathcal{C}'}$, since for Q in $\hat{\mathcal{C}}$ and $x \in \Omega$, $g_{\hat{F}Q,x} = \mathrm{Id}_{k'} \otimes_k g_{\hat{Q},x}?$ is an epimorphism whenever $g_{\hat{Q},x}$ is. The standard family $\mathbf{P} = \{P_{x,i}\}_{x,i}$ of projectives in $\hat{\mathcal{C}}$ therefore maps to a standard family (with respect to families of standard objects $\{N'_{x,i}\}_i$) $\mathbf{P}' = \{P'_{x,i}\}_{x,i}$ of projectives in $\widehat{\mathcal{C}'}$, where $P'_{x,i} = \hat{F}P_{x,i}$. We have

$\operatorname{end}(\mathbf{P}')^{\mathrm{op}} = \varprojlim_{\Gamma \in I} \mathcal{A}(\Gamma)'$ where $\mathcal{A}(\Gamma)' \cong \operatorname{end}(\mathbf{P}'_\Gamma)^{\mathrm{op}}$; by abuse of notation we write $\mathcal{A}' := \operatorname{end}(\mathbf{P}')^{\mathrm{op}}$. Then $(\mathcal{A}', \{\mathcal{A}(\Gamma)'\}_{\Gamma \in I_1})$ is a left stratified ring, which one readily checks is stratified if $\mathcal{A}$ is stratified.

Define the full subcategory $\mathcal{E}'$ of objects of $\mathcal{A}'$-mod which are $\mathcal{A}(\Gamma)'$-modules for some $\Gamma \in I_1$. It is easy to check from 8.9 that base change change $k' \otimes_k ?$ induces the exact functors $F_* : \hat{\mathcal{E}} \to \widehat{\mathcal{E}'}$ and $F_\dagger : \mathcal{E} \to \mathcal{E}'$.

Observe that the $\boldsymbol{\Delta}$-(resp., $\boldsymbol{\nabla}$-) modules in $\mathcal{E}'$ can be taken to be the modules $\hom_{A'}(\mathbf{P}'_{\leq x}, N'_{x,i}) = \Delta'_{x,i}$ and $\hom_{A'}(\mathbf{P}_{\leq x}(x)', N'_{x,i}) = \nabla'_{x,i}$.

Remarks. We assume that Ω is finite and briefly discuss flat base change in relation to tilting modules. Assume that $\mathbf{Q} = \{Q_{x,i}\}_{x,i}$ is a standard family of injective objects in $\mathcal{C}$ as in Section 12, and define $\mathcal{B}$, $\mathbf{T}$, $\mathcal{T}$ as there. Then $\mathbf{Q}' = \{Q'_{x,i}\}_{x,i}$ is a standard family of injective objects in $\mathcal{C}'$. One has $\operatorname{end}(\mathbf{Q}')^{\mathrm{op}} \cong \mathcal{B}'$ where $\mathcal{B} = \operatorname{end}(\mathbf{Q})^{\mathrm{op}}$. Also, $\hom_{A'}(\mathbf{P}', \mathbf{Q}') \cong \mathcal{T}'$, and $\mathbf{T}' := \{\mathcal{T}' e_{x,i}\}_{x,i}$ is a full family of tilting modules for $\mathcal{A}'$. We have $\operatorname{end}_{\mathcal{A}'}(\mathbf{T}') \cong \operatorname{end}_{A'}(Q') \cong (\mathcal{B}')^{\mathrm{op}}$.

13.4. **Proof of Theorem 1.35.** Assume that each Δ_x is k-flat. We define $\mathcal{A}'_\Gamma$, $\Delta'_{x,i}$, $\boldsymbol{\Delta}'_x$ etc as in 1.35. The assumptions and long exact tor^k-sequences imply that any object M of $\mathcal{C}$ is k-flat. Moreover, if M in $\mathcal{C}$ has a finite filtration $M = M_0 \supseteq \ldots \supseteq M_n = 0$ with successive subquotients M^{i-1}/M^i in $\operatorname{add} \boldsymbol{\Delta}_{x_i}$, then $M' = k' \otimes_k M$ has the finite filtration $M' = M'_0 \supseteq \ldots \supseteq M'_n = 0$ with successive subquotients $M'_{i-1}/M'_i = (M_{i-1}/M_i)'$ in $\operatorname{add} \boldsymbol{\Delta}'_{x_i}$. This implies that for $\Gamma \subseteq \Lambda$ in I, the kernel V of the natural surjection $\mathcal{A}(\Lambda) \to \mathcal{A}(\Gamma)$ is k-flat, since $V = \oplus_{x,i} V e_{x,i}$ with $V e_{x,i}$ in $\mathcal{C}$. In particular, taking $\Gamma = \emptyset$, $\mathcal{A}(\Lambda)$ is k-flat for all $\Lambda \in I_1$. Now it follows readily using the definition 1.22 that $\mathcal{A}'$ is a left stratified ring.

Now we may define $\mathcal{C}' = k' \otimes_k \mathcal{C}$ and make the standard identifications described in 1.32. Thus, $\mathcal{E}' = k' \otimes_k \mathcal{E} \cong (\mathcal{C}')^\dagger$ identifies with the full abelian subcategory of $\mathcal{A}'$-mod consisting of modules which are $\mathcal{A}'_\Gamma$-modules for some $\Gamma \in I_1$, $\mathcal{C}'$ is the smallest extension closed subcategory of $\mathcal{E}'$ containing $\operatorname{add}\{\Delta'_{x,i}\}_{x,i}$, and Δ'_x is a set of standard objects in $\mathcal{C}'$ (and also a set $\boldsymbol{\Delta}$-objects in $\mathcal{E}'$) corresponding to $x \in \Omega$. The assertion 1.35(b) follows easily using k-flatness of objects of $\mathcal{C}$. Note $\hat{F} : \mathcal{C} \to \widehat{\mathcal{C}'}$ is exact and preserves convergent direct sums. For $\Gamma \in I_1$, $F(\mathcal{A}(\Gamma) e_{x,i}) = \mathcal{A}(\Gamma)' e_{x,i}$ is projective in $\mathcal{A}(\Gamma)'$-mod and hence projective in $\mathcal{C}'_\Gamma$, so $\hat{F}$ clearly maps $P_{x,i}$ in $\mathbf{P}$ to a projective object of $\widehat{\mathcal{C}'}$. Since $\hat{F}$ clearly preserves convergent direct sums, 1.35(c) follows. The formulae in 8.9 imply 1.35(d).

If $\Gamma \in I_1$, P is projective in $\mathcal{C}_\Gamma$ and M is in $\mathcal{A}_\Gamma$-mod, then

$$\hom_{\mathcal{E}'}(P', M') = \hom_{\mathcal{E}}(P, M)'. \tag{13.4.1}$$

Indeed, it is enough to check this for $P = \mathcal{A}_\Gamma e_{x,i}$ when it is trivial. It follows in particular that $\operatorname{end}(\Delta'_x)^{\mathrm{op}} = R'_x$, from which it follows directly that $\mathcal{A}'$ is stratified if $\mathcal{A}$ is stratified. Also,

$$\nabla'_{x,i} \cong \oplus_{y,j} \hom_{\mathcal{E}}(\mathcal{A}(x) e_{y,j}, \Delta_{x,i})' = \oplus_{y,j} \hom_{\mathcal{A}'}(\mathcal{A}'(x) e_{y,j}, \Delta'_{x,i})$$

(since $\mathcal{A}(x) e_{y,j}$ is in $\operatorname{add} \boldsymbol{\Delta}_x$ and hence is projective in $\mathcal{C}_{\leq x}$). This proves the remaining parts of 1.35(a)–(e).

Before proving 1.35(f), we record the following useful fact concerning the exact sequences $(*_{P,x})$ of 7.9:

13.4.2. For any $y \in \Omega$ and any projective object Q of $\hat{\mathcal{C}}$, if we denote $Q' := \hat{F}Q$ then there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} \hom(Q(\leq y), \mathbf{\Delta}_y)' & \xrightarrow{f'_{Q,y}} & \hom(Q(y), \mathbf{\Delta}_y)' & \xrightarrow{g'_{Q,y}} & \operatorname{ext}^1(Q(< y), \mathbf{\Delta}_y)' & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ \hom(Q'(\leq y), \mathbf{\Delta}'_y) & \xrightarrow{f_{Q',y}} & \hom(Q'(y), \mathbf{\Delta}'_y) & \xrightarrow{g_{Q',y}} & \operatorname{ext}^1(Q'(< y), \mathbf{\Delta}'_y) & \longrightarrow & 0. \end{array}$$

In this diagram, ext^i is taken in $\mathcal{E}$ in the top row, and in $\mathcal{E}'$ in the bottom row. The top exact row arises by applying $k' \otimes_k ?$ to part of $(*_{Q,y})$. The bottom (exact) row is part of $(*_{Q',y})$; $g_{Q',y}$ is surjective since Q' is projective in $\hat{\mathcal{C}}$. To prove that the left and middle vertical maps are isomorphisms, one may assume by 9.2 that y is a maximum element of Ω; then $Q(\leq y)$ and $Q(y)$ are both projective in $\mathcal{C}$ and one has isomorphisms as required by (13.4.1). By the five lemma, the right vertical map is an isomorphism too.

Now make the assumptions of 1.35(f), and define $\mathcal{A}$, $\mathcal{B}$, $\mathbf{T}$, $\mathcal{T}$, $\mathcal{C}^{\Delta}$, $\mathcal{C}^{\nabla}$ etc as in Section 12. Let $\mathcal{C}^{\Delta'}$ (resp., $\mathcal{C}^{\nabla'}$) denote the smallest extension-closed additive subcategory of $\mathcal{E}'$ containing all $\operatorname{add} \mathbf{\Delta}'_x$ (resp., $\operatorname{add} \{\mathbf{\nabla}'_x\}$) for $x \in \Omega$, regarded as a perfectly exact subcategory of $\mathcal{E}'$. Let $\mathcal{F}'$ be the full additive subcategory of $\mathcal{E}'$ consisting of objects in both $\mathcal{C}^{\Delta'}$ and $\mathcal{C}^{\nabla'}$.

Our assumptions imply that objects of $\mathcal{C}^{\Delta}$ or $\mathcal{C}^{\nabla}$ are k-flat and so base change $F_{\dagger} = k' \otimes_k ?$ induces exact functors $\mathcal{C}^{\Delta} \to \mathcal{C}^{\Delta'}$ and $\mathcal{C}^{\nabla} \to \mathcal{C}^{\nabla'}$. Hence $F_{\dagger}$ induces an additive functor $\mathcal{F} \to \mathcal{F}'$. Now from 12, $\mathcal{F} = \operatorname{add} \mathbf{Q}$ and $\mathcal{F}'$ consists of injective objects of $\mathcal{C}' = \mathcal{C}^{\mathbf{\Delta}'}$; it follows that $\mathbf{Q}' = \{Q'_{x,i}\}$ is a family of injectives in $\mathcal{C}'$. Clearly, $\mathbf{Q}'$ is a standard family of injectives in $\mathcal{C}'$ and it follows $\mathcal{F}' = \operatorname{add} \mathbf{Q}'$. We now make the standard identification of the family of tilting modules $\mathbf{T}'$ corresponding to $\mathbf{Q}'$ as $\mathbf{T}' = \mathbf{Q}'$.

We assert that

$$\text{(13.4.3)} \qquad \hom_{\mathcal{E}'}(\mathbf{\Delta}'_x, N') \cong \hom_{\mathcal{E}}(\mathbf{\Delta}_x, N)'$$

for any N in $\mathcal{C}^{\nabla}$. In fact, using the 5-lemma and 1.26 for $\mathcal{C}$ and $\mathcal{C}'$, one sees the class of modules N for which the natural map from right to left is an isomorphism contains $\nabla_{x,i}$ and is closed under extensions. We claim also that

$$\text{(13.4.4)} \qquad \hom_{\mathcal{E}'}(M', \mathbf{T}') \cong \hom_{\mathcal{E}}(M, \mathbf{T})'$$

for all M in $\mathcal{C}$. For the class of modules M in $\mathcal{C}$ for which the natural map from right to left is an isomorphism contains $\Delta_{x,i}$ by (13.4.3) and is closed under extension; for since $\mathbf{T}'$ (resp., $\mathbf{T}$ consists of injective objects of $\mathcal{C}'$ (resp., $\mathcal{C}$) applying $\hom_{\mathcal{E}'}(?, \mathbf{T}')$ (resp., $\hom_{\mathcal{E}}(?, \mathbf{T})$) to a short exact sequence in $\mathcal{C}'$ (resp., $\mathcal{C}$) gives a short exact sequence of projective k'-modules (resp., projective k-modules) which is necessarily split, and one can finish with the five lemma. As a consequence of (13.4.4), we obtain $\operatorname{end}(\mathbf{T}')^{\mathrm{op}} \cong \operatorname{end}(\mathbf{T})'$, completing the proof of 1.35(f).

13.5. **Adjoint functors.** Let $\mathcal{C}^i$ (i=1,2) be split stratified exact categories over G. We extend all standard notation for $\mathcal{C}$ to $\mathcal{C}^i$, denoting by X^i the standard object associated to $\mathcal{C}^i$ in the same way X is associated to $\mathcal{C}$ (so associated to $\mathcal{C}^i$, we have a standard family of projectives $\mathbf{P}^i$ in $\hat{\mathcal{C}}^i$, the endomorphism ring $\mathcal{A}^i := \operatorname{end}(\mathbf{P}^i)^{\mathrm{op}}$, abelian categories $\hat{\mathcal{E}}^i \cong \mathcal{C}^{i*}$ and $\mathcal{E}^i \cong \mathcal{C}^{i\dagger}$ etc). We extend this notation, defining X^i

for such X for all integers i by setting $X^j = X^i$ whenever i and j have the same parity (so $\cdots = \mathcal{C}^{-2} = \mathcal{C}^0 = \mathcal{C}^2 = \cdots$ and $\cdots = \mathcal{C}^{-1} = \mathcal{C}^1 = \mathcal{C}^3 = \cdots$ etc).

Now suppose $F^i \colon \mathcal{C}^i \to \mathcal{C}^{i+1}$ are exact functors with F^i left adjoint to F^{i+1} for all integers i and each F^i stable forwards for f.g. ideals (we do not assume $F^i \cong F^{i+2}$). Recall the restriction $F^i_\dagger$ of F^i_* is defined under these conditions; it is left adjoint to $F^{i+1}_\dagger$. For use in the proof of 1.36, we shall give a module-theoretic description of an adjunction between $F^i_\dagger$ and $F^{i+1}_\dagger$, similar to the results in 8.9 (cf. [23, 1.9] for the much less technical case of finite weight posets).

Let $\Lambda \in I^i$ and $\Gamma \in I^{i+1}$. We define the graded $(\mathcal{A}^{i+1}, \mathcal{A}^i)$-bimodule (actually, even a $(\mathcal{A}^{i+1}_\Gamma, \mathcal{A}^i_\Lambda)$-bimodule)

$$ {}_\Gamma M^i_\Lambda := \hom_{\mathcal{C}^{i+1}}\Big(\mathbf{P}^{i+1}_\Gamma, \sigma_\Gamma F^i(\mathbf{P}^i_\Lambda)\Big). \tag{13.5.1} $$

There is a natural inverse system $\{{}_\Gamma M^i_\Lambda\}_{\Lambda \in I^i, \Gamma \in I^{i+1}}$, in which the canonical maps are all epimorphisms. If $\Gamma \supseteq F^i(\Lambda)$, then one has isomorphisms

$$ \hom_{\mathcal{C}^i_\Lambda}(\sigma_\Lambda F^{i-1} M, N) \cong \hom_{\mathcal{C}^i}(F^{i-1}M, N) \cong \hom_{\mathcal{C}^{i+1}_\Gamma}(M, F^i N) $$

naturally for M in $\mathcal{C}^{i+1}_\Gamma$ and N in $\mathcal{C}^i_\Lambda$. Hence

13.5.2. $\sigma_\Lambda F^{i-1} \colon \mathcal{C}^{i-1}_\Gamma \to \mathcal{C}^i_\Lambda$ is left adjoint to $F^i \colon \mathcal{C}^i_\Lambda \to \mathcal{C}^{i-1}_\Gamma$ if $\Gamma \supseteq F^i(\Lambda)$; in particular, since F^i is exact, $\sigma_\Lambda F^{i-1}$ takes projective objects of $\mathcal{C}^{i-1}_\Gamma$ to projectives in $\mathcal{C}^i_\Lambda$ if $\Gamma \supseteq F^i(\Lambda)$.

Thus, if $\Gamma \supseteq F^i(\Lambda)$ then

$$ {}_\Gamma M^i_\Lambda \cong \hom_{\mathcal{C}^{i+1}}\Big(\mathbf{P}^{i+1}_\Gamma, F^i(\mathbf{P}^i_\Lambda)\Big) \cong \hom_{\mathcal{C}^i}\Big(\sigma_\Lambda F^{i-1}(\mathbf{P}^{i+1}_\Gamma), \mathbf{P}^i_\Lambda\Big) \tag{13.5.3} $$

and so

13.5.4. each $e_{y,j}\,({}_\Gamma M^i_\Lambda)$ is a f.g. projective right $\mathcal{A}^i_\Lambda$-module provided that $\Gamma \supseteq F^i\Lambda$.

From 13.5.2, it follows also that

13.5.5. each $({}_\Gamma M^i_\Lambda)e_{x,l}$ is a f.g. projective $\mathcal{A}^{i+1}_\Gamma$-module if $\Lambda \supseteq F^{i+1}\Gamma$.

We claim now that if $\Lambda' \supseteq \Lambda$, then there are short exact sequences

$$ 0 \to \sum_{z \in \Sigma, j} ({}_\Gamma M^i_{\Lambda'})e_{z,j}\mathcal{A}^i e_{x,l} \xrightarrow{\sigma} {}_\Gamma M^i_{\Lambda'} e_{x,l} \xrightarrow{\rho} {}_\Gamma M^i_\Lambda e_{x,l} \to 0 \tag{13.5.6} $$

for all x, l where ρ is the induced by the restriction map ${}_\Gamma M^i_{\Lambda'} \to {}_\Gamma M^i_\Lambda$ and Σ is the finite coideal of Λ' defined by $\Sigma := \{\, z \in \Lambda' | z \geq x,\ z \notin \Lambda \,\}$. For obviously, $\operatorname{Im}\sigma$ is contained in $\ker\rho$. For the reverse inclusion, choose an admissible epimorphism $\oplus_{z\in\Sigma} Q^i_z(\Lambda') \to P^i_{x,l,\Lambda'}(\Lambda' \setminus \Lambda)$ with Q^i_z in $\operatorname{add}\{P^i_{y,j}\}_j$. Applying the exact functors $\hom_{\mathcal{C}^{i+1}}(P^{i+1}_{y,j,\Gamma}, \sigma_\Gamma F^i ?)$ to the diagram

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & P^i_{x,l,\Lambda'}(\Lambda' \setminus \Lambda) & \longrightarrow & P^i_{x,l,\Lambda'} & \longrightarrow & P^i_{x,l,\Lambda} & \longrightarrow & 0 \\ & & \uparrow & & & & & & \\ & & \oplus_{z\in\Sigma}\,(Q^i_z)(\Lambda'). & & & & & & \end{array} $$

gives $\ker\rho \subseteq \operatorname{Im}\sigma$. From this short exact sequence (13.5.6), it follows that

13.5.7. For a $\mathcal{A}^i_\Lambda$-module N, the natural epimorphisms

$$_\Gamma M^i_{\Lambda'} \otimes_{\mathcal{A}^i} N \to {}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} N$$

are isomorphisms if $\Lambda' \supseteq \Lambda$.

Now define the inverse system $\{F^i_\Lambda\}_{\Lambda \in I^i}$ of graded $(\mathcal{A}^{i+1}, \mathcal{A}^i)$-bimodules with $F^i_\Lambda := \varprojlim_{\Gamma \in I^{i+1}} \hom\big(\mathbf{P}^2_\Gamma, F(\mathbf{P}^1_\Lambda)\big)$. Note $F^i_\Lambda = \varprojlim_{\Gamma \in I^{i+1}} {}_\Gamma M_\Lambda$, in fact $F^i_\Lambda \cong {}_\Gamma M_\Lambda$ canonically provided $\Gamma \supseteq F^i(\Lambda)$. By 8.9, $F^i_*\colon \hat{\mathcal{E}}^i \to \hat{\mathcal{E}}^{i+1}$ is given by

$$F^i_* := \varprojlim_{\Lambda \in I^i} F_\Lambda \otimes_{\mathcal{A}^i} \big(\oplus_{x,i} \hom_{\mathcal{A}^i}(\mathcal{A}^i_\Lambda e_{x,i}, ?)\big).$$

Hence $F^i_\dagger \cong G_i$ where

$$G_i := \varprojlim_{\Lambda \in I^i} \varprojlim_{\Gamma \in I^{i+1}} {}_\Gamma M^i_\Lambda \otimes ?\colon \mathcal{E}^i \to \mathcal{E}^{i+1}. \tag{13.5.8}$$

It follows using 13.5.7 that

$$_\Gamma M_\Lambda \otimes_{\mathcal{A}^i} N \cong {}_\Gamma M_\Sigma \otimes_{\mathcal{A}^i} N \cong F^i_\Sigma \otimes_{\mathcal{A}^i} N \cong G^i(N)$$

canonically provided N is in $\mathcal{A}_\Sigma$-mod, $\Lambda \supseteq \Sigma$ and $\Gamma \supseteq F^i(\Sigma)$.

For M in $\mathcal{A}^i_\Lambda$-mod and N in $\mathcal{A}^{i+1}_\Gamma$-mod there is adjointness

$$\hom_{\mathcal{A}^{i+1}_\Gamma}({}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} M, N) \cong \hom_{\mathcal{A}^i_\Lambda}\Big(M, \oplus_{x,l} \hom_{\mathcal{A}^{i+1}_\Gamma}({}_\Gamma M^i_\Lambda e_{x,l}, N)\Big). \tag{13.5.9}$$

Assume that $\Lambda \supseteq F^{i+1}(\Gamma)$. Then one has isomorphisms of $(\mathcal{A}^{i+1}, \mathcal{A}^i)$-bimodules

$$\begin{aligned} &\oplus_{x,l} \hom_{\mathcal{A}^{i+1}_\Gamma}\Big(({}_\Gamma M^i_\Lambda) e_{x,l}, \oplus_{y,j} \mathcal{A}^{i+1}_\Gamma e_{y,j}\Big) \\ &\cong \hom_{\mathcal{A}^{i+1}_\Gamma}\big(\varphi^{i+1}_\Gamma(\sigma_\Gamma F^i(\mathbf{P}^i_\Lambda)), \varphi^{i+1}_\Gamma(\mathbf{P}^{i+1}_\Gamma)\big) \\ &\cong \hom_{\mathcal{C}^{i+1}}\big(\sigma_\Gamma F^i(\mathbf{P}^i_\Lambda), \mathbf{P}^{i+1}_\Gamma\big) \cong {}_\Lambda M^{i+1}_\Gamma \end{aligned}$$

using finite generation of ${}_\Gamma M^i_\Lambda e_{x,l}$, 1.19 and (13.5.3) i.e. we have an an isomorphism

$$\oplus_{x,l} \hom_{\mathcal{A}^{i+1}_\Gamma}\Big(({}_\Gamma M^i_\Lambda) e_{x,l}, \mathcal{A}^{i+1}_\Gamma\Big) \cong {}_\Lambda M^{i+1}_\Gamma. \tag{13.5.10}$$

By (13.5.5), one has for $\Lambda \supseteq F^{i+1}(\Gamma)$ an isomorphism of $(\mathcal{A}^i, \mathcal{A}^{i+1})$-bimodules

$$\oplus_{x,l} \hom_{(\mathcal{A}^{i+1}_\Gamma)^{\mathrm{op}}}\Big(e_{x,l}({}_\Lambda M^{i+1}_\Gamma), \mathcal{A}^{i+1}_\Gamma\Big) \cong {}_\Gamma M^i_\Lambda \tag{13.5.11}$$

and a standard isomorphism

$$\oplus_{x,l} \hom_{\mathcal{A}^{i+1}_\Gamma}\Big(({}_\Gamma M^i_\Lambda) e_{x,l}, N\Big) \cong {}_\Lambda M^{i+1}_\Gamma \otimes_{\mathcal{A}^{i+1}} N \tag{13.5.12}$$

for N in $\mathcal{E}^{i+1}_\Gamma$. Hence for $\Lambda \supseteq F^{i+1}(\Gamma)$, we obtain an adjunction

$$\hom_{\mathcal{E}^{i+1}_\Gamma}({}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} M, N) \cong \hom_{\mathcal{E}^i_\Lambda}(M, {}_\Lambda M^{i+1}_\Gamma \otimes_{\mathcal{A}^{i+1}} N). \tag{13.5.13}$$

Now ${}_\Gamma M^i_\Lambda := \hom_{\mathcal{C}^{i+1}}\big(\mathbf{P}^{i+1}_\Gamma, \sigma_\Gamma F^i(\mathbf{P}^i_\Lambda)\big)$ and if $\Lambda \supseteq F^{i+1}(\Gamma)$ then we have ${}_\Lambda M^{i+1}_\Gamma = \hom_{\mathcal{C}^{i+1}}\big(\sigma_\Gamma F^i(\mathbf{P}^i_\Lambda), \mathbf{P}^{i+1}_\Gamma\big)$ so composition induces a linear map

$${}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} {}_\Lambda M^{i+1}_\Gamma \to \mathcal{A}^{i+1}_\Gamma. \tag{13.5.14}$$

In terms of the map (13.5.14), the adjunction (13.5.13) is given by the composite

$$\begin{aligned}
&\hom_{\mathcal{A}^i_\Lambda}(M, {}_\Lambda M^{i+1}_\Gamma \otimes_{\mathcal{A}^{i+1}} N)\\
&\to \hom_{\mathcal{A}^{i+1}_\Gamma}\Big({}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} M,\ {}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} {}_\Lambda M^{i+1}_\Gamma \otimes_{\mathcal{A}^{i+1}_\Gamma} N\Big)\\
&\to \hom_{\mathcal{A}^{i+1}_\Gamma}\Big({}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} M, \mathcal{A}^{i+1}_\Gamma \otimes_{\mathcal{A}^{i+1}_\Gamma} N\Big)\\
&\cong \hom_{\mathcal{A}^{i+1}_\Gamma}({}_\Gamma M^i_\Lambda \otimes_{\mathcal{A}^i} M, N)
\end{aligned} \tag{13.5.15}$$

Now suppose that we have $\Lambda \subseteq \Lambda' \in I^i$ and $\Gamma \subseteq \Gamma' \in I^i$ with $\Lambda' \supseteq F^{i+1}(\Gamma')$ and $\Lambda \supseteq F^{i+1}(\Gamma)$. One can check that the evident diagram

$$\begin{array}{ccc}
{}_{\Gamma'} M^i_{\Lambda'} \otimes_{\mathcal{A}^{i+1}} {}_{\Lambda'} M^{i+1}_{\Gamma'} & \longrightarrow & \mathcal{A}^{i+1}_{\Gamma'}\\
\downarrow & & \downarrow\\
{}_{\Gamma} M^i_{\Lambda} \otimes_{\mathcal{A}^{i+1}} {}_{\Lambda} M^{i+1}_{\Gamma} & \longrightarrow & \mathcal{A}^{i+1}_{\Gamma}
\end{array} \tag{13.5.16}$$

with horizontal maps induced by composition of maps is commutative. It follows readily from (13.5.15)–(13.5.16) that the adjunctions (13.5.13) are compatible with the maps defining the inverse systems M^i, M^{i+1}, and give an adjunction between the functors G^i and G^{i+1} as given by (13.5.8).

Remarks. It should be possible to give a module theoretic description of an adjunction between F^* and F_* in 8.9 and use it to give conditions more general than those in 1.36 under which one can perform "base change" on the adjunction between F^* and F_*.

13.6. **Proof of Theorem 1.36.** Maintain the notation of the previous subsection, but make the additional assumptions in 1.36. Most of the argument doesn't require bistability of the F^i but just that they are stable forwards for f.g. ideals.

Set ${}_\Gamma M^{i\prime}_\Lambda = k' \otimes_k {}_\Gamma M^{i\prime}_\Lambda$. Now the facts 13.5.4–13.5.16 used in constructing the functors G^i, G^{i+1} and proving their adjointness give corresponding statements with ${}_\Lambda M^i_\Gamma$ replaced by ${}_\Lambda M^{i\prime}_\Gamma$, F^i_Λ replaced by $F^{i\prime}_\Lambda$, $\mathcal{A}^i$ replaced by $\mathcal{A}^{i\prime}$ etc (note that the modules in 13.5.6 are in $\mathcal{C}^{i+1}$ and hence are k-flat by our assumptions). We obtain functors

$$G^{i\prime} := \varprojlim_{\Lambda\in I^i} \varprojlim_{\Gamma\in I^{i+1}} {}_\Gamma M^{i\prime}_\Lambda \otimes ?\colon \mathcal{E}^i \to \mathcal{E}^{i+1}$$

with ${}_\Gamma M^{i\prime}_\Lambda \otimes_{\mathcal{A}^i} N \cong G^{i\prime}(N)$ canonically provided N is in $\mathcal{A}^{i\prime}_\Sigma$-mod, $\Lambda \supseteq \Sigma$ and $\Gamma \supseteq F^i(\Sigma)$. Moreover, $G^{i\prime}$ is exact and is left adjoint to $G^{i+1\,\prime}$ and one has a diagram

$$\begin{array}{ccc}
\mathcal{E}^i & \xrightarrow{G^i} & \mathcal{E}^{i+1}\\
{\scriptstyle L_i}\downarrow & & \downarrow{\scriptstyle L_{i+1}}\\
k' \otimes_k \mathcal{E}^i & \xrightarrow{G^{i\prime}} & k' \otimes_k \mathcal{E}^{i+1}
\end{array}$$

which is commutative up to natural isomorphism. The diagram shows that $G^{i\prime}$ takes any object $k' \otimes_k M$ with $M \in \mathcal{C}^i$ to an object of $\mathcal{C}^{i+1\,\prime}$. This applies in particular to $k' \otimes_k \Delta^i_{x,l}$, and by exactness of $G^{i\prime}$, it follows that $G^{i\prime}$ restricts to an exact functor $\mathcal{C}^{i\prime} \to \mathcal{C}^{i+1\,\prime}$ which we denote by $H^i = k' \otimes_k F^i$. One has $H^{i+1}L^i \cong L^{i+1}F^i$ from the diagram. One easily sees using the diagram that if F^i is stable forwards or backwards for f.g. ideals or f.g. coideals, then H^i has the same property. In particular, if F^i is bistable, then so is H^i and so $\hat{H}^i$, $\hat{H}^i_*$, $\hat{H}^i_\dagger$ are defined.

Finally, suppose given $\eta^i \colon F^i \to K^i$. Define the inverse systems N^j in the same way as M^j above but using K in place of F i.e. ${}_\Gamma N^i_\Lambda := \hom_{\mathcal{C}^{j+1}}\big(\mathbf{P}^{j+1}_\Gamma, \sigma_\Gamma K^i(\mathbf{P}^j_\Lambda)\big)$. Then η^i induces a morphism of inverse systems $M^i \to N^i$ and hence a natural transformation $k' \otimes_k \eta^i$ as required.

Remarks. Using 8.9, one can show that

$$H^i_* \cong \varprojlim_{\Lambda \in I^i} F^{i\prime}_\Lambda \otimes_{\mathcal{A}^{i\prime}} \big(\oplus_{x,l} \hom_{\mathcal{A}^{i\prime}}(\mathcal{A}^{i\prime}_\Gamma e_{x,l}, ?)\big) \colon \hat{\mathcal{E}}^{i\prime} \to \hat{\mathcal{E}}^{i+1\,\prime}$$

where $F^{i\prime}_\Lambda := k' \otimes_k F^i_\Lambda$.

14. Grothendieck group of $\mathcal{C}$

In this section we establish the main facts about Grothendieck groups of $\mathcal{C}$ and related categories, as summarized in Theorem 1.38.

14.1. **Proof of Theorem 1.38(a)–(d).** The maps described in 1.38(a)–1.38(c) are well-defined homomorphisms. It is easy to check that

$$([N'_x]_{\mathcal{C}_x})_{x \in \Omega} \mapsto [\oplus_{x \in \Omega} N'_x]_{\mathcal{C}}$$

gives an inverse for the map in 1.38(a).

The inverse to the map in 1.38(b) (resp., the map in the first assertion in 1.38(c)) sends $[Q]$ to $\sum_{i \geq 0} (-1)^i [Q_i]_{\mathcal{P}}$ where

$$0 \to Q^n \to \ldots \to Q^0 \to Q \to 0$$

is a finite projective resolution of Q in $\mathcal{C}$ (resp., finite projective resolution of Q by f.g. projective $\mathcal{A}$-modules). The fact these are well-defined and inverses to the given maps follows by Schanuel's lemma (which is applicable in 1.38(b) because of the functor φ, for instance). See [2, Ch VII, Prop 1.3] for details of this standard argument.

It remains to prove 1.38(d). If x is a maximal element of an ideal Γ of Ω, we have $R_x \cong \oplus_{i,j} e_{x,i} \mathcal{A}_\Gamma e_{x,j}$. The assumptions imply that then

$$\mathcal{A}(x) \cong \oplus_i \mathcal{A}_\Gamma e_{x,i} \mathcal{A}_\Gamma \cong (\oplus_i \mathcal{A}_\Gamma e_{x,i}) \otimes_{R_x} (\oplus_i e_{x,i} \mathcal{A}_\Gamma)$$

is projective as both left and right $\mathcal{A}_\Gamma$-module. Hence the last assertion follows inductively from (C.12.2).

Remarks. Observe that Theorem 1.38(a) and its proof continue to hold for a small weakly stratified exact category. Using the exactness and resolution theorems in [45], one sees that the analogues of 1.38(a)–1.38(c) also hold for Quillen's higher K-groups.

14.2. The proof of 1.38(e) is similar to that of (its special case) 1.38(b); it uses a version for $\hat{\mathcal{C}}$ of Schanuel's lemma.

For the proof, we shall find it convenient to fix for each isomorphism class $\{N\}$ of objects in $\mathcal{C}_x$ an object $Q = \mathrm{Proj}(N)$ in $\hat{\mathcal{C}}$ with $Q(x) = N$ and $Q(y) = 0$ unless $y \geq x$ (and $\mathrm{Proj}(0) = 0$). For M in $\hat{\mathcal{C}}$, one may construct as in 7.11 an admissible epimorphism $P^0 \to M \to 0$ with $P^0 = \oplus_x \mathrm{Proj} M(x)$ such that $\big(\mathrm{Proj} M(x)\big)(x) \to M(x)$ is an isomorphism. Observe that $\ker(P^0 \to M)(x) \cong \oplus_{y \neq x} (\mathrm{Proj} M(y))(x)$.

Recursively, one obtains a projective resolution $\cdots \to P^1 \to P^0 \to M \to 0$ in which for $i \geq 0$, $P^i \cong \oplus_x \mathrm{Proj}(\ker(P^{i-1} \to P^{i-2})(x)$, with $P^{-1} = M$ and $P^{-2} = 0$.

We call such a projective resolution $P^\bullet \to M \to 0$ a standard projective resolution of M. The above remarks imply that

14.2.1. In a standard projective resolution $P^\bullet \to M \to 0$, P^i depends up to isomorphism only on the family $\{M(x)\}_{x\in\Omega}$, and the support of P^i is contained in Γ_i as defined in 1.18.

14.3. **Schanuel's lemma.** Given M in $\hat{\mathcal{C}}$, we shall call a projective resolution $P^\bullet \to M \to 0$ in $\hat{\mathcal{C}}$ a convergent projective resolution if $\oplus_i P^i$ is a convergent direct sum. For example, standard projective resolutions and finite projective resolutions are convergent. We now have the following analogue of Schanuel's lemma for $\hat{\mathcal{C}}$.

Lemma. *Let $P^\bullet \to M \to 0$ and $Q^\bullet \to M \to 0$ be two convergent projective resolutions of M in $\hat{\mathcal{C}}$. Then*

$$\oplus_i(P^{2i} \oplus Q^{2i+1}) \cong \oplus_i(Q^{2i} \oplus P^{2i+1})$$

in $\hat{\mathcal{C}}$.

Proof. Let $P^\infty := \oplus_i(P^{2i} \oplus Q^{2i+1})$ and $Q^\infty := \oplus_i(Q^{2i} \oplus P^{2i+1})$. Note that for $\Gamma \in I$, application of $\hat{\sigma}_\Gamma$ to the convergent projective resolutions give finite projective resolutions of $\hat{\sigma}_\Gamma(M)$, and the usual Schanuel's lemma gives isomorphisms $\hat{\sigma}_\Gamma(P^\infty) \cong \hat{\sigma}_\Gamma(Q^\infty)$. (If the $\mathbf{N}_x$ are Krull-Schmidt families, this is already sufficient to give the assertion, by the arguments in 15.5).

In order to sketch a proof of the assertion in general, we first recall the usual proof [2] of Schanuel's lemma. Define $L^0 = M^0 = R^0 = M$ and decompose the resolutions into short $\hat{\mathcal{C}}$-exact sequences $0 \to M^{i+1} \to P^i \to M^i \to 0$ and $0 \to L^{i+1} \to Q^i \to L^i \to 0$. Define projective objects $P^{(0)} = P^0$, $Q^{(0)} = Q^0$, $P^{(i+1)} = P^{i+1} \oplus Q^{(i)}$ and $Q^{(i+1)} = Q^{i+1} \oplus P^{(i)}$ for $i \in \mathbb{N}$. Recursively define objects R^i and isomorphisms $L^i \oplus P^{(i-1)} \cong R^i \cong M^i \oplus Q^{(i-1)}$ by considering the commutative diagram

$$\begin{array}{ccccc}
 & & M^{i+1} & = & M^{i+1} \\
 & & \downarrow & & \downarrow \\
L^{i+1} & \longrightarrow & R^{i+1} & \longrightarrow & P^{(i)} \\
\| & & \downarrow & & \downarrow \\
L^{i+1} & \longrightarrow & Q^{(i)} & \longrightarrow & R^{(i)}
\end{array}$$

in which the bottom right hand square is a pullback square, and the rows and columns are exact (note the middle row and column are split). Let $\Gamma \in I$ and let n_x denote the smallest integer such that $P^j_{\leq x} = Q^j_{\leq x} = 0$ for all $j \geq n_x$. Note that for $x \in \Omega$ and $j \geq n_x$, one has $P^{(j)}_{\leq x} \cong P^\infty_{\leq x}$ and $Q^{(j)}_{\leq x} \cong Q^\infty_{\leq x}$ for even j and $P^{(j)}_{\leq x} \cong Q^\infty_{\leq x}$ and $Q^{(j)}_{\leq x} \cong P^\infty_{\leq x}$ for odd j, canonically. To prove the lemma, one uses Zorn's lemma to show that one may choose the splittings of the middle row and column in such a way that for any $x \in \Omega$, the isomorphism $\theta_{x,i} \colon P^{(i)}_{\leq x} \cong Q^{(i)}_{\leq x}$ induced by the isomorphism $L^{i+1} \oplus P^{(i)} \cong M^{i+1} \oplus Q^{(i)}$ is independent of i for $i \geq n_x$ (note that if $i \geq n_x$ then $L^i_{\leq x} = M^i_{\leq x} = 0$). The (obviously compatible) family of isomorphisms

$\theta_{x,n_x}\colon P^{(\infty)}_{\le x} \cong Q^{(\infty)}_{\le x}$ give an isomorphism $P^\infty = \{P^\infty_{\le x}\}_{x\in\Omega} \cong \{Q^\infty_{\le x}\}_{x\in\Omega} = Q^\infty$ as required. □

14.4. **Proof of Theorem 1.38(e).** It is known (e.g. see [2]) that if C is a small split exact category, every element of the Grothendieck group $\mathrm{K}_0(C)$ can be written in the form $[L]-[M]$ for some objects L, M of C. Moreover, $[L]-[M]=[L']-[M']$ iff there is an object N such that $L\oplus M'\oplus N \cong L'\oplus M\oplus N$. These facts apply in particular to the split exact categories $\mathcal{C}_x$ and $\hat{\mathcal{P}}$.

We define $\mathbb{Z}[G]$-module homomorphisms $\pi'\colon \mathrm{K}_0(\hat{\mathcal{C}}) \to \mathrm{K}_0(\hat{\mathcal{P}})$ and $\iota'\colon \widehat{\mathrm{K}}_0(\mathcal{C}) \to \mathrm{K}_0(\hat{\mathcal{C}})$ as follows. For M in $\hat{\mathcal{C}}$, choose a convergent projective resolution $P^\bullet \to M \to 0$ and define $\pi'(M) = [\oplus_i P^{2i}] - [\oplus_i P^{2i+1}]$; this map is well-defined by Schanuel's lemma and the horse-shoe lemma, and clearly $\pi'\iota = \mathrm{Id}$. For a family $\{[M_x]_{\mathcal{C}_x}\}_{x\in\Omega}$ in $\widehat{\mathrm{K}}_0(\mathcal{C})$ with M_x an object of $\mathcal{C}_x$, define $\iota(\{[M_x]\}_{x\in\Omega}) = \oplus_{x\in\Omega}\theta(M_x)$ (a convergent direct sum); this induces a well-defined map ι' by the preceding remarks on split exact categories, and clearly $\pi\iota' = \mathrm{Id}$.

Finally, we check that $\pi\iota\pi'\iota' = \mathrm{Id}$ and $\pi'\iota'\pi\iota = \mathrm{Id}$. For the first of these, consider $M = \{[M_x]_{\mathcal{C}_x}\}_{x\in\Omega}$ in $\widehat{\mathrm{K}}_0(\mathcal{C})$ with M_x in $\mathcal{C}_x$ and only finitely many $M_x \neq 0$ for $x\in\Gamma$, for any $\Gamma\in I_1$. Choose a standard projective resolution $P^\bullet \to \oplus_x M_x$. Then $\pi\iota\pi'\iota'(M) = \{[\oplus_i P^{2i}(x)]_{\mathcal{C}_x} - [\oplus_i P^{2i+1}(x)]_{\mathcal{C}_x}\}_x = M$ since for each x, $P^\bullet(x) \to M_x \to 0$ is a finite resolution of M_x in the (split) exact category $\mathcal{C}_x$. On the other hand, suppose Q is in $\hat{\mathcal{P}}$. Choose for each x in Ω a standard projective resolution $Q^\bullet_x \to \theta(Q(x)) \to 0$. Then $R^\bullet := \oplus_x Q^\bullet_x \to \oplus_x\theta(Q(x)) \to 0$ is a convergent projective resolution of $\oplus_x\theta(Q(x))$, so $\pi'\iota'\pi\iota([Q]) = [\oplus_i R^{2i}] - [\oplus_i R^{2i+1}]$. By 14.2.1 a standard projective resolution $S^\bullet$ of Q also has $R^i = S^i$, so by Schanuel's lemma $\pi'\iota'\pi\iota([Q]) = [Q]$ as required.

15. Projective covers and highest weight structure

Throughout this section we suppose in addition to our standing assumptions that each family $\mathbf{N}_x$ for $x\in\Omega$ is a Krull-Schmidt family. Then $R_{x,i}$ is a graded local ring with trivially graded residue ring and R_x is a basic semiperfect ring.

15.1. **Proof of Theorem 1.40 in the case** $\Omega\in I_1$**.** Identify $\mathcal{C} = \hat{\mathcal{C}}$. We first prove 1.40(d) by induction on $\sharp(\{\,x\in\Omega | x\ge y\,\})$. Define the ring $S_{y,i}$ and homomorphism α. If y is maximal in Ω, 1.40(d) clearly holds. If not, choose some maximal element $x>y$ of Ω and consider a homogeneous element $f\in\mathrm{end}(P_{y,i})$. Its enough to show f is a unit in $\mathrm{end}(\mathbf{P}_{y,i})$ iff $\alpha(f)\notin J_{y,i}$. If $\alpha(f)\in J_{y,i}$ then clearly f is not a unit in $S_{y,i}$. On the other hand, if $\alpha(f)\notin J_{y,i}$, then, firstly, f is homogeneous of degree 1_G. By induction $\sigma_{\neq x}(f)$ is a unit in $\mathrm{End}(P_{y,i}(\neq x))$ and there is a diagram

(15.1.1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{y,i}(x) & \longrightarrow & P_{y,i} & \longrightarrow & P_{y,i}(\neq x) & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle \sigma_x(f)} & & \big\downarrow{\scriptstyle f} & & \cong\big\downarrow{\scriptstyle \sigma_{\neq(x)}(f)} & & \\
0 & \longrightarrow & P_{y,i}(x) & \longrightarrow & P_{y,i} & \longrightarrow & P_{y,i}(\neq x) & \longrightarrow & 0
\end{array}$$

giving a morphism of extensions in $\mathcal{C}$. In turn, this gives a commutative diagram

(15.1.2)
$$\begin{array}{ccccc} \hom(P_{y,i}(x),\mathbf{N}_x) & \longrightarrow & \operatorname{ext}^1(P_{y,i}(\neq x),\mathbf{N}_x) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \hom(P_{y,i}(x),\mathbf{N}_x) & \longrightarrow & \operatorname{ext}^1(P_{y,i}(\neq x),\mathbf{N}_x) & \longrightarrow & 0 \end{array}$$

of R_x^{op}-modules in which the rows are projective covers of $\operatorname{ext}^1(P_{y,i}(\neq x),\mathbf{N}_x)$. Applying $? \otimes_{R_x} R_x/J_x$ to this diagram gives a commutative diagram $(15.1.2)'$ in which the horizontal arrows and right vertical arrow are isomorphisms, and so the left vertical arrow in $(15.1.2)'$ is an isomorphism also. Since the the left vertical map in (15.1.2) is between f.g. projective right R_x-modules, this map is an isomorphism too. But this map is induced by $\sigma_x(f)$ in (15.1.1), and the equivalence C.9 implies $\sigma_x(f)$ is an isomorphism. Finally, the 5-lemma applied to (15.1.1) implies that f is an automorphism of $P_{y,i}$, completing the proof of 1.40(d).

Since $\Omega \in I_1$, convergent direct sums are just finite direct sums. Clearly, $\mathcal{P} = \operatorname{add}\mathbf{P}$. Observe that $P_{x,i} \cong P_{y,j}\langle g\rangle$ implies $(x,i) = (y,j)$. For $0 \neq N_{x,i} = P_{x,i}(x) \cong P_{y,j}(x)\langle g\rangle$ implies $y \leq x$, so $x = y$ by symmetry and then $N_{x,i} \cong N_{x,j}\langle g\rangle$ so $i = j$ and $g = 1_G$. It now follows from 1.40(d) that $\mathbf{P}$ is a Krull-Schmidt family, and hence 1.40(c), (e) hold (in fact, $\mathcal{A}$ is basic semiperfect). We now identify $\mathcal{C}$ with its strict image $\mathcal{C}^{\Delta}$ under φ, noting that idempotents split in $\mathcal{C}$ by Remark 9.6. If M is in $\mathcal{C}$, it is a finitely generated $\mathcal{A}$-module so there is a projective cover $P \to M$ with P a finitely generated projective $\mathcal{A}$-module. We have P in $\mathcal{C}$, hence $P \to M$ is an admissible epimorphism in $\mathcal{C}$ by 1.2. It now follows that $P \to M$ is a projective cover in $\mathcal{C}$, proving 1.40(a). Since $P_{x,i} \to N_{x,i}$ is an admissible epimorphism in $\mathcal{C}$ with $P_{x,i}$ indecomposable projective, 1.40(a) implies $P_{x,i} \to N_{x,i}$ is a projective cover in $\mathcal{C}$, proving 1.40(b).

15.2. We sketch a more explicit construction (independent of the functor φ) of a projective cover $g\colon Q' \to N$ of an object N of $\mathcal{C}$, leaving the proof to the reader.

First, fix a maximal element x of Ω, choose a projective cover $b\colon Q_1 \to N(\neq x)$ in $\mathcal{C}_{\neq x}$ by induction on the cardinality of the support of N, and set $Q := \mathbf{N}_x * Q$ in $\mathcal{P}$. Identify $Q_1 = Q(\neq x)$ and form the following commutative diagram with exact rows:

(15.2.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(x) & \xrightarrow{i} & Q & \longrightarrow & Q(\neq x) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & N(x) & \longrightarrow & N & \longrightarrow & N(\neq x) & \longrightarrow & 0. \end{array}$$

Now form another commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hom(\mathbf{N}_x, Q(x)) & \longrightarrow & \hom(\mathbf{N}_x, Q'(x)) & \longrightarrow & \hom(\mathbf{N}_x, Q'_x) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} & & \\ & & \hom(\mathbf{N}_x, Q(x)) & \xrightarrow{d} & \hom(\mathbf{N}_x, N(x)) & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

Here, d is induced by a, and we choose the bottom row so it is exact as a sequence of R_x-modules. Then choose Q'_x in $\mathcal{C}_x$ and e so e is a projective cover of H as right R_x-module, using C.9. Define $Q' = Q \oplus Q'_x$, so $Q'(x) = Q(x) \oplus Q'_x$ and one has the evident split short exact sequence as the top row of the diagram. Finally, the map f

may be inserted since $\hom(\mathbf{N}_x, Q'_x)$ is projective in k_x-mod. Note f is a (necessarily split) epimorphism by Nakayama's lemma. We identify $Q'(\neq x) = Q(\neq x)$.

Now we construct the map $g\colon Q' = Q_x \oplus Q \to N$ with component maps $Q \to N$ from (15.2.1) and $Q_x \hookrightarrow Q'(x) \to N(x) \to N$, where the middle map $Q'(x) \to N(x)$ corresponds to f by C.9. It may be checked that g is a projective cover of N.

15.3. We next give two lemmas for use in the proof of 1.40 in general.

Lemma. *Suppose that $\Omega \in I_1$, that P, Q in $\mathcal{P}$ are isomorphic and $x \in \Omega$ is maximal. Then any isomorphism $P(\neq x) \cong Q(\neq x)$ in $\mathcal{C}$ can be lifted to an isomorphism $P \cong Q$*

Proof. Let $P' \to P(\neq x)$ and $Q' \to Q(\neq x)$ be projective covers of $P(\neq x)$ and $Q(\neq x)$ in $\mathcal{C}$ respectively. Clearly, $P' \cong Q'$ and there is N in $\operatorname{add} \mathbf{N}_x$ such that $P \cong P' \oplus N$ and $Q \cong Q' \oplus N$. Now the given isomorphism $P(\neq x) \cong Q(\neq x)$ lifts to an isomorphism $P' \cong Q'$ which is necessarily an isomorphism. Extending this isomorphism in the obvious way to a map $P \cong P' \oplus N \cong Q' \oplus N \cong Q$ gives an isomorphism $P \to Q$ extending the given one $P(\neq x) \to Q(\neq x)$. □

15.4. Recall Lemma 7.7.

Lemma. *For a morphism $f\colon P \to M$ in $\hat{\mathcal{C}}$, the following conditions* (i)–(iii) *are equivalent:*

(i) *f is a projective cover of M in $\hat{\mathcal{C}}$*
(ii) *for all $\Gamma \in I$, $f_\Gamma\colon P_\Gamma \to M_\Gamma$ is a projective cover of M_Γ in $\mathcal{C}_\Gamma$*
(iii) *for all $x \in \Omega$, $f_{\leq x}\colon P_{\leq x} \to M_{\leq x}$ is a projective cover of $M_{\leq x}$ in $\mathcal{C}_{\leq x}$.*

Proof. We show that (i) implies (ii). Assume f is a projective cover of M; in particular, it is an admissible epimorphism. Suppose that $g\colon Q' \to P_\Gamma$ is a map in $\mathcal{C}_\Gamma$ such that Q' is projective in $\mathcal{C}_\Gamma$ and $f_\Gamma g\colon Q' \to M_\Gamma$ is an admissible epimorphism; we must show g is an admissible epimorphism. We may assume without loss of generality by 7.10.3 and 6.2 that $Q' = Q_\Gamma$ and $g = h_\Gamma$ for some $Q \in \hat{\mathcal{P}}$ and some map $h\colon Q \to P$. Choose a projective object $Q'' \cong \hat{\sigma}_{\Omega\setminus\Gamma} Q''$ in $\hat{\mathcal{C}}$ and an admissible epimorphism $l'\colon Q'' \to \hat{\sigma}_{\Omega\setminus\Gamma} P$. This gives a map $(h, l)\colon Q \oplus Q'' \to P$ where $l\colon Q'' \to \hat{\sigma}_{\Omega\setminus\Gamma} P \to P$ is the obvious composite map, and one readily verifies $f(h, l)$ is an admissible epimorphism. Hence (h, l) is an admissible epimorphism since f is a projective cover, so $(h_\Gamma, l_\Gamma)\colon Q_\Gamma \oplus Q''_\Gamma \to P_\Gamma$ is an admissible epimorphism in $\mathcal{C}_\Gamma$. Since $Q''_\Gamma = 0$, g is an admissible epimorphism as required.

It is trivial that (ii) implies (iii). Finally, assume (iii) and let $g\colon Q \to P$ be a map in $\hat{\mathcal{C}}$ such that fg is an admissible epimorphism. Then $f_{\leq x} g_{\leq x}$ is an admissible epimorphism in $\mathcal{C}_{\leq x}$ for all $x \in \Omega$, which implies $g_{\leq x}$ is an admissible epimorphism in $\mathcal{C}_{\leq x}$ since $f_{\leq x}$ is a projective cover. The standard equivalence $\hat{\mathcal{C}} \cong \hat{\mathcal{C}}_{I_0}$ now implies that g is an admissible epimorphism, as required. □

Remarks. As a corollary of the lemma, it follows that if $P \to M$ is a projective cover in $\mathcal{C}_\Gamma$ for some $\Gamma \in I$, then for any ideal $\Lambda \subseteq \Gamma$, $\sigma_\Lambda P \to \sigma_\Lambda M$ is a projective cover in $\mathcal{C}_\Lambda$. (This fact is also obvious from the explicit construction of projective covers.)

15.5. **Proof of Theorem 1.40 in general.** There is a natural epimorphism of rings $S_{x,i} = \mathrm{end}_{\hat{\mathcal{C}}}(P_{x,i}) \to \mathrm{end}_{\mathcal{C}}(P_{x,i,\Gamma})$ by projectivity of $\mathbf{P}_x$. A homogeneous element of $S_{x,i}$ is a unit iff its image under each of these epimorphisms is a unit, using 6.2. Now 1.40(d) follows from 15.1. As in 15.1, $\mathbf{P}$ is a Krull-Schmidt family, $\mathcal{A}$ is basic semiperfect and 1.40(e) follows.

Next we prove 1.40(c). Consider an object Q of $\hat{\mathcal{P}}$. Define an ideal Ω_0 of Ω by $\Omega_0 := \{ z \in \Omega | Q_{\leq z} = 0 \}$. It is easily seen from 15.1 that there are unique up to isomorphism objects P_x in add$\{P_{x,i}\}_i$, for $x \in \Omega$, such that $P_x = 0$ if $x \in \Omega_0$ and there are isomorphisms $\rho_y \colon (\oplus_{x\in\Omega} P_x)_{\leq y} \cong Q_{\leq y}$ for all $y \in \Omega$. An application of Zorn's lemma in $\hat{\mathcal{C}}_{I_0}$ shows that there is an inclusion-maximal ideal $\Sigma \supseteq \Omega_0$ of Ω on which the ρ_y for $y \in \Sigma$ can be chosen compatibly with the maps defining the inverse systems $\oplus_x P_x$ and Q. By 15.3, it follows that in fact $\Sigma = \Omega$. The exact category equivalence $\hat{\mathcal{C}} \cong \hat{\mathcal{C}}_{I_0}$ implies $\oplus P_x \cong Q$ in $\hat{\mathcal{C}}$. Uniqueness of the multiplicities is clear from the definition of the P_x.

Now we prove 1.40(a). Let M be in $\hat{\mathcal{C}}$, $\Omega_0 = \{ x | M_{\leq x} = 0 \}$. Choose for each $\Gamma \in I$ a projective cover $f_\Gamma \colon P_\Gamma \to M_\Gamma$. By the preceding remark, one may identify $f_\Gamma = \sigma_\Gamma f_\Lambda$ for $\Gamma \subseteq \Lambda$ in I. The canonical epimorphisms $P_\Lambda \to \sigma_\Gamma(P_\Lambda) = P_\Gamma$ so defined make $P := \{P_\Gamma\}_{\Gamma\in I}$ into an object of $\hat{\mathcal{C}}$ and $f := \{f_\Gamma\}_{\Gamma\in I}$ a morphism in $\hat{\mathcal{C}}$. By the preceding lemma, f is a projective cover in $\hat{\mathcal{C}}$. Finally, 1.40(b) follows from 1.40(a) as in 15.1.

15.6. **Proof of Theorem 1.41 and Lemma 1.42.** Clearly, $\Delta_{x,i} = \mathcal{A}_{\leq x} e_{x,i}$ is a highest weight module of highest weight (x, i) and degree 1_G. Let V be another such module, generated by $v \in e_{x,i} V_{1_G}$. Suppose V is a $\mathcal{A}_\Lambda$-module for $x \in \Lambda \in I_1$. Then one has an epimorphism $\alpha \colon \mathcal{A}_\Lambda e_{x,i} \to V$ mapping $e_{x,i} \to V$. Since $\mathcal{A}$ is left stratified, there is an epimorphism $\mathcal{A}_\Lambda e_{x,i} \to \Delta_{x,i}$ with kernel K having a finite filtration with successive subquotients in add$\,\boldsymbol{\Delta}_y$ for various $y > x$. Now $\hom(\Delta_{y,j}, V) = \hom(\mathcal{A}_{\leq y} e_{y,j}, V) = 0$ for $y > x$ since $e_{y,j} V = 0$. Thus, $\alpha(K) = 0$ and V is a quotient of $\Delta_{x,i}$, proving 1.41(a).

We have already seen that $\mathcal{A}$ is a basic semiperfect ring. In particular, $\mathcal{A}e_{x,i}$ has a unique simple quotient module in $\mathcal{A}$-Mod, namely $L_{x,i} := \mathcal{A}e_{x,i}/\mathrm{rad}\, \mathcal{A}e_{x,i}$ and the assertions of 1.41(c)–(d) hold in $\mathcal{A}$-mod in place of $\mathcal{E}$, using 1.40(e). Fix an ideal $\Gamma \in I$ of Ω with x as a maximal element. Since $\Delta_{x,i} \cong \mathcal{A}_\Gamma e_{x,i}$ is a quotient module of $\mathcal{A}e_{x,i}$, it follows that $\Delta_{x,i} := \varphi(N_{x,i})$ has a unique maximal graded submodule with simple graded quotient module $L_{x,i}$. Now $L_{x,i}$ is a $\mathcal{A}_\Gamma$-module and thus in $\mathcal{E}$. Since any object of $\mathcal{E}$ is a $\mathcal{A}_\Lambda$-module for some $\Lambda \in I_1$, we see 1.41(b)–(d) hold.

Note that $L_{y,i}\langle g\rangle$ appears as a simple subquotient of an object M of $\mathcal{E}$ iff $e_{y,i} M_g \cong \hom(\mathcal{A}e_{y,i}, M)_g \neq 0$. Since $\mathcal{A}_\Gamma$ is basic semiperfect, we have using 1.40(e) that the maximal graded submodule of $\Delta_{x,i} = \mathcal{A}_\Gamma e_{x,i}$ is

$$\mathrm{rad}\ \Delta_{x,i} = e_{x,i} \Delta_{x,i} J_{x,i} + \sum_{(y,j)\neq(x,i)} e_{y,j} \Delta_{x,i}.$$

The remaining assertions of 1.41 follow easily. One also sees that 1.42 holds, recalling 1.26(a) for $\boldsymbol{\nabla}$ in 1.42(b) and noting for 1.42(c) that $Q^{\mathrm{op}}_{x,i}$ is projective in $\mathcal{C}^{\mathrm{op}}$, $\mathrm{end}_{\mathcal{C}}(Q_{x,i})^{\mathrm{op}} = \mathrm{end}_{\mathcal{C}^{\mathrm{op}}}(Q^{\mathrm{op}}_{x,i})$, and $\mathrm{end}(\mathcal{T} e_{x,i})^{\mathrm{op}} \cong \mathrm{end}(Q_{x,i})^{\mathrm{op}}$.

15.7. We define local composition series of objects of $\mathcal{E}_{\mathrm{fin}}$ and discuss their relationship with composition factor multiplicities as defined in 1.43.

For M in $\mathcal{E}_{\mathrm{wfin}}$ and $x \in \Omega$, define a local composition series of M at x to be a finite filtration $M = M^0 \supseteq M^1 \supseteq \dots \supseteq M^n = 0$ such that for each i, M^{p-1}/M^p is either simple or satisfies $e_{y,j}M^{p-1}/M^p = 0$ for all $y \geq x$ and j. For instance, a composition series of M in the usual sense (if it exists) is a local composition series of M at any $x \in \Omega$. If M has a local composition series at x, then $[M : L_{x,i}\langle g\rangle]$ is clearly the number of factors M^{i-1}/M^i isomorphic to $L_{x,i}\langle g\rangle$.

Lemma. *If the index sets I_y are all finite (for instance, all singletons), then for all $x \in \Omega$, every M in $\mathcal{E}_{\mathrm{fin}}$ has a local composition series at x.*

Proof. The quantity $\sum_{y\geq x}\sum_j\sum_g [M : L_{y,j}\langle g\rangle]$ is finite. One proves the lemma by an induction on this quantity using 1.41; the argument is essentially the same as in [34, 9.6], so we omit it. □

15.8. **Proof of Proposition 1.44.** We show both sides are equal to $\{\{e_{x,i}\nabla_{y,j}\}\}$. First, $L_{x,i} = (e_{x,i}L_{x,i})_{1_G}$ is a quotient of

$$(e_{x,i}\Delta_{x,i})_{1_G} = (e_{x,i}\mathcal{A}_{\leq x}e_{x,i})_{1_G} = (e_iR_xe_i)_{1_G}$$

and is therefore finite dimensional over k. Note that by 1.25,

$$e_{x,i}\nabla_{y,j} = \mathrm{hom}(\mathcal{A}_\Gamma e_{x,i}, \nabla_{y,j}) = \mathrm{hom}(\mathcal{A}_\Gamma e_{x,i}(y), \nabla_{y,j})$$

provided $x, y \in \Gamma \in I_1$. Now $\mathcal{A}_\Gamma e_{x,i}(y)$ is a direct sum of $n_{g,l}$ copies of $T_g\Delta_{y,l}$ for varying g and l, with $n_{g,l}$ defined as in 1.44 and by 1.26, $\mathrm{hom}(\Delta_{y,l}, \nabla_{y,j}) \cong e_lR_ye_j$. A simple computation shows that the left hand side of the formula in Proposition 1.44 is just $\{\{e_{x,i}\nabla_{y,j}\}\}$; it follows immediately that $\nabla_{y,j}$ is in $\mathcal{E}_{\mathrm{wfin}}$ since $(e_{x,i}\nabla_{y,j})_g$ is finite-dimensional over k. On the other hand, $\{\{e_{x,i}\nabla_{y,j}\}\}$ is clearly equal to the right hand side of the formula as well by definition of composition factor multiplicity.

15.9. We conclude this section with a formula for the multiplicities of the projective indecomposables $P_{x,i}$ as direct summands of a projective object Q of $\hat{\mathcal{C}}$, in terms of the maps $g_{Q,y}$ associated to Q. We assume for simplicity in the formulation that G is the trivial group (the result in this case is equivalent to that for general G by C.6).

Fix Q in $\hat{\mathcal{P}}$ and consider for $z \in \Omega$ the exact sequence

$$\mathrm{Hom}(Q(\leq z), \mathbf{N}_z) \xrightarrow{f_z} \mathrm{Hom}(Q(z), \mathbf{N}_z) \xrightarrow{g_z} \mathrm{Ext}^1(Q(< z), \mathbf{N}_z) \to 0$$

of right R_z-modules. Applying $? \otimes_{R_z} R_z/J_z$ gives a right exact sequence of R_z/J_z-modules. Since $\mathrm{Hom}(Q(z), \mathbf{N}_z)$ is a f.g. projective right R_z-module, it follows that $\mathrm{Hom}(Q(z), \mathbf{N}_z) \otimes_{R_z} R_z/J_z$ is a f.g. right R_z/J_z-module. Now the modules e_iR_z/J_z form a full set of representatives of isomorphism classes of simple right R_z/J_z modules, and R_z/J_z is semisimple so any f.g. R_z/J_z module N is isomorphic to a direct sum of such modules with (uniquely determined) finite multiplicities which we denote $[N : e_iR_z/J_z]$.

Proposition. *Define natural numbers $c_{z,i} := [\mathrm{Im}\,(f_z \otimes_{R_z} \mathrm{Id}_{R_z/J_z}) : e_i(R_z/J_z)]$. Then $Q \cong \oplus_{z,i}\, c_{z,i}P_{z,i}$, where for $n \in \mathbb{N}$ and an object P of $\hat{\mathcal{P}}$, nP denotes the direct sum of n copies of P.*

Proof. We may assume $\Omega \in I$ without loss of generality. The Grothendieck group $\mathrm{K}_0[\mathcal{C}]$ is a free abelian group on basis $\{[N_{x,i}]\}_{x,i}$. For a f.g. right R_z-module N, define a "generating function" $\mathrm{Poinc}_z(N) \in \mathrm{K}_0[\mathcal{C}]$ by

$$\mathrm{Poinc}_z(N) := \sum_i [N \otimes_{R_z} R_z/J_z : e_i R_z/J_z][N_{z,i}].$$

For any M in $\mathcal{C}$ one has the identity

$$[M] = \sum_z \mathrm{Poinc}_z\big(\mathrm{Hom}(M(z), \mathbf{N}_z)\big)$$

in $\mathrm{K}_0[\mathcal{C}]$. Indeed, since $[M] = \oplus_z [M(z)]$ one may assume $M = M(z)$ for some z, and then note both sides are additive in M and the identity holds for $M = N_{z,i}$.

By 1.40(ii), $\mathrm{Poinc}_z\big(\mathrm{Hom}(P_{x,i}(z), \mathbf{N}_z)\big) = \mathrm{Poinc}_z\big(\mathrm{Ext}^1(P_{x,i}(< z), \mathbf{N}_z)\big)$ for $z \neq x$ since $P_{x,i}$ is indecomposable projective. Hence

$$[P_{x,i}] = [N_{x,i}] + \sum_{z > x,i} \mathrm{Poinc}_z\big(\mathrm{Ext}^1(P_{x,i}(< z), \mathbf{N}_z)\big).$$

Write $Q \cong \oplus_{x,i} d_{x,i} P_{x,i}$ for some $d_{x,i} \in \mathbb{N}$. Then

$$\begin{aligned}[Q] - \sum_{x,i} d_{x,i}[N_{x,i}] &= \sum_{x,i} d_{x,i}\big([P_{x,i}] - [N_{x,i}]\big)\\ &= \sum_{z > x,i} d_{x,i} \mathrm{Poinc}_z\big(\mathrm{Ext}^1(P_{x,i}(< z), \mathbf{N}_z)\big)\\ &= \sum_z \mathrm{Poinc}_z\big(\mathrm{Ext}^1(Q(< z), \mathbf{N}_z).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\sum_{x,i} d_{x,i}[N_{x,i}] &= \sum_z \Big(\mathrm{Poinc}_z\big(\mathrm{Hom}(Q(z), \mathbf{N}_z)\big) - \mathrm{Poinc}_z\big(\mathrm{Ext}^1(Q(< z), \mathbf{N}_z)\Big)\\ &= \sum_z \mathrm{Poinc}_z\big(\ker\,(g_z \otimes_{R_z} \mathrm{Id}_{R_z/J_z})\big)\\ &= \sum_z \mathrm{Poinc}_z\big(\mathrm{Im}\,(f_z \otimes_{R_z} \mathrm{Id}_{R_z/J_z})\big)\end{aligned}$$

so $d_{x,i} = c_{x,i}$ as required. □

16. Blocks of $\mathcal{C}$

This section describes a notion of "blocks" for the stratified exact category $(\mathcal{C}, \{\mathcal{C}_x\}_{x \in \Omega})$. Here, our "blocks" give a partition of the weight poset Ω; variants giving a partition of the indexing set $\{(x,i)\}_{x,i}$ of the standard objects $N_{x,i}$ could be given.

16.1. Define the finest equivalence relation $\equiv$ on Ω (i.e. the one with the smallest equivalence classes) such that $x \equiv y$ if $x \leq y$ and either $\hom(\mathbf{N}_x, \mathbf{N}_y) \neq 0$ or $\mathrm{ext}^1(\mathbf{N}_x, \mathbf{N}_y) \neq 0$. The equivalence classes for $\equiv$ will be called "blocks" of Ω; let us denote them by $\{\Omega_j\}_{j \in J}$. Let Ω' be a new poset equal to Ω as set, but with partial order $\leq'$ given by $x \leq' y$ iff $x \leq y$ and x, y are in the same block of Ω. In the order $\leq'$, each "block" is both an ideal and a coideal, and elements of different "blocks" are incomparable in $\leq'$. Then $(\mathcal{C}, \{\mathcal{C}_x\}_{x \in \Omega'})$ is also a split stratified exact category, by 2.2(c). For the remainder of this section, we use notation $M(\Gamma)$ for M in $\mathcal{C}$ and locally closed subsets Γ of Ω'.

Any object M of $\mathcal{C}$ has a canonical "block decomposition" $M = \oplus_j M(\Omega_j)$ with $M(\Omega_j) = 0$ for all but finitely many j, by Lemma 2.4, for instance. If $M' = \oplus_j M'(\Omega_j)$ is another object of $\mathcal{C}$, then any map $f\colon M \to M'$ is the direct sum of the induced maps $f = \oplus \sigma_{\Omega_j}(f)$. Thus, $\mathcal{C}$ is the direct sum $\mathcal{C} \cong \oplus_j \mathcal{C}_{\Omega_j}$ of (statified) exact categories. In particular, one may choose the projective objects $P_{x,i}$ in $\hat{\mathcal{C}}$ so that $P_{x,i}(y) = 0$ unless x and y are in the same block; note this condition holds automatically if the $P_{x,i}$ are indecomposable.

For the remainder of this subsection, we assume that $\mathbf{P}$ is chosen in this way, so $P_{x,i}(y)$ is zero unless x and y are in the same block. Then $\mathcal{A} = \oplus_j \mathcal{A}(\Omega_j)$ where $\mathcal{A}(\Omega_j) = \oplus_{x,y\in\Omega_j,p,q}\, e_{x,p}\mathcal{A}e_{y,q}$ is a two-sided ideal of $\mathcal{A}$. For any M in $\mathcal{A}$-mod, we have canonically $M = \oplus M_j$ where $M_j = \oplus_{x\in\Omega_j,i}\, e_{x,i}M$ is a $\mathcal{A}$-mod annihilated by $\mathcal{A}(\Omega_k)$ for $k \neq j$. This provides an identification of $\mathcal{A}$-mod with the product of categories $\mathcal{A}(\Omega_j)$-mod. For $\mathcal{A}(\Omega_j)$-modules M_j, the direct sum $\oplus_j M_j$ is in $\hat{\mathcal{E}}$ (resp., $\mathcal{E}$) iff all M_j are in $\hat{\mathcal{E}}$ (resp., there is a $\Gamma \in I$ so all M_j are $\mathcal{A}(\Gamma)$-modules).

Remarks. If $\mathcal{C}'$ is obtained by base change from $\mathcal{C}$ under the conditions of 13.3 or 1.35, each "block" of $\mathcal{C}$ is a union of "blocks" of $\mathcal{C}'$.

16.2. Observe that $\hom(P_{x,i}, P_{y,j}) \neq 0$ iff $\hom(P_{x,i,\Gamma}, P_{y,j,\Gamma}) \neq 0$ for some $\Gamma \in I$; then $\hom(P_{x,i,\Lambda}, P_{y,j,\Lambda}) \neq 0$ for all $\Lambda \supseteq \Gamma$ in I.

Proposition. *Let $\sim$ denote the finest equivalence relation on Ω such that $x \sim y$ if* $\hom(P_{x,i}, P_{y,j}) \neq 0$ *for some i and j. Then $\sim$ coincides with $\equiv$.*

Proof. Suppose first that $\hom(P_{x,i}, P_{y,j}) \neq 0$. Then there exist $z, w \in \Omega$ such that $\hom(P_{x,i}(z), P_{y,j}(w)) \neq 0$. This implies that $\hom(\mathbf{N}_z, \mathbf{N}_w) \neq 0$, so $z \equiv w$. Moreover, by choice of $\mathbf{P}$, we have $x \equiv z$ and $y \equiv w$, so $x \equiv y$. Hence if $x \sim y$, then $x \equiv y$.

For the converse implication, it will be sufficient to assume that $x < y$ and $\mathrm{ext}^p(N_{x,i}, N_{y,j}) \neq 0$ for $p = 0$ or $p = 1$, and show that $x \sim y$.

Choose $\Gamma \in I$ containing x and y, and a projective resolution $P^\bullet \to N_{x,i} \to 0$ of $N_{x,i}$ in $\mathcal{C}_\Gamma$, with $P^0 = P_{x,i,\Gamma}$. Since $\hom_{\mathcal{C}_\Gamma}(P_{v,k,\Gamma}, P_{w,l,\Gamma}) = 0$ unless $v \sim w$, one may assume that each P^q is an object of add $\oplus_{z\sim x,k} P_{z,k,\Gamma}$. Computing $\mathrm{ext}^p(N_{x,i}, N_{y,j}) \neq 0$ using the projective resolution shows that $\hom(P^p, N_{y,j}) \neq 0$, so there exists $z \sim x$ and k with $\hom(P_{z,k,\Gamma}, N_{y,j}) \neq 0$. This implies that $\hom(P_{z,k}, P_{y,j}) \neq 0$ and so $z \sim y$. This gives $x \sim y$ as required. □

Remarks. Suppose each $\mathbf{N}_x$ is a Krull-Schmidt family and $P_{x,i}$ is chosen to be indecomposable. Then the above shows that $x \equiv y$ iff there is some $\Gamma \in I$, and a sequence $x = x_0, x_1, \ldots, x_n = y$ in Ω such that for each $p = 1, \ldots n$, there are i and j such that $\mathcal{A}(\Gamma)e_{x,i}$ and $\mathcal{A}(\Gamma)e_{y,j}$ have simple subquotient objects in $\mathcal{E}$ which are isomorphic up to degree shift.

16.3. Assume in this subsection that for $\Gamma \in I$, $x, y \in \Omega$ and all i, j, one has $\hom(P_{x,i,\Gamma}, P_{y,j,\Gamma}) \neq 0$ iff $\hom(P_{y,j,\Gamma}, P_{x,i,\Gamma}) \neq 0$. This holds for instance if there is a suitable duality functor on $\hat{\mathcal{P}}$ fixing the $P_{x,i}$ up to translation.

Proposition. *Under the above assumptions, the block equivalence relation $\equiv$ is the finest equivalence relation $\backsim$ on Ω such that $x \backsim y$ if* $\hom(P_{x,i}, \theta(N_{y,j})) \neq 0$ *for some i and j.*

Proof. If $\hom(P_{x,i}, N_{y,j}) \neq 0$, then $\hom(P_{x,i}, P_{y,j}) \neq 0$ and so $x \sim y$ i.e. $x \equiv y$.

Conversely, it will suffice to show that if $\Gamma \in I$ and $H := \hom(P_{x,i,\Gamma}, P_{y,j,\Gamma}) \neq 0$, then $x \backsim y$. Note that x and y are in Γ. Suppose that $H \neq 0$ and $x \not\backsim y$. Then $\hom(P_{x,i,\Gamma}, N_{y,j}) = 0$, so by 9.5, there is some $\Gamma \ni y_1 > y$ and k_1 with $\hom(N_{y_1,k_1}, P_{y,j}(y_1)) \neq 0$ and $\hom(P_{x,i,\Gamma}, N_{y_1,k_1}) \neq 0$. These imply that $x \backsim y_1$ and $\hom(P_{y_1,k_1,\Gamma}, P_{y,j,\Gamma}) \neq 0$. So $y < y_1 \in \Gamma$, $\hom(P_{y,j,\Gamma}, P_{y_1,k_1,\Gamma}) \neq 0$ and $y \not\backsim y_1$. Repeating the above argument, we obtain a sequence $y = y_0 < y_1 < y_2 < \cdots$ in Γ with $y_{i-1} \not\backsim y_i$ for all i. This contradicts the finiteness of $\{ z \in \Gamma | z \geq y \}$, completing the proof. $\square$

Remarks. Suppose above that each $\mathbf{N}_x$ is a Krull-Schmidt family and the $\mathbf{P}_x$ are indecomposable in $\mathcal{P}$. Then the above result implies that $x \equiv y$ iff there is a sequence $x = x_0, x_1, \ldots, x_n = y$ in Ω such that for each $i = 1, \ldots n$, $\Delta_{x_{i-i}}$ and Δ_{x_i} have irreducible subquotient modules which are isomorphic up to degree shift.

17. Preservation of indecomposability under base change and ungrading functors

Here, we prove Propositions 1.46 and 1.47, and some related results. For this section only, rings and modules are assumed to be trivially graded unless otherwise indicated.

17.1. **Proof of Proposition 1.46.** Set $\mathbf{\Delta}'_x := \{\Delta'_{x,i}\}_i$. We have $\operatorname{end}(\mathbf{\Delta}'_x)^{\mathrm{op}} \cong k' \otimes_k R_x = R_x/JR_x$. Now $e_i R_x e_j \subseteq \operatorname{rad} R_x$ if $i \neq j$ since R_x is graded basic semiperfect. By the assumptions, we therefore have $JR_x \subseteq \operatorname{rad} R_x$. Hence

$$\operatorname{end}(\mathbf{\Delta}'_x)^{\mathrm{op}}/\operatorname{rad} \operatorname{end}(\mathbf{\Delta}'_x)^{\mathrm{op}} \cong R_x/\operatorname{rad} R_x \cong \oplus_i R_{x,i}/\operatorname{rad} R_{x,i}$$

and $\{\Delta'_{x,i}\}_{x,i}$ is a Krull-Schmidt family over G. In particular, $\Delta'_{x,i} \neq 0$.

Observe that if $x \in \Gamma \in I$, then $\mathcal{A}_\Gamma e'_{x,i}$ is graded projective in $\mathcal{A}'_\Gamma$-mod and has $\Delta'_{x,i} \neq 0$ as a quotient. The endomorphism ring $\operatorname{end}(\mathcal{A}'_\Gamma e_{x,i})^{\mathrm{op}} = e_{x,i}\mathcal{A}'_\Gamma e_{x,i}$ is a graded quotient ring of the local ring $\operatorname{end}(\mathcal{A}_\Gamma e_{x,i})^{\mathrm{op}} = e_{x,i}\mathcal{A}_\Gamma e_{x,i}$ and so it is itself a graded local ring. Hence $\mathcal{A}_\Gamma e'_{x,i}$ is indecomposable in $\mathcal{C}'$, which implies that $P'_{x,i}$ is the projective cover of $\Delta'_{x,i}$ in $\hat{\mathcal{C}}'$. The assertions that various objects are Krull-Schmidt families over G now follows from 1.42 applied to $\mathcal{C}'$ in place of $\mathcal{C}$ (note that $k' \otimes_k \overline{\Delta_{x,i}} \cong \overline{\Delta'_{x,i}}$, where the right hand side is the analogue for $\mathcal{C}'$ of $\overline{\Delta_{x,i}}$). To complete the proof, it suffices to note that $L'_{x,i} \cong \overline{\Delta'_{x,i}} / \oplus_{(y,j)\neq(x,i)} e_{y,j}\overline{\Delta'_{x,i}}$, so by Theorem 1.41 the $L'_{x,i}$ are a set of representatives of isomorphism classes of simple object of $\mathcal{E}'$ up to translation.

17.2. We will actually prove Proposition 1.47 for a slightly more general class of ungrading functors considered below.

Assume that $F \colon \mathcal{B} \to \mathcal{B}'$ is a weak ungrading functor, as defined in 1.37. Let $\{N_{x,i}\}_{x\in\Omega,i}$ be a family of objects of $\mathcal{B}$ with Ω an interval finite poset. Assume that $\operatorname{Ext}^p_{\mathcal{B}'}(FN_{x,i}, FN_{y,j}) = 0$ for $p = 0$ unless $x \leq y$ and for $p = 1$ unless x and y are comparable. Then one may form as in 1.7 a stratified exact category $(\mathcal{C}', \{\mathcal{C}'_x\}_{x\in\Omega})$ with $\mathcal{C}'_x$ equal to the split exact category $\operatorname{Add}\{FN_{x,i}\}_i$ and with $\mathcal{C}'$ a full additive subcategory of $\mathcal{B}'$.

The assumptions imply that $\operatorname{ext}^p_{\mathcal{B}}(N_{x,i}, N_{y,j}) = 0$ for $p = 0$ unless $x \leq y$ and for $p = 1$ unless x and y are comparable, so one may form similarly a stratified exact category $(\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})$ over G where $\mathcal{C}_x = \operatorname{add}\{N_{x,i}\}_i$ and with $\mathcal{C}$ a full additive subcategory of $\mathcal{B}$.

It is easy to check that F restricts to a (trivially bistable) exact functor $F\colon \mathcal{C} \to \mathcal{C}'$, inducing in turn an exact functor $\hat{F}\colon \hat{\mathcal{C}} \to \hat{\mathcal{C}}'$. If $\mathbf{P} = \{P_{x,i}\}_{x,i}$ is a standard family of projective objects in $\hat{\mathcal{C}}$, then by 1.18 and the assumptions, $\mathbf{P}' := \{\hat{F}P_{x,i}\}_{x,i}$ is a standard family of projective objects in $\hat{\mathcal{C}}'$. Note that since one may compute $\operatorname{ext}^p_{\mathcal{C}}(M,N)$ using projective resolutions in $\mathcal{C}_\Gamma$ provided M and N are in $\mathcal{C}_\Gamma$, and similarly in $\mathcal{C}'$, it follows that $F\colon \mathcal{C} \to \mathcal{C}'$ is an ungrading functor.

We have $\mathcal{A} := \operatorname{end}(\mathbf{P})^{\mathrm{op}} = \varprojlim_{\Gamma\in I} \operatorname{end}(\mathbf{P}(\Gamma))^{\mathrm{op}}$ (inverse limit of G-graded, diagonalizable rings) and $\mathcal{A}' := \operatorname{End}(\mathbf{P}')^{\mathrm{op}} = \varprojlim_{\Gamma\in I} \operatorname{End}(\mathbf{P}'(\Gamma))^{\mathrm{op}}$ (inverse limit as diagonalizable rings) with $\mathcal{A}'(\Gamma) = \operatorname{End}(\mathbf{P}'(\Gamma))^{\mathrm{op}}$ isomorphic to the diagonalizable ring (with G-grading forgotten) underlying $\mathcal{A}(\Gamma) = \operatorname{end}(\mathbf{P}_\Gamma)^{\mathrm{op}}$. We identify $\mathcal{A}'(\Gamma)$ with the underlying ungraded ring of $\mathcal{A}(\Gamma)$, for $\Gamma \in I_1$.

It follows from 8.9 that the natural forgetful functors (forgetting G-grading) $\mathcal{A}(\Gamma)$-mod $\to \mathcal{A}(\Gamma)'$-Mod induce the exact functor $F_\dagger\colon \mathcal{E} \to \mathcal{E}'$ between the abelian subcategory $\mathcal{E}$ of $\mathcal{A}$-mod associated to $\mathcal{C}$ and the abelian subcategory $\mathcal{E}'$ of $\mathcal{A}'$-Mod associated to $\mathcal{C}'$. Clearly, $F_\dagger$ maps the standard $\boldsymbol{\Delta}$-modules (resp., $\boldsymbol{\nabla}$-modules) for $\mathcal{A}$ to standard $\boldsymbol{\Delta}$-modules (resp., $\boldsymbol{\nabla}$) modules for $\mathcal{A}'$. Also, if Ω is finite and $\mathcal{C}$ has sufficiently many injective objects, then $F_\dagger$ takes a standard family of graded tilting modules $\mathbf{T}$ for $\mathcal{A}$ as graded ring to a standard family of tilting modules for $\mathcal{A}'$ as ungraded ring.

Observe that the ungrading functor associated to a graded stratified ring in 1.37 may be regarded as a special case of the construction of this subsection.

17.3. **Proof of Proposition 1.47.** We maintain the notation of the previous subsection. It will suffice to show the following.

Proposition. *Suppose that for each x, $\mathbf{N}_x$ is a Krull-Schmidt family over G and that the $P_{x,i}$ are chosen to be indecomposable. Assume also that for each x, i the natural isomorphism* $\operatorname{end}_{\mathcal{B}}(N_{x,i})^{\mathrm{op}} \to \operatorname{End}(FN_{x,i})^{\mathrm{op}}$ *(forgetting the grading on the first factor) maps the graded Jacobson radical of* $\operatorname{end}_{\mathcal{B}}(N_{x,i})^{\mathrm{op}}$ *isomorphically onto the ungraded Jacobson radical of* $\operatorname{End}(FN_{x,i})^{\mathrm{op}}$*. Then all statements of Proposition* 1.47 *hold for the ungrading functor F constructed in the previous subsection.*

Proof. The hypotheses imply that the underlying ungraded ring $\operatorname{End}(F_\dagger\Delta_{x,i})^{\mathrm{op}}$ of $R_{x,i}$ is a local ring with the (underlying ungraded ring of) $R_{x,i}/J_{x,i}$ as its residue ring. We claim that if there is an isomorphism $f\colon F_\dagger\Delta_{x,i} \cong F_\dagger\Delta_{x,j}$ then $i = j$. Indeed, write $f = \sum_{g\in G} f_g$ with f_g homogeneous of degree $g \in G$, and similarly write $h = f^{-1}$ as $h = \sum_g h_g$. Then the map $\sum_g f_g h_{g^{-1}}$ is the identity in the graded local ring $\operatorname{end}(\Delta_{x,j})^{\mathrm{op}}$ so some $f_g h_{g^{-1}}$ is a homogeneous unit in that ring. Hence the homomorphism f_g has a right inverse k of degree g^{-1}, satisfying $f_g k^{-1} = \mathrm{Id}$. The map kf_g is a homogeneous idempotent in the local ring $\operatorname{end}(\Delta_{x,i}\langle g\rangle)^{\mathrm{op}}$, hence it is the identity and we get an isomorphism $\Delta_{x,i}\langle g\rangle \cong \Delta_{x,j}$ which implies since $\mathbf{N}_x$ is a Krull-Schmidt family over G that $i = j$ and $g = 1_G$.

By 1.39 and the assumptions, it now follows that $\{F_\dagger\Delta_{x,i}\}_i$ is a Krull-Schmidt family (over the trivial group) and $\operatorname{Rad}\operatorname{End}(\{F_\dagger\Delta_{x,i}\}_i)^{\mathrm{op}}$ identifies with the underlying ungraded ideal of J_x in R_x. For $x \in \Gamma \in I$, the maps $g_{P_{x,i},y} \otimes_{R_y} \mathrm{Id}_{R_y/J_y}$ from

$$\hom_{\mathcal{C}}(\mathcal{A}_\Gamma e_{x,i}(y), \boldsymbol{\Delta}_y) \otimes_{R_y} R_y/J_y \to \operatorname{ext}^1_{\mathcal{C}}(\mathcal{A}_\Gamma e_{x,i}(<y), \boldsymbol{\Delta}_y) \otimes_{R_y} R_y/J_y$$

are isomorphisms of graded right R_y/J_y-modules by 1.40(ii). Forgetting the grading, this shows $\mathcal{A}(\Gamma)' e_{x,i}$ is indecomposable projective as an object of $\mathcal{C}'_\Gamma$ and

$\hat{F}: \hat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ maps each projective indecomposable object $P_{x,i}$ in $\mathcal{C}$ to a projective indecomposable object $\hat{F}P_{x,i}.$. Now 1.47(a)–(b) follow in a similar way to 1.46(a)–(b).

Now to prove 1.47(c), it is enough by 1.39 to show rad $e_{x,i}\mathcal{A}_\Gamma e_{x,i} = \text{Rad } e_{x,i}\mathcal{A}e_{x,i}$ for all x, i with $x \in \Gamma \in I_1$; this holds since by 1.40, the underlying abelian group of both sides is the kernel of the ring epimorphism

$$\text{end}(\mathcal{A}_\Gamma e_{x,i})^{\text{op}} \to \text{end}(\Delta_{x,i})^{\text{op}} = R_{x,i} \to R_{x,i}/J_{x,i}$$

induced by the natural epimorphism $\mathcal{A}_\Gamma e_{x,i} \to \Delta_{x,i}$.

Let M be in $\mathcal{A}_\Gamma$-mod for $\Gamma \in I_1$. Note that the unique (up to degree shift) simple graded $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module is simple even as ungraded $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module by (c). If $e_{x,i}M$ has a composition series as ungraded $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module then it is Artinian and Noetherian as ungraded module, hence also as graded module, and so has a composition series as graded $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module; on the other hand, a composition series as graded $e_{x,i}\mathcal{A}_\Gamma e_{x,i}$-module is also one as ungraded module, and 1.47(d) follows.

Finally we prove 1.47(e). If M is a graded $\mathcal{A}_\Gamma$-module, its socle in $\mathcal{E}$ is $\{\, m \in M | (\text{rad } \mathcal{A}_\Gamma)m \,\} = 0$ which equals $\{\, m \in M | (\text{Rad } \mathcal{A}_\Gamma)m \,\} = 0$ as ungraded module. Similarly, if M is f.g. one has rad $M = (\text{rad } \mathcal{A}_\Gamma)M = (\text{Rad } \mathcal{A}_\Gamma)M = \text{Rad } M$ as ungraded module. □

17.4. In this subsection, we consider another set of conditions under which indecomposability of projectives in $\hat{\mathcal{C}}$ is preserved by ungrading functors.

Suppose that $F: \mathcal{C} \to \mathcal{C}'$ is an ungrading functor constructed as in 17.2. We continue to suppose in this subsection that the $\mathbf{N}_x$ are Krull-Schmidt families and the $P_{x,i}$ are chosen to be indecomposable, but now we assume also that $\mathcal{C}$ is a k-category over G for some commutative unital graded ring k.

Proposition. *Assume that that each $e_{y,i}\Delta_{x,j}$ is a f.g. projective k-module and that there is some graded Artinian quotient ring k' of k such that for each non-zero f.g. projective k-module M one has $k' \otimes_k M \neq 0$ (for instance, k is graded local with trivially graded residue field k'). Then $\hat{F}(P_{x,i})$ is indecomposable in $\widehat{\mathcal{C}'}$.*

Proof. It is clearly enough to show that $FP_{x,i,\Gamma}$ is indecomposable in $\mathcal{C}'$ for all $\Gamma \in I_1$. For that, we may assume Ω is finite and $\Gamma = \Omega$. Then it is enough to show that $\mathcal{A}e_{x,i}$ is indecomposable as ungraded $\mathcal{A}$-module. For any M in $\mathcal{C}$, the assumptions imply $e_{y,j}\varphi(M)$ is a f.g. projective k-module; in particular, $e_{y,j}\mathcal{A}e_{x,i}$ is a f.g. projective k-module.

Suppose $\mathcal{A}e_{x,i} = Q_1 \oplus Q_2$ as an ungraded $\mathcal{A}$-module with $Q_i \neq 0$. Writing $M' = k' \otimes_k M$, one gets $\mathcal{A}e'_{x,i} = Q'_1 \oplus Q'_2$ where Q'_1, Q'_2 are non-zero by assumption on k'. On the other hand, $\text{end}(\mathcal{A}e'_{x,i})^{\text{op}} = e_{x,i}\mathcal{A}'e_{x,i}$ is f.g. as graded k'-module, hence it is a graded Artinian ring. As a (non-zero) quotient ring of $e_{x,i}\mathcal{A}e_{x,i}$, $e_{x,i}\mathcal{A}'e_{x,i}$ is graded local with trivially graded residue ring. By C.19, the underlying ungraded ring $\text{end}(\mathcal{A}e'_{x,i})^{\text{op}}$ of $e_{x,i}\mathcal{A}'e_{x,i}$ is local, contrary to decomposability of $\mathcal{A}e'_{x,i}$ as ungraded $\mathcal{A}$-module. □

17.5. Suppose in this subsection that k is a commutative, local ring over G with graded Jacobson radical J such that k/J is a trivially graded field. We assume that k is Noetherian as ungraded ring. Let k' denote the localization $k' := k_J$ of k at J, and denote base change $k' \otimes_k ?$ by $M \mapsto M'$.

Suppose given families of objects $\mathbf{N}_x$ in A-modfg so the full additive subcategories $\mathcal{C}_x := \operatorname{add} \mathbf{N}_x$ satisfy the conditions 1.4(i) and 1.4(ii). Then we have the stratified exact category $(\mathcal{C}, \{\mathcal{C}_x\}_{x\in\Omega})$ with $\mathcal{C}$ a full additive subcategory of A-modfg as constructed in 1.7. By forgetting the grading and applying flat base change $k'\otimes_k ?$, we have also a split exact category $(\mathcal{C}', \{\operatorname{Add} \mathbf{N}'_x\}_{x\in\Omega})$ with $\mathcal{C}'$ a full additive subcategory of A-Modfg, where $\mathbf{N}'_x = \{N'_{x,i}\}_i$.

If we choose a standard family of projective pro-objects $\mathbf{P} = \{P_{x,i}\}_{x,i}$ in $\hat{\mathcal{C}}$, then the obvious family $\mathbf{P}' = \{P'_{x,i}\}_{x,i}$ with $(P'_{x,i})_\Gamma = P'_{x,i,\Gamma}$ is a standard family of projective pro-objects in $\hat{\mathcal{C}}'$. We have corresponding abelian categories $\hat{\mathcal{E}}$ contained in $\mathcal{A}$-mod and $\hat{\mathcal{E}}'$ contained in $\mathcal{A}'$-Mod where $\mathcal{A}' := \varprojlim_\Gamma \mathcal{A}'_\Gamma$. The corresponding $\mathbf{\Delta}$-modules $\Delta'_{x,i}$ for $\mathcal{C}'$ are obtained by forgetting the grading and applying base change to the $\mathbf{\Delta}$-modules $\Delta_{x,i}$ i.e. $\Delta'_{x,i} = k' \otimes_k \Delta_{x,i}$.

Proposition. (a) *The $\mathbf{\Delta}$-module weightspaces $e_{y,j}\Delta_{x,i}$ are f.g. graded projective k-modules iff the weightspaces $e_{y,j}\Delta'_{x,i}$ are f.g. projective k'-modules.*

(b) *Suppose each I_x is a singleton, that the structural morphism $k \to R_x$ is an isomorphism, and that the $P_{x,i}$ are chosen indecomposable in $\hat{\mathcal{C}}$. Then $P'_{x,i}$ is indecomposable in $\widehat{\mathcal{C}'}$.*

Proof. The first claim follows immediately from C.21.2, since $e_{y,j}\Delta_{x,i}$ is a f.g. k-module. The second claim follows using condition 1.40(ii). □

Appendix A. Poset terminology

A.1. Let Ω be a poset (partially ordered set). For any $x \le y$ in Ω, we define the (closed) interval $[x, y] := \{\, z \in \Omega | x \le z \le y \,\}$. We say that Ω is interval finite if each interval $[x, y]$ with $x \le y$ in Ω is a finite set.

If $*$ denotes one of the relations $=, \ge, \le, >, <, \ne, \not\ge$ etc on Ω, we sometimes abuse notation by writing $*x$ or $(*x)$ in place of $\{\, z \in \Omega | z * x \,\}$, for $x \in \Omega$; for example, we commonly write $M(\ge x)$ instead of $M(\{\, z \in \Omega | z \ge x \,\})$. We also commonly write x instead of the singleton $\{x\}$.

The opposite poset Ω^{op} coincides with Ω as a set, but is endowed with an order $\le^{\mathrm{op}}$ satisfying $x \le y$ iff $y \le^{\mathrm{op}} x$.

We give Ω the topology for which the sets $\le x$ for $x \in \Omega$ form a basis. The open (resp., closed) sets of this topology are called order ideals or just ideals (resp., order coideals or coideals) of Ω. The intersection of the ideals (resp., coideals) of Ω which contain a subset $\Gamma \subseteq \Omega$ is called the ideal (resp., coideal) of Ω generated by Γ; the closure of a subset coincides with the coideal it generates. An ideal (resp., coideal) is called finitely generated if it is generated by some finite subset of Ω. We write "f.g." for short for "finitely generated" (we also refer to f.g. modules, f.g. projectives etc).

We often have occasion to refer to locally closed subsets of Ω i.e. subsets which are open in their closure, or equivalently, those which are the intersection of an ideal and a coideal. A subset Γ of Ω is locally closed iff for all $x, y \in \Gamma$ and $z \in \Omega$ with $x \le z \le y$, one has $z \in \Gamma$.

A.2. We sometimes regard a poset Ω as the set of objects of a small category, with a unique morphism from x to y if $x \le y$ in Ω and no other morphisms besides these. An inverse (resp., direct) system on a category A (e.g. Ω) with values in a category C is then defined to be a contravariant (resp., covariant) functor $A \to C$.

Usually, we denote an inverse or direct system $F\colon A \to C$ simply as $\{F(x)\}_{x\in A}$, suppressing notation for the values of the functor F on the morphisms of A when these are obvious from the context. Of course, direct systems on A are the same as inverse systems on the opposite category A^{op}.

One defines limits and colimits of inverse and direct systems as usual; $\varprojlim F$ (resp., $\varinjlim F$) is a terminal (resp., initial) object in the category with objects consisting of an object c of C with morphisms $c \to F(x)$ (resp., $F(x) \to c$) for all objects x in A satisfying the obvious compatibility conditions.

Appendix B. Exact categories

We record here for the reader's convenience the axioms and basic facts we shall use concerning exact categories in the sense of Quillen [45]. Most important for our applications is the functoriality of the Gabriel-Quillen embedding of a small exact category in an abelian category.

B.1. We begin with some general terminology we shall use concerning categories. A full subcategory B of a category A is called a strict subcategory if any object of A isomorphic to an object of B is also in B; the strict image of a functor $F\colon A \to B$ is the smallest strict full subcategory of B containing all objects $F(a)$ for a in A. A category is called small if it has only a set of objects, and it is called svelte if it has only a set of isomorphism classes of objects.

We say that idempotents split in an additive category C if each idempotent endomorphism e of an object M has a kernel (so e is the composite morphism $M \to \ker(\mathrm{Id} - e) \to M$ induced by the projections and inclusions of a direct sum decomposition $M = \ker e \oplus \ker(\mathrm{Id}_M - e)$).

B.2. Let C be an additive category endowed with a class D of sequences

(B.2.1) $$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$$

in $\mathcal{C}$ to be called short exact sequences. One calls the maps i (resp., j) occurring in some member of D an admissible monomorphism (rep., admissible epimorphism). (Some references e.g. [37] call the maps i inflations, the maps j deflations and pairs (i, j) conflations). We say that C is an exact category (with short exact sequences D) if the following self-dual system of axioms holds:

(i) Any sequence (B.2.1) in C isomorphic to a sequence in D is in D.
(ii) For M, M' in C, the split exact sequence $0 \to M \to M \oplus M' \to M' \to 0$ is in D.
(iii) For (B.2.1) in D, $i = \ker\ j$ and $j = \mathrm{coker}\ i$ in C.
(iv) Admissible epimorphisms are closed under composition. Dually for admissible monomorphisms.
(v) Admissible epimorphisms are closed under base change by arbitrary maps in C. Dually for admissible monorphisms.

Note (i) and (ii) above trivially imply that Id_0 is an admissible epimorphism. Hence by [37, Appendix A], axioms (i)–(v) imply that

B.2.2. If a map $M \to M'$ in C has a kernel in C and there is a map $N \to M$ in C such that $N \to M \to M'$ is an admissible epimorphism, then $M \to M'$ is an admissible epimorphism, and dually.

The axioms for an exact category in the sense of Quillen [45] are precisely (i)–(v) and B.2.2, so the axioms (i)–(v) are equivalent to those of Quillen. (Note that by [37, Appendix A], axioms (i)–(v) still contain some redundancy; we mention also [16, Appendix A], which gives a system of axioms for exact categories in which all retractions admit kernels).

B.3. We explicitly record the following alternative description of pullbacks in exact categories (cf. the proof of [37, Proposition A.1]). Let C be an exact category, $f\colon M \to N$ be an admissible epimorphism and $k\colon L \to N$ be an arbitrary map. The axioms imply there is a commutative square with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & K & \xrightarrow{g} & L & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & H & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

in which the right hand square is a pullback square.

B.3.1. The pullback square is determined up to isomorphism by the requirement that $(g, -h)^t$ is a kernel of the admissible epimorphism (k, f).

To prove this, one has only to see that the sequence

$$0 \to K \xrightarrow{(g,-h)^t} L \oplus M \xrightarrow{(k,f)} N \to 0$$

is exact in C; this holds by B.2.2, since (k, f) has a kernel $(g, -h)^t$ by definition of pullbacks and the composite map $f\colon M \xrightarrow{(0,1)^t} L \oplus M \xrightarrow{(k,f)} N$ is an admissible epimorphism.

B.4. An admissible subobject of an object M of an exact category is by definition an isomorphism class of admissible monomorphisms $M' \to M$; we usually pick one monomorphism from the isomorphism class and write $M' \subseteq M$ or $M \supseteq M'$, calling M' itself a subobject by abuse of terminology. We write M/M' for a cokernel (object) of $M' \to M$. One subobject M'' of M is contained in another subobject M'' of M, written $M'' \subseteq M' \subseteq M$, if $M'' \to M$ factors as $M'' \to M' \to M$ for some (necessarily unique) admissible monomorphism $M'' \to M'$. Finite decreasing filtrations $M = M^0 \supseteq M^1 \supseteq \ldots \supseteq M^n = 0$ (or increasing filtrations) of an object M are defined similarly. Admissible quotient objects are defined dually to admissible subobjects. The admissible subobjects of admissible quotients of an object M of an exact category coincide with the admissible quotient objects of admissible subobjects of M; we call these objects the admissible subquotients of M.

B.5. An additive subcategory A of an exact category C is said to closed under extensions if in a short exact sequence (B.2.1) in C, M is in A whenever both M' and M are in A.

In an exact category C, if a map $f\colon M \to N$ factorizes as $f = uv$ where u is an admissible monomorphism and v is an admissible epimorphism, then this factorization is unique up to isomorphism. One says that a sequence $M \xrightarrow{f} N \xrightarrow{g} P$ in C is exact (at N) in C if there exist an admissible epimorphism $M \xrightarrow{f_1} L_1$, an admissible monomorphism $L_2 \xrightarrow{g_1} P$ and a short exact sequence

$$0 \to L_1 \xrightarrow{f_2} N \xrightarrow{g_2} L_2 \to 0$$

such that $f = f_2 f_1$ and $g = g_1 g_2$. Exactness of longer sequences is defined in the usual way. For example, a sequence (B.2.1) in C is exact in this sense iff it is in D. An exact sequence $M' \to M \to M'' \to 0$ (resp., $0 \to M' \to M \to M''$) is called a left (resp., right) exact sequence.

An additive functor $F\colon C \to D$ between exact categories is said to reflect exactness provided that a sequence (B.2.1) in C is exact in C whenever F(B.2.1) is exact in D. We shall say F is perfectly exact if F is full and faithful, exact, reflects exactness and the strict image of F is closed under extension in D. If F is a perfectly exact inclusion functor, we say C is a perfectly exact subcategory of D.

An equivalence of exact categories is a category equivalence which is exact and reflects exactness (i.e. the equivalence and an inverse equivalence are both exact).

B.6. A covariant functor $F\colon C \to D$ between exact categories is called right exact (resp., left exact) if it is additive and carries short exact sequences in C to right exact (resp., left exact) sequences in D. The axioms imply that F is left (resp., right) exact iff it carries left exact (resp., right exact) sequences in C to left (resp., right) exact sequences in D; in particular, composites of left (resp., right) exact functors are left (resp., right) exact. These definitions are extended to contravariant functors $C \to D$ by regarding such functors as covariant functors $C^{\text{op}} \to D$. For example, the covariant (resp., contravariant) functor $\operatorname{Hom}_C(M, ?)$ (resp., $\operatorname{Hom}_C(?, M)$) is left exact for any object M of C.

An object M of an exact category C is called projective if $\operatorname{Hom}(M, ?)$ is an exact functor from C to abelian groups; equivalently (cf [16, Lemma 2.1]), every admissible epimorphism $p\colon P \to M$ is split (i.e. there is a map $s\colon M \to P$ with $ps = \mathrm{Id}_M$). One says C has sufficiently many projectives if for each M in C there is an admissible epimorphism $P \to M$ with P projective. The full subcategory of projective objects of C is closed under formation of finite direct sums and taking direct summands. Injective objects etc in exact categories are defined dually.

The following technical fact is included here simply to make the compatibility of terminology between [52, Appendix A] and [37, Appendix A] more explicit.

B.6.1. For an exact category C, a contravariant additive functor $G\colon \mathcal{C} \to \mathbb{Z}\text{-Mod}$ is left exact iff for each admissible epimorphism $g\colon E \to F$ in C, the two canonical projections $p_0, p_1\colon L \to E$ from the pullback $L = E \times_F E$ to E give a left exact sequence $0 \to G(F) \xrightarrow{G(f)} G(E) \xrightarrow{h} G(L)$ of abelian groups, where $h = G(p_0 - p_1)$.

To see this, first form the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & L & \xrightarrow{p_1} & E & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & E & \xrightarrow{g} & F & \longrightarrow & 0
\end{array}$$

The top row is split, giving an isomorphism $(i, \Delta)\colon A \oplus E \to L$ where $\Delta\colon E \to L$ is the diagonal map given by $p_0\Delta = p_1\Delta = \mathrm{Id}_E$. Using this isomorphism, the sequence $L \xrightarrow{p_0 - p_1} E \xrightarrow{g} F \to 0$ identifies with $A \oplus E \xrightarrow{(f,0)} E \xrightarrow{g} F \to 0$. Hence $G(g)$ is a kernel of $G(p_0 - p_1)$ in $\mathbb{Z}$-Mod iff $G(g)$ is a kernel of $G(f)$, and the assertion follows.

B.7. Parts (a)–(c) of the following proposition (cf [45], [37, Appendix A]) indicate some of the most common ways in which exact categories arise in practice. The Gabriel-Quillen embedding theorem ([25], [45]) described in the next subsection implies that any svelte exact category is equivalent to one arising as in (c), and then (d) often permits one to establish results in exact categories using standard diagram lemmas (9-lemma, 5-lemma, snake lemma etc) for abelian categories.

Proposition. (a) *Any abelian category with the usual short exact sequences is an exact category.*

(b) *Any additive category with the split exact sequences as short exact sequences is an exact category.*

(c) *Any full, extension closed additive subcategory of an exact category has a natural structure of exact category, so the inclusion is perfectly exact.*

(d) *An additive category C with a class D of short exact sequences is an exact category iff for each svelte subcategory A of C there is a svelte exact category B with exact sequences E such that B is a full subcategory of C containing A and E is a subclass of D containing each short exact sequence $0 \to L \to M \to N \to 0$ in C with L, M and N in A.*

We say that an exact category is split if each of its short exact sequences is split exact i.e. all of its objects are projective (and injective). A Serre subcategory B of an exact category C is defined to be a full additive subcategory B of C which is closed under extensions and formation of admissible subquotients in C; we then regard B as a perfectly exact subcategory of C.

B.8. **Gabriel-Quillen embedding theorem.** For any small exact category C, let let $\tilde{C}$ be the abelian category of contravariant additive functors $F\colon C \to \mathbb{Z}$-Mod, with natural transformations as morphisms, and let C^* denote the full additive subcategory of $\tilde{C}$ consisting of left exact contravariant functors.

Theorem. (a) *The category C^* is an abelian category with all limits, exact filtered colimits, and a generator (so C^* is therefore a "Grothendieck category"; it has injective envelopes, an injective cogenerator and is well-powered i.e. there is only a set of subobjects of any object).*

(b) *The Yoneda representation functor $\phi_C = \phi\colon C \to C^*$ which is determined by $\phi(M) = \mathrm{Hom}_C(?, M)$ is perfectly exact.*

Proof. Proofs of this can be found in [52, Appendix A] (using the equivalent characterization of left exact functors given in B.6.1) and [37, Appendix A]. We recall in outline the proof sketched in [37], emphasizing some additional points which we shall require.

First, $\tilde{C}$ itself is an abelian category with all limits, exact filtered colimits and a family of generators $\mathrm{Hom}(?, M)$ for M in C (i.e. for any F in $\tilde{C}$, $\mathrm{Hom}(\phi(M), F) \cong F(M)$ is non-zero for some M in C). Limits and colimits in $\tilde{C}$ are computed pointwise. Limits (e.g. kernels or products) and direct sums in $\tilde{C}$ of left exact functors are also left exact, so C^* has all limits and direct sums, and they are computed pointwise as well.

A functor F in $\tilde{C}$ is called effaceable if for each M in C and each $x \in F(M)$, there is an admissible epimorphism $f\colon N \to M$ in C such that $F(f)(x) = 0$. One checks that the full subcategory of effaceable functors forms a Serre subcategory of $\tilde{C}$ and the corresponding quotient (abelian) category [25] is equivalent to C^*. Letting

$j_*\colon C^* \to \tilde{C}$ denote the inclusion functor, the quotient functor provides an exact left adjoint $j^*\colon \tilde{C} \to C^*$ and the counit $j^*j_* \to \mathrm{Id}$ is an isomorphism. Given $\eta\colon F \to G$ in C^*, the cokernel of η in C^* may be calculated as $\operatorname{coker}_{C^*}(\eta) = j^*(\operatorname{coker}_{\tilde{C}}(\eta))$.

The fact that ϕ is full and faithful follows from the Yoneda lemma. In verifying that ϕ is perfectly exact, one establishes the following (cf [52, A.7.15])

B.8.1. For F in C^* and M in C, a map $f\colon F \to \phi(M)$ is an epimorphism in C^* iff there is an admissible epimorphism $h\colon N \to M$ in C and a map $g\colon \phi(N) \to F$ such that $\phi(h) = fg$.

The assertions that C^* has exact filtered colimits and a generator follow from the analogous facts for $\tilde{C}$ by general facts about quotient categories [25]; the other claims on the nature of the category C^* are well-known consequences of these facts together with the existence of limits in C^* (cf loc cit and [40]). □

B.9. In general, let us say that an object M of a cocomplete abelian category B is small if $\mathrm{Hom}_B(M, ?)$ preserves infinite direct sums. Observe that for any object M of a small exact category C, $\phi(M)$ is small since by the Yoneda lemma,

$$\mathrm{Hom}(\phi(M), \oplus F_i) \cong (\oplus F_i)(M) \cong \oplus(F_i(M)) \cong \oplus \mathrm{Hom}(\phi(M), F_i).$$

Corollary. (a) *If P is a projective object of C, then $\phi(P)$ is a small projective in C^*.*

(b) *If $\mathbf{Q} = \{Q_i\}_i$ is a family of projective objects of C such that every object of C is an admissible quotient of an object of* $\operatorname{Add}\mathbf{Q}$, *then $\phi(\mathbf{Q}) := \{\phi(P_i)\}_i$ is a generating set of small projective objects of C^*.*

Proof. Using B.8.1 and the fact that admissible epimorphisms $M \to P$ in C are split, one sees that any epimorphism $f : A \to \phi(P)$ in C^* is split, so (a) holds. For (b), it is enough to show that given A in C^*, there is a non-zero map $\phi(P') \to A$ for some P' in $\mathbf{Q}$; but there is a non-zero map $\phi(M) \to A$ for some object M of C and an admissible epimorphism $P \to M$ with P in $\operatorname{Add}\mathbf{Q}$, and then the composite $\phi(P) \to \phi(M) \to A$ is non-zero. □

Remarks. See C.8.

B.10. **Extensions in exact categories.** Given an exact category C, one may define Yoneda Ext-groups $\mathrm{Ext}^i_C(M, N)$ for M, N in C as equivalence classes of i-fold extensions as for abelian categories (see e.g. [40] and [30]), provided the classes of extensions form sets, as we assume (this assumption always holds for svelte C). The Yoneda Ext-groups for exact categories have the usual properties of Yoneda Ext in abelian categories. If C has enough projectives (resp., injectives) the Yoneda Ext-groups exist and can be computed as usual using projective (resp., injective) resolutions in the first (resp., second variable). Proofs of these assertions given for abelian categories in [40] for instance apply mutatis mutandis to exact categories. (One may even define derived categories $D^*(C)$, for $* \in \{\emptyset, +, -, b\}$ if C is svelte and has split idempotents [43], though we shall not use this).

An exact functor $F\colon A \to C$ between exact categories induces a natural map on i-fold extensions and hence natural maps $\mathrm{Ext}^i_A(M, N) \to \mathrm{Ext}^i_C(FM, FN)$. If F is a perfectly exact functor, these maps are isomorphisms for $i \le 1$.

Lemma. *If C is a small exact category, then the natural maps $\mathrm{Ext}^i_C(M, N) \to \mathrm{Ext}^i_{C^*}(\phi(M), \phi(N))$ for M, N in C are isomorphisms.*

Proof. If C has enough projectives, this follows since one may choose a projective resolution $P^\bullet \to M \to 0$ in C and compute

$$\mathrm{Ext}^i(M,N) = \mathrm{H}^i(\mathrm{Hom}(P^\bullet, N) = \mathrm{H}^i(\mathrm{Hom}(\phi(P^\bullet), \phi(N)) = \mathrm{Ext}^i(\phi(M), \phi(N))$$

noting $\phi(P^\bullet) \to \phi(M) \to 0$ is a projective resolution in $\mathcal{C}^*$ since ϕ is exact and preserves projectives. This case is adequate for the applications in this paper.

In general, for $i \leq 1$ this follows since ϕ is perfectly exact. For $i \geq 1$, use the terminology of "morphisms with fixed ends between i-fold extensions" and related facts from [40]. One first deduces from B.8.1, that if E is an i-fold extension of $\phi(M)$ by $\phi(N)$, there is an i-fold extension E' of M by N in $\mathcal{C}$ and a morphism $\phi(E') \to E$ with fixed ends. This implies at once that the given maps between Ext^i-groups are epimorphisms. To show they are monomorphisms, suppose E is an i-fold extension of M by N in C such that the class of $\phi(E)$ is zero. By [40], there are morphisms of i-fold extensions $0 \leftarrow F \to \phi(E)$ with fixed ends where

$$0\colon 0 \to \phi(N) \to \phi(N) \to 0 \to \cdots \to 0 \to \phi(M) \to \phi(M) \to 0$$

represents the zero element of $\mathrm{Ext}^i(M,N)$. By above, there is a morphism $\phi(F') \to F$ with fixed ends and hence morphisms $0 \xleftarrow{f'} \phi(F') \xrightarrow{g'} \phi(E)$ with fixed ends. Since ϕ is perfectly exact, this gives a sequence of morphisms $0 \xleftarrow{f} F' \xrightarrow{g} E$ in C with fixed ends which satisfies $\phi(f) = f'$ and $\phi(g) = g'$. This implies the class of E is zero. ☐

B.11. For the following facts about splitting idempotents in exact categories, cf [52, A.9.1].

Any additive category C has a "Karoubianization" $F\colon C \to C'$ i.e. an additive functor F from C to an additive category C' in which all idempotents split, such that any additive functor $G\colon C \to D$ where D is additive with splitting idempotents factors as $G = G'F$ for some additive functor $G'\colon C' \to D$ which is unique up to natural isomorphism. Moreover, F is full and faithful, and any object in C' is a direct summand of an object of the strict image of F. (Because of a misprint in [52, A.9.1], we recall the construction of F. The objects of C' are pairs (M,e) with M an object of C and e an idempotent morphism of M. A morphism $g\colon (M,e) \to (N,f)$ in C' is a morphism $g\colon M \to N$ in C satisfying $fg = ge = g$, and F is determined on objects by $F(M) = (M, \mathrm{Id}_M)$ for M in C.)

If C is an exact category, C' has a natural structure of exact category making F perfectly exact (regarding F as an inclusion, the exact sequences in C' are those which are direct summands of exact sequences in C). If C is a small exact category, one can take C' to be the full additive subcategory of all direct summands of objects in the strict image of $\phi\colon C \to C^*$, and $F\colon C \to C'$ to be the restriction of ϕ (as follows from the proof in loc cit).

Remarks. If C is small with splitting idempotents, then by [52, A.7.16], a morphism $f\colon M \to N$ in C is an admissible epimorphism iff $\phi(f)\colon \phi(M) \to \phi(N)$ is an epimorphism in C^*.

B.12. **Functoriality of the Gabriel-Quillen embedding.** In the next several subsections, we discuss functoriality of the Gabriel-Quillen embedding with respect to right exact functors between small exact categories (cf [52, A.8.2], where the discussion is limited to exact functors).

Suppose that $T\colon C \to D$ is a covariant right exact functor between small exact categories C and D. For F in D^*, the composite (contravariant) functor $T^*(F) := FT\colon C \to \mathbb{Z}$ is left exact i.e. is an object of C^*, by the discussion in B.5. Given a morphism (natural transformation) $\eta\colon F \to G$ in D^*, define the natural transformation $T^*(\eta) = \eta T\colon F^* \to G^*$ with components $T^*(\eta)_c = \eta_{Tc}$. This makes $T^*\colon D^* \to C^*$ into a functor. We claim that T^* is left exact. Indeed, suppose that $0 \to H \xrightarrow{\eta} K \xrightarrow{\epsilon} L$ is a left exact sequence in D^* i.e. for any object d of D, $\eta_d = \ker \epsilon_d$ in $\mathbb{Z}$-Mod. Then for an object c of C, $\eta_{Tc} = \ker \epsilon_{Tc}$ which implies that $0 \to T^*H \xrightarrow{T^*\eta} T^*K \xrightarrow{T^*\epsilon} T^*L$ is exact in C^*.

Now given another right exact functor $S\colon D \to E$ between small exact categories, one has $(ST)^* = T^*S^*$. To summarize,

B.12.1. The assignment $X \to X^*$ described above defines a contravariant functor from the category of small exact categories and right exact functors to the "category" of abelian categories and left exact functors (we write "category" for a category possibly without small Hom-sets).

It is not difficult to check the following fact (cf [40, V.4])

B.12.2. If $S\colon D \to C$ is left adjoint to $T\colon C \to D$ and both S, T are right exact, then S^* is left adjoint to T^*. If the unit $1 \to TS$ (resp., counit $ST \to 1$) of the adjoint pair (S,T) is a natural isomorphism (and hence S (resp., T) is a full embedding) then the unit (resp., counit) of the adjoint pair (S^*,T^*) is a natural isomorphism.

In fact, the triangular identities for the unit and counit of the adjoint pair (S,T) imply the triangular identities for (S^*,T^*). We also record:

B.12.3. Let $R\colon C \to D$ be another right exact functor and $\nu\colon R \to T$ be a natural transformation. For $F \in D^*$, one has a natural transformation $F\nu\colon T^*(F) \to R^*(F)$ of functors in C^* with components $(F\nu)_c = F(\nu_c)\colon FT(c) \to RT(c)$. Moreover, these natural transformations $F\nu$ for $F \in D^*$ form the components of a natural transformation $\nu^*\colon T^* \to R^*$.

B.13. The Proposition below follows from the special adjoint functor theorem exactly as in [52, A.8.2], so we do not repeat the proof.

Proposition. *Suppose that $T\colon C \to D$ is a right exact functor between small exact categories. Then $T^*\colon D^* \to C^*$ has a left adjoint $T_*\colon C^* \to D^*$ and the diagram*

$$\begin{array}{ccc} C & \xrightarrow{F} & D \\ \downarrow{\scriptstyle \phi_C} & & \downarrow{\scriptstyle \phi_D} \\ C^* & \xrightarrow{F_*} & D^* \end{array}$$

is commutative up to natural isomorphism.

B.14. It follows from the essential uniqueness of adjoints that the maps $C \mapsto C^*$ and $T \mapsto T_*$ defines a covariant functor from the category of small exact categories with right exact functors (up to natural isomorphism) as morphisms to the "category" of abelian categories with right exact functors (up to natural isomorphism) as morphisms. Properties of adjoints also imply that a natural transformation $\eta\colon S \to T$ between right exact functors $C \to D$ induces a natural transformation $\eta_*\colon S_* \to T_*$ (the "conjugate" of $\eta^*\colon T^* \to S^*$, see [38]) in a natural way.

B.15. We find it convenient to extend $T \mapsto T^*$ and $T \mapsto T_*$ to functors defined on the category of svelte exact categories, with morphisms right exact functors up to natural isomorphism. This can be accomplished by choosing for each svelte exact category C an equivalent small exact category C_0 and defining $C^* := C_0^*$. By abuse of notation, we write ϕ_C for the composite $C \xrightarrow{\cong} C_0 \to C_0^* = C^*$ and still call ϕ_C the Gabriel-Quillen embedding. Similarly, if D is a svelte exact category equivalent to the small exact category D_0, replace a right exact functor $T\colon C \to D$ with a right exact functor $T_0\colon C_0 \to D_0$ compatible with T under the equivalences $C \cong C_0$, $D \cong D_0$ and set $T^* = (T_0)^*$, $T_* = (T_0)_*$. In the body of the paper, rather than do this explicitly, we always tacitly assume that any svelte exact category we consider is to be replaced by an equivalent small exact category.

B.16. **Grothendieck Group.** Let C be a svelte exact category. The Grothendieck group $\mathrm{K}_0(C)$ of C is defined to be the abelian group with generators $[M]_C$ for (isomorphism class of) objects M of C with relations $[M]_C = [M']_C + [M'']_C$ for each short exact sequence $0 \to M' \to M \to M'' \to 0$ in C. We often abbreviate $[\ ]_C$ by $[\ ]$ when the category C is fixed.

An exact functor $F\colon C \to D$ between exact categories induces a group homomorphism $\mathrm{K}_0(F)\colon \mathrm{K}_0(C) \to \mathrm{K}_0(D)$ determined by $[M]_C \mapsto [FM]_D$.

Appendix C. Appendix: Diagonalizable rings and modules, and categories with automorphisms

This appendix contains a more detailed discussion of categories over G and diagonalizable G-graded rings and modules, as introduced in 1.10. Generally, we do not provide proofs when these are straightforward modifications of proofs of corresponding facts about graded or ungraded unital rings as can be found in standard references such as [12] or [2]. However, the following remarks may be helpful to the reader. Many general facts about unital G-graded rings can be found in [42]. The notion of trivially graded J-diagonalizable ring is equivalent to the notion of a small additive category with J as its set of objects, in such a way that diagonalizable left (resp., right) modules correspond to covariant (resp., contravariant) functors on the category with values in abelian groups; the references [26] and [41] actually give, in this alternative language, most of the facts we use about trivially graded diagonalizable rings and their modules. In general, for a G-graded J-diagonalizable ring A, the category A-mod of graded diagonalizable A-modules is equivalent to the category B-Mod of diagonalizable B-modules for some ungraded $G \times J$-diagonalizable ring B (see C.6). Most of the results we give for G-graded diagonalizable rings are actually equivalent to their special cases for trivially graded diagonalizable rings, so the reader may wish to assume $G = \{\mathrm{Id}\}$ in reading the paper. However, we state most of our results in the graded setting since the description in terms of B-Mod loses track of the natural automorphisms provided by grading shifts on A-mod (which are essential for deeper study of the motivating examples) and is inconvenient for applications in that it uses the unnecessarily "large" ring B.

C.1. **Categories over G.** Let B be an additive category, G a multiplicative group (always abelian in applications, though we do not assume this) and $\{T_g\}_{g\in G}$ be a fixed family of functors $T_g\colon B \to B$ with $T_e = \mathrm{Id}_B$ and $T_gT_h = T_{gh}$ (we say for short that B is an additive category over G). For a class C of objects in B, or a subcategory C of B, $\mathrm{add}_B\, C = \mathrm{add}\, C$ denotes the full additive subcategory over G

of objects M in B such that $M \oplus N \cong \oplus_{i=1}^{n} M_i\langle g_i\rangle$ for some objects M_i in C, N in B and some $g_i \in G$.

For M, N in B, we define we define the G-graded $\mathbb{Z}$-module $\hom_B(M,N)$ with $\hom(M,N)_g = \operatorname{Hom}_{\mathcal{B}}(M, T_g N)$. One has associative, bilinear composition maps

$$\hom(N,P)_g \times \hom(M,N)_h \to \hom(M,P)_{hg}$$

induced by the usual maps

$$\operatorname{Hom}(T_h N, T_h T_g P) \times \operatorname{Hom}(M, T_h N) \to \operatorname{Hom}(M, T_{hg} P).$$

These maps make $\operatorname{end}(M)^{\mathrm{op}} := \hom(M,M)^{\mathrm{op}}$ a G-graded ring and $\hom(M,N)$ a graded $(\operatorname{end}(M), \operatorname{end}(N))$-bimodule.

We say that B is an abelian or exact category over G if the T_g are exact functors. We may than define similarly $\operatorname{ext}^i_B(M,N)$ with $\operatorname{ext}^i(M,N)_g = \operatorname{Ext}^i_B(M\langle g\rangle, N)$ and associative, bilinear "graded Yoneda product" maps

$$\operatorname{ext}^j(N,P)_g \times \operatorname{ext}^i(M,N)_h \to \operatorname{ext}^{i+j}(M,P)_{hg}$$

induced by the usual Yoneda product.

A covariant functor F between categories over G will be called a functor over G if $FT_g = T_g F$ for all F.

Remarks. Instead of assuming strict equalities $T_g T_h = T_{gh}$, $T_{1_G} = \mathrm{Id}$ in the definition of category over G, it would be enough to require the existence of "coherent" natural isomorphisms $T_g T_h \cong T_{gh}$ and $T_{1_G} \cong \mathrm{Id}_B$. The coherence conditions stipulate that for all g, h, k in G, the evident composite natural isomorphisms $T_g T_h T_k \to T_{gh} T_k \to T_{ghk}$ and $T_g T_h T_k \to T_g T_{hk} \to T_{ghk}$ coincide, that $T_g T_e \cong T_g \mathrm{Id} = T_g$ coincides with $T_g T_e \cong T_{ge} = T_g$ and that $T_e T_g \cong \mathrm{Id} T_g = T_g$ coincides with $T_e T_g = T_{eg} = T_g$. Similarly, one can extend the notion of functor over G.

All the constructions and results of this paper extend to categories and functors over G in this more general sense. Though this extension may be strictly necessary in some applications, we leave details to the interested reader.

C.2. **Diagonalizable rings and modules.** Let J be a set and G be a group. We identify the opposite group G^{op} with G as a set. By a G-graded, J-diagonalizable ring A, we mean a (possibly non-unital, associative) G-graded ring A ($A = \oplus_g A_g$, $A_g A_h \subseteq A_{gh}$) with a given family $\{e^A_j\}_{j\in J} \subseteq A_{1_G}$ of homogeneous orthogonal idempotents of A such that $A = \oplus_{i,j} e^A_i A e^A_j$. We usually abbreviate $e_i = e^A_i$. The family $\{e_j\}_{j\in J}$ is regarded as part of the data defining A, but we often will not introduce a specific notation for the index set J unless required for clarity. The opposite ring A^{op} has a natural structure of G^{op}-graded, J-diagonalizable ring with $(A^{\mathrm{op}})_g = A_g$ as sets. A homomorphism $A \to B$ of G-graded J-diagonalizable rings is a ring homomorphism $f\colon A \to B$ satisfying $f(A_g) \subseteq B_g$ and $f(e^A_j) = e^B_j$.

Let A be any G-diagonalizable, J-diagonalizable ring A. Note that if $\{f_k\}_k$ is another family of orthogonal idempotents of A such that $A = \oplus f_k A f_l$ and $e_i f_k = f_k e_i$ for all i, k, then a graded A-module M satisfies $M = \oplus_i e_i M$ iff it satisfies $M = \oplus_k f_k M$, since both conditions are equivalent to $A = \oplus_{i,k} e_i f_k M$. A G-graded A-module M (i.e. $M = \oplus_{g\in G} M_g$ with $A_h M_g \subseteq M_{hg}$) will be called diagonalizable if $M = \oplus_j e_j M$. Graded modules M over A not necessarily satisfying $M = \oplus_{i\in J, g\in G} e_i M_g$ will be called possibly non-diagonalizable graded A-modules. We denote the full abelian subcategory of the category of G-graded A-modules consisting of the diagonalizable G-graded A-modules by A-mod. A morphism $f\colon M \to N$ in A-mod is an A-module homomorphism preserving the G-grading i.e. $f(M_g) \subseteq N_g$. In general, a G-graded diagonalizable ring A has an underlying diagonalizable ring (forgetting the G-grading on A) and we shall write A-Mod for the category of diagonalizable A-modules.

The category A^{op}-mod will frequently be identified with the category of G-graded right A-modules M satisfying $M = \oplus_j Me_j$.

Observe that if J is finite, then $1_A = \sum_{j\in J} e_j$ is an identity element of A, that homomorphisms of G-graded, J-diagonalizable rings preserve their identity elements and that A-mod (resp., A-Mod) is the usual category of unitary G-graded A-modules.

A graded A-module is said to be graded Noetherian if it satisfies the ascending chain condition in A-mod, and A itself is said to be (left) Noetherian if finitely generated graded A-modules are graded Noetherian (or equivalently, Ae_i is graded Noetherian for all i). Recall that

C.2.1. If $M' \to M' \to M''$ is an exact sequence in A-mod for a ring A, then M is Noetherian if M' and M'' are both Noetherian.

C.3. Let A be a G-graded, J-diagonalizable ring. For $g \in G$, there is an exact grade-shift functor $T_g\colon A\text{-mod} \to A\text{-mod}$ defined on objects by setting T_gM equal to M as ungraded A-module, with gradation $(T_gM)h = M_{hg}$. Then $T_e = \mathrm{Id}$ and $T_gT_h = T_{gh}$. We often write $M\langle g\rangle = T_{g^{-1}}M$. The family $\{T_g\}_{g\in G}$ makes A-mod into an abelian category over G, so we may define $\hom_{A\text{-mod}}(?,?)$ and $\mathrm{ext}^{\bullet}_{A\text{-mod}}(?,?)$. Observe also that $\mathrm{add}_A\{Ae_j\}$ is the full subcategory of A-mod consisting of f.g. graded projective A-modules.

If K is another set and B is another G-graded K-diagonalizable ring, then by a (A,B)-bimodule we mean a G-graded abelian group N with commuting left A and right B-module structures making N a graded (A,B)-bimodule (i.e. $A_gN_hB_k \subseteq N_{ghk}$) and satisfying $N = \oplus_{j,k} e^A_j N e^B_k$. Using hom in place of Hom, many standard properties of modules and bimodules extend to this setting, and will be used without special comment. For example, $N\otimes_B ?\colon B\text{-Mod} \to A\text{-Mod}$ has a right adjoint given by $\oplus_k \hom_A(Ne^B_k, ?)$, and $A\otimes_A ? \cong \mathrm{Id} \cong \oplus_j \hom(Ae_j, ?)\colon A\text{-Mod} \to A\text{-Mod}$.

C.4. Let A be a G-graded, J-diagonalizable ring. Note that the exact inclusion functor from A-mod to the category of all G-graded A-modules has an exact right adjoint given on objects by $M \mapsto \oplus_j e_jM = \sum_j Ae_jM$. This implies that A-Mod is complete; limits in A-mod may be computed by first computing the limit in the category of all G-graded A-modules and then applying the above right adjoint.

Observe also that a direct sum in the category of all G-graded A-modules for a family of objects in A-Mod lies in A-mod and hence gives their direct sum in A-mod. It follows that A-mod has exact filtered colimits. Since A is clearly a projective generator for A-mod, one may conclude by general properties of abelian categories (see e.g. [40]) that A-Mod has injective envelopes and an injective generator.

C.5. Many of the diagonalizable rings and modules we consider arise in the following way. Given a family $\mathbf{M} := \{M_j\}_{j\in J}$ of objects of a fixed additive category C over G, the abelian group $A := \oplus_{i,j\in J} \hom_C(M_i, M_j)$ has a structure of J-diagonalizable ring with $e_i = \mathrm{Id}_{M_i}$, $e_iAe_j = \hom(M_i, M_j)$ and multiplication map $e_iAe_j \times e_jAe_k \to e_iAe_k$ given by composition $(f,g) \mapsto gf$ in hom-sets. By abuse of notation, we denote this ring by $A = \mathrm{end}_C(\mathbf{M})^{\mathrm{op}}$. If H is a (covariant, say) functor from our fixed additive category to another additive category, we sometimes write $H(\mathbf{M}) := \{H(M_j)\}_{j\in J}$.

If $\mathbf{N} = \{N_k\}_{k\in K}$ is another family of objects of C, $B := \mathrm{end}(\mathbf{N})$ is a K-diagonalizable ring and $\oplus_{j,k}\,\mathrm{hom}(M_j, N_k)$ has a natural structure of graded (A, B)-bimodule L with $e_j^A L e_k^B = \mathrm{hom}(M_j, N_k)$. Again by abuse of notation, write $L = \mathrm{hom}(\mathbf{M}, \mathbf{N})$. Similarly, we have an (A, B)-bimodule $\mathrm{ext}^i(\mathbf{M}, \mathbf{N})$ if B is an abelian or exact category over G. In particular, if K (resp., J) is a singleton, we may regard $\mathrm{ext}^i(\mathbf{M}, ?)$ (resp., $\mathrm{ext}^i(?, \mathbf{N})$ as a covariant (resp., contravariant) functor $C \to A$-mod (resp., $C \to B$-mod).

If G above is trivial, we usually write $L = \mathrm{Hom}_C(\mathbf{M}, \mathbf{N})$, $A = \mathrm{End}_C(\mathbf{M})^{\mathrm{op}}$, $\mathrm{Ext}^i(\mathbf{M}, \mathbf{N})$ etc.

C.6. Let A be a G-graded, J-diagonalizable ring. Consider the family $\mathbf{M} := \{Ae_j\langle g\rangle\}_{(j,g)\in J\times G}$ of objects of A. We have a $J \times G$-diagonalizable ring $B := \mathrm{End}_{A\text{-mod}}(\mathbf{M})^{\mathrm{op}}$ with

$$e_{p,g} B e_{q,h} = \mathrm{Hom}_{A\text{-mod}}(Ae_p\langle g\rangle, Ae_q\langle h\rangle) \cong (e_p A e_q)_{gh^{-1}}.$$

Regarding this as an identification, the multiplication map

$$e_{p,g} B e_{q,h} \times e_{q,h} B e_{r,k} \to e_{p,g} B e_{r,k}$$

identifies with the multiplication

$$(e_p A e_q)_{gh^{-1}} \times (e_q A e_r)_{hk^{-1}} \to (e_p A e_r)_{gk^{-1}}$$

in A.

We define a B-module structure on any object M of A-mod, making M an object of B-Mod as follows. Note $M = \oplus_{j,g} e_j M_g$. Set $e_{j,g} M = e_j M_g$, and define the B-module multiplication by the map $e_{i,h} B e_{j,g} \times e_{j,g} M \to e_{i,h} M$ which identifies with A-module multiplication $(e_i A e_j)_{hg^{-1}} \times e_j M_g \to e_i M_h$. One can easily check that every B-module arises in this way from a uniquely determined A-module structure on its underlying set, and that if N is another object of A-mod, a function $f\colon M \to N$ is a morphism in A-mod iff it is a morphism in B-Mod. Hence one has an isomorphism of categories A-mod $=$ B-Mod. (Note that non-isomorphic G-graded J-diagonalizable rings A, A' may produce isomorphic $G \times J$-diagonalizable rings B, B'; this simply corresponds to the possibility of different actions of G on B-Mod.)

In particular, suppose that C is an additive category over G and $\mathbf{M} = \{M_j\}_{j\in J}$ is a family of objects of C. We have another family $\mathbf{M}' := \{M_j\langle g\rangle\}_{(j,g)\in J\times G}$ in C. If $A := \mathrm{end}_C(\mathbf{M})^{\mathrm{op}}$ above, then one sees immediately that $B \cong \mathrm{End}(\mathbf{M}')$ as $J \times G$-diagonalizable ring. Hence $\mathrm{end}(\mathbf{M})^{\mathrm{op}}$-mod $\cong$ $\mathrm{End}(\mathbf{M}')^{\mathrm{op}}$-Mod in this situation; regarding this as an identification, the functor $\mathrm{hom}(\mathbf{M}, ?)\colon C \to \mathrm{end}(\mathbf{M})^{\mathrm{op}}$-mod identifies with $\mathrm{Hom}(\mathbf{M}', ?)\colon C \to \mathrm{End}(\mathbf{M}')^{\mathrm{op}}$-Mod, etc.

C.7. If A is a G-graded diagonalizable ring, then A-mod has infinite direct sums and a set $\{Ae_i\}_i$ of small projective objects, the family of translates of which form a set of generators. Conversely, we have the following result, which, in case $\mathbf{Q}$ is a singleton and G is trivial, reduces to a standard characterization [2, Chapter II, (1.3)] due to Mitchell and Gabriel of the categories of unitary modules over unital rings, amongst the abelian categories. The proof in general is essentially the same as in loc cit so it won't be given here.

Proposition. *Suppose that B is a cocomplete abelian category over G with a set of small projective objects $\mathbf{Q} = \{Q_i\}_i$ such that the set of translates of projectives in $\mathbf{Q}$ forms a set of generators for B. Then the functor $\hom(\mathbf{Q}, ?)\colon B \to \operatorname{end}(\mathbf{Q})^{\mathrm{op}}$-mod is an equivalence of abelian categories.*

C.8. The following general fact is of basic importance in this paper.

Proposition. *Suppose that C is an exact category over G with a family $\mathbf{Q} = \{Q_j\}_{j\in J}$ of projective objects such that for any M in C, there is an admissible epimorphism $Q \to M$ with Q in $\operatorname{add}\mathbf{Q}$. Then there is an equivalence $\alpha\colon C^* \cong A$-mod where $A := \operatorname{end}(\mathbf{Q})^{\mathrm{op}}$ under which ϕ_C identifies with $\varphi = \varphi_C := \hom_C(\mathbf{Q}, ?)$.*

Proof. The translates of objects in $\mathbf{P} := \phi(\mathbf{Q})$ form a generating set of small projectives in C^* by B.9(a). By C.7, $\alpha := \hom(\mathbf{P}, ?)\colon C^* \to A$-mod is an equivalence. Then ϕ corresponds to $\varphi := \alpha\phi = \hom_{C^*}(\phi(\mathbf{Q}), \phi?) \cong \hom_C(\mathbf{Q}, ?)$ since ϕ is full and faithful.

It is convenient to have more explicit descriptions of α and an inverse equivalence β. For G in $\mathcal{C}^*$, let $G(\mathbf{Q})$ be the object of A-mod with $e_jG(\mathbf{Q})_h = (GT_{h^{-1}})(Q_j)$ and module structure

$$(e_kAe_j)_g \times e_jG(\mathbf{Q})_h \mapsto e_jG(\mathbf{Q})_{gh}$$

given by $(f, m) \mapsto \big((GT_{h^{-1}})(f)\big)(m)$ if $f\colon T_{g^{-1}}Q_k \to Q_j$ and $m \in (GT_{h^{-1}})(Q_j)$. By the Yoneda lemma, one has

$$\alpha(G) = \hom(\phi(\mathbf{Q}), G) \cong G(\mathbf{Q}) \tag{C.8.1}$$

for any $G \in C^*$. Define $\beta_{\mathcal{C}} = \beta\colon A\text{-mod} \to C^*$ by

$$\beta(N)(M) = \operatorname{Hom}_A(\varphi(M), N) \tag{C.8.2}$$

for N in A-mod and M in C (note $\beta(N)$ is left exact since φ is exact and $\operatorname{Hom}(?, N)$ is left exact). Then

$$\alpha(\beta(N)) = (\beta(N))(\mathbf{Q}) = \oplus_j \hom(\varphi(Q_j), N) \cong \oplus_j \hom(Ae_j, N) \cong \oplus_j e_jN \cong N$$

naturally in N, and β is an inverse equivalence to α. □

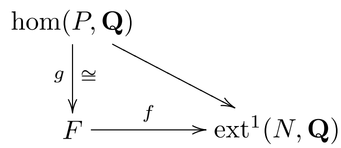
C.9. Let C be a split exact category over G with a class $\mathbf{Q} = \{Q_k\}_{k\in K}$ of objects such that $C = \operatorname{add}\mathbf{Q}$. Set $A = \operatorname{end}(\mathbf{Q})^{\mathrm{op}}$. There is an equivalence $\varphi = \hom(\mathbf{Q}, ?)$ of C with a full additive subcategory of $\operatorname{add}\{Ae_k\}_{k\in K}$. (This is a special case of C.8, but also follows more directly since the natural map $\hom_C(M, N) \to \hom_A(\varphi M, \varphi N)$ is an isomorphism for $M = Q_k$ and $N = Q_{k'}$.) Dually, $\varphi' = \hom(?, \mathbf{Q})$ induces a contravariant equivalence between C and a full additive subcategory of $\operatorname{add}\{e_kA\}_{k\in K}$ in A^{op}-mod. Letting $\rho\colon \operatorname{add}\{Ae_k\}_{k\in K} \to \operatorname{add}\{e_kA\}_{k\in K}$ be the standard contravariant equivalence $\oplus_k\operatorname{Hom}(?, Ae_k)$ between f.g. left and f.g. right projective graded A-modules, we have

$$\varphi' \cong \rho\varphi.$$

Indeed, the right hand side is just $\oplus_k \hom_A(\varphi?, \varphi Q_k) \cong \oplus_k \hom(?, Q_k)$.

C.10. Let C be a full additive subcategory over G of an exact category B over G. We regard C as a split exact category and assume $C = \operatorname{add}\mathbf{Q}$ for some set $\mathbf{Q} = \{Q_k\}_{k\in K}$ of its objects. Set $A = \operatorname{end}(\mathbf{Q})$. The following simple fact is basic for the construction of projective objects in stratified exact categories.

C.10.1. Let N be in B and $f\colon F \to \mathrm{ext}^1(N, \mathbf{Q})$ be a map in A-mod, with F f.g. projective. Assume there is an isomorphism $g\colon \hom(P, \mathbf{Q}) \cong F$ for some P in $\mathrm{add}\,\mathbf{Q}$ (this holds if F is isomorphic to a finite direct sum of translates of modules Ae_j for various $j \in J$, or if idempotents split in C). Then there is a short exact sequence $0 \to P \to N' \to N \to 0$ in B inducing a commutative diagram



in A-mod.

To see this, note that for any N, P in B,there are homomorphisms

$$\mathrm{ext}^1(N, P) \to \hom_A(\hom(P, \mathbf{Q}), \mathrm{ext}^1(N, \mathbf{Q})) \tag{C.10.2}$$

natural in N and P. For $P = Q_j$, this map is the isomorphism

$$e_j M \cong \hom_{A\text{-mod}}(Ae_j, M)$$

for $M = \mathrm{ext}^1(N, \mathbf{Q})$. It follows that in general the above homomorphism is an isomorphism for all P in $\mathrm{add}\,\mathbf{Q}$ and all N in B, and this proves the fact.

C.11. Let R be a commutative, unital Z-graded unital ring where Z is a subgroup of the center of a group G. We regard R as a G-graded ring with $R_g = 0$ for $g \in G \setminus Z$. We say that a G-graded, J-diagonalizable ring A is an R-algebra if there is a given G-graded R-module structure on each set $e_j A e_k = \oplus_g (e_j A e_k)_g$ making the multiplication $e_j A e_k \times e_k A e_l \to e_j A e_l$ a R-bilinear map for all j, k and l in J.

An additive category C over G will be said to be a R-category if all groups $\hom(M, N)$ with M, N in C have a given structure of graded R-module (compatible with their natural structure of graded abelian groups) making composition maps R-bilinear. In particular, $\mathrm{end}(\mathbf{M})^{\mathrm{op}}$ is an R-algebra for any family of objects $\mathbf{M}$ of C. If C is an abelian or exact category, then $\mathrm{ext}^i(M, N)$ is also a graded R-module and the graded Yoneda product is R-bilinear.

If A is a G-graded, J-diagonalizable R-algebra, then any object of A-mod has a natural graded R-module structure and homomorphisms in A-mod are automatically homomorphisms of graded R-modules. It follows that A-mod has a natural structure of R-category.

C.12. **Projective dimension and extensions.** Let $A = \oplus_{i,j \in L, g \in G} e_i A_g e_j$ be a G-graded ring. The graded projective dimension of a graded left A-module M is the minimum length of all graded projective resolutions of M; we denote it by proj.dim. M. The left graded global dimension gr.gl.dim. A of A is defined as the supremum of the graded projective dimensions of all graded left A-modules. If A is left and right graded Noetherian, the left and right graded global dimensions of A coincide.

Proofs of the following facts are essentially the same as those in the ungraded or $\mathbb{Z}$-graded cases to be found in [14], [15] and [21].

Fix $I \subseteq L$. Abbreviate $eM = \sum_{i \in I} e_i M$, $Me = \sum_i M e_i$ for a left or right A-module, $AeM = \oplus_i A e_i M$, $AeA = \oplus_{i \in I} A e_i A$, $eAe = \oplus_{i,j \in I} e_i A e_j$ etc.

C.12.1. For a graded left A-module M, AeM is a graded projective A-module iff eM is a graded projective eAe-module and the multiplication map $Ae\otimes_{eAe} eM \to AeM$ is an isomorphism of graded A-modules.

C.12.2. Suppose that the ideal $J = AeA$ is projective as both left and right graded A-module. Then gr.gl.dim. $(A) \leq \max(2 + \text{gr.gl.dim. } A/J, \text{gr.gl.dim. } eAe)$.

C.12.3. If $J = AeA$ with J projective as graded left A-module, the natural maps $\theta_i\colon \text{ext}^i_{A/J}(M,N) \to \text{ext}^i_A(M,N)$ are isomorphisms for all A/J-modules M, N.

C.13. **Tilting modules.** A family $\mathbf{T} := \{Te_j\}_{j\in L}$ of graded modules for a G-graded ring $A = \oplus_{i,j\in L,g\in G} e_iA_ge_j$ will be called a full family of tilting modules for A if there exists an integer N such that each Te_j has a graded projective dimension of length at most N by f.g. projective A-modules, if for each $j \in L$ there is an exact sequence $0 \to Ae_j \to T^0 \to \ldots \to T^N \to 0$ with $T^i \in \text{add}\,\{Te_j\}_j$, and if $\text{ext}^p_A(Te_i, Te_j) = 0$ for $p > 0$. If L is finite, then $\oplus_{j\in L}Te_j$ is a full tilting module for the unital ring A in the standard sense.

Let $\mathbf{T}$ be a full family of graded tilting modules for A, and define the G-graded ring $B := \text{end}(\mathbf{T})^{\text{op}}$, so $\mathcal{T} := \oplus_j T_j$ is naturally a graded (A,B)-bimodule. Let $\text{D}^b(A\text{-mod})$, $\text{D}^b(B\text{-mod})$ be the bounded derived category of A-mod and B-mod respectively; they are naturally triangulated categories over G, with automorphisms G induced by grading shifts. The argument [8] from the ungraded setting applies mutatis mutandis to show that the right derived functor $\text{Rhom}_A(\mathbf{T}, ?)$ of $\oplus_j \text{hom}_A(Te_j, ?)$ induces an equivalence $\text{D}^b(A\text{-mod}) \to \text{D}^b(B\text{-mod})$ of triangulated categories over G, with inverse equivalence induced by the left derived functor $\mathcal{T}\otimes^L_B ?$. Moreover, the family $\mathbf{T}' := \{\oplus_j e_j\mathcal{T}\}$ of B^{op}-modules is a full family of graded tilting module for B^{op}, with $\text{end}_{B^{\text{op}}}(\mathbf{T}') \cong A^{\text{op}}$.

C.14. In our applications, tilting modules arise in the following way.

Proposition. *Let C be a small exact category over G for which there exists an integer N such that every object of C has a projective resolution of length at most N and an injective resolution of length at most N. Choose a family of projective (resp., injective) objects $\mathbf{P}' := \{P_j\}$ (resp., $\mathbf{Q}' := \{Q_k\}$) so every object of $\mathcal{C}$ has a projective (resp., injective) resolution of length at most N by objects of* add $\mathbf{P}'$ *(resp.,* add $\mathbf{Q}'$*). By* C.8, *$C^* \cong A$-Mod where $A := \text{end}(\mathbf{P}')^{\text{op}}$ and $(C^{\text{op}})^* \cong B^{\text{op}}$-mod where $B := \text{end}_{C^{\text{op}}}(\mathbf{Q}) = \text{end}_C(\mathbf{Q}') = \text{end}(\mathbf{T}')^{\text{op}}$. Then the (A,B)-bimodules $\mathbf{T}' := \{\text{hom}(\mathbf{P}, Q_k)\}_k$ constitute a full family of tilting modules inducing equivalences $D^b(A\text{-mod}) \cong D^b(B\text{-mod})$ and $D^b(B^{\text{op}}\text{-mod}) \cong D^b(A^{\text{op}}\text{-mod})$.*

Proof. We need only verify that $\mathbf{T}'$ is a full family of tilting modules in the sense of C.13. The projective (resp., injective) objects of C coincide with the objects of add $\mathbf{P}$ (resp., add $\mathbf{Q}$). Therefore, applying $\text{hom}(\mathbf{P}, ?)$ to an injective resolution of P_j and a projective resolution of Q_k in C gives existence of resolutions as required. Now for any object M of $\mathcal{C}$, B.10 implies that $\text{ext}^p_{\mathcal{A}}(\text{hom}(\mathbf{P}', (M), \mathcal{T}e_k) = \text{ext}^p_{\mathcal{C}}(M, Q_k) = 0$ for $p > 0$ since Q_k is injective in $\mathcal{C}$. In particular, this implies that $\text{ext}^p_{\mathcal{A}}(\mathcal{T}e_{x,i}, \mathcal{T}e_{y,j}) = 0$ for $p > 0$. □

C.15. **Radicals of graded rings and modules.** For a G-graded diagonalizable ring $A = \oplus_{i,j\in L,g\in G}\, e_iA_ge_l$ and graded A-module $M = \oplus M_g$, the radical rad M of M is defined as the intersection of all maximal graded submodules of M. One has rad $(M\langle g\rangle) = (\text{rad } M)\langle g\rangle$. A homomorphism $f\colon M \to N$ of graded A-modules restricts to a homomorphism rad $M \to$ rad N, and so rad $(M \oplus M') =$ rad $M \oplus$ rad M'. If N is a submodule of rad M, then rad $(M/N) =$ (rad $M)/N$.

The graded Jacobson radical of A is the intersection of all maximal graded left ideals of A, i.e. $J :=$ rad ${}_A A$. We call A/J the residue ring of A (even if it is not a division ring).

Note that any simple object in A-mod is isomorphic to $(Ae_i/I)\langle g\rangle$ for some maximal submodule I of Ae_i and some $g \in G$. Moreover, any maximal left ideal of A is of the form $\oplus_{j\neq i} Ae_j \oplus I$ for some i and some maximal left ideal I of Ae_i, so $J = \oplus_i$rad Ae_i. Using these observations, standard facts about the Jacobson radicals of unital ungraded rings (and their proofs, see e.g. [12]) apply mutatis mutandis in this situation. For example, J is the intersection of the maximal graded right ideals of A, so J is a two sided ideal of A. Also, J is the intersection of the annihilators of all simple graded left A-modules.

Note that for $i \in L$, $e_i Ae_i$ is a G-graded ring with identity e_j. We will use frequently the following characterizations of the radical:

C.15.1. for $i, j \in L$ and $g \in G$, $e_i J_g e_j$ consists of those elements x of $e_i A_g e_j$ such that $e_j - ax$ (resp., $e_l - axb$) has a left inverse in $e_j A_1 e_j$ (resp., inverse in $e_l A_1 e_l$) for all $a \in e_j A_{g^{-1}} e_i$ (resp., $a \in e_l A_p e_i$, $b \in e_j A_q e_l$ with $pgq = 1$ where 1 is the identity of G).

We will also use various special cases of the following general fact:

C.15.2. Suppose that $\{f_k\}_{k\in K}$ is a family of homogeneous orthogonal idempotents of $\mathcal{A}$ with $f_k e_i = e_i f_k$ for all i, and H is a subgroup of G. Then one has a H-graded K-diagonalizable ring $B = \oplus_{i,j\in K} \oplus_{h\in H} f_i A_h f_j$ and rad $B = \oplus_{i,j\in K} \oplus_{h\in H} f_i A_h f_j$.

Indeed, the characterization of radicals in terms of left inverses proves the claim in the special case in which $\{f_k\} \subseteq \{e_i\}$. In general, this proves that for $K' \subseteq K$, one has $\oplus_{i,j\in K'} \oplus_{h\in H} f_i(\text{rad } B)_h f_j = \text{rad } (\oplus_{i,j\in K'} f_i B_h f_j)$ and shows it is enough to prove the claim with K replaced by any of its finite subsets (of cardinality 1 or 2). So now we assume that K is finite. But then one may regard A (resp., rad B) as a diagonalizable ring with respect to the orthogonal idempotents $\{e_i f_k\}_{i,k} \cup \{e_i - \sum_k e_i f_k\}_i$ (resp., $\{e_i f_k\}_{i,k}$) without changing A-mod or rad A (resp., B-mod or rad B) and the claim reduces to its previously considered special case (with K possibly infinite again).

In particular, the above fact implies that J_{1_G} is the Jacobson radical of the trivially graded L-diagonalizable ring A_{1_G}, and $e_i J e_i$ is the Jacobson radical of the G-graded unital ring $e_i A e_i$. If G is an additive ordered ordered group (e.g. $\mathbb{Z}$) and $A = \oplus_{g\geq 0} A_g$ is positively graded, then $J = J_1 \oplus A_{>0}$ where $A_{>0} = \oplus_{g>0} A_g$, and similarly for the radical of a positively graded module $M = \oplus_{g\geq 0} M_g$ over A.

C.16. **Nakayama's lemma.** For a G-graded ring A with graded Jacobson radical J, the Jacobson radical of A/J is zero, and for any A-module M, we have $JM \subseteq$ rad M. Hence one gets Nakayama's lemma: if M is a f.g. graded A-module with rad $M = M$ (e.g. $JM = M$) then $M = 0$.

It follows that if $f\colon M \to N$ is a map of f.g. graded A-modules with $M/JM \to N/JN$ an epimorphism, then f is an epimorphism.

This in turn implies (see e.g. [2]) that the functor $\theta\colon \operatorname{add} A \to \operatorname{add} A/J$ given on objects by $Q \to Q/JQ$ has the following property: if $f\colon Q \to Q'$ with Q, Q' in $\operatorname{add} A$ and $\theta(f)$ is an isomorphism then f is an isomorphism. In particular, θ is injective on isomorphism types of f.g. graded projective A-modules.

C.17. **Basic semiperfect rings and projective covers.** A graded diagonalizable ring $A = \oplus_{i,j\in L, g\in G} e_i A_g e_j$ will be called a basic semisimple ring if for each i, $Ae_iA = e_iA = Ae_i = e_iA_{1_G}e_i$ and this ring is a trivially-graded division ring. Clearly, the Ae_i then form a full set of isomorphism classes of simple A-modules, and any module in A-mod is completely reducible. We will say that $A = \oplus_{i,j,g} e_i A_g e_j$ is a basic semiperfect ring if A/J is basic semisimple, where J is the Jacobson radical of A.

Fix a G-graded basic semiperfect ring A. By the discussion in C.16, any f.g. projective A-module is a direct sum of copies of modules $\mathcal{A}e_i\langle g\rangle$ with uniquely-determined finite multiplicities. Moreover, $\mathrm{end}_A(Ae_i) \cong (e_iAe_i)^{\mathrm{op}}$ has graded radical $(e_iJe_i)^{\mathrm{op}}$, and $(e_iAe_i/e_iJe_i)^{\mathrm{op}} \cong \mathrm{end}_{A/J}(Ae_i/Je_i) \cong e_i(A/J)e_i$ is a trivially graded division ring. i.e. $\mathrm{end}_A(Ae_i)^{\mathrm{op}}$ is a local ring with trivially graded residue ring.

Now since A/J is semisimple, $\mathrm{rad}\ M = JM$ for any graded A-module M. If M is a f.g. A-module, a standard argument shows that a map $Q \to M$ with Q in $\mathrm{add}\, A$ is a projective cover for M if the induced map $Q/JQ \to M/JM$ is an isomorphism; in particular, one may choose an isomorphism $Q/JQ \cong M/JM$ for some Q in $\mathrm{add}\, A$ and lift the composite $Q \to Q/JQ \to M/JM$ to a projective cover $Q \to M$ of M. Hence any f.g. graded module for a basic semiperfect ring has a projective cover.

C.18. **Local rings.** Let R be a unital G-graded ring. We shall say that R is a graded division ring if every non-zero homogeneous element of R is invertible. We call R a graded local ring if the following three equivalent conditions hold

(a) R/J is a graded division ring, where J is the graded Jacobson radical of R
(b) R has a unique maximal graded left ideal
(c) the homogeneous non-units of R generate a proper ideal of R.

Note that R is a graded local ring iff R_{1_G} is local as an ungraded ring.

C.19. We have the following simple relationships between graded and ungraded modules over Artinian rings.

Lemma. *Let R be a graded unital ring which is left Artinian as graded ring. Let J denote the Jacobson radical of R as graded ring. Assume that R/J is a trivially graded semisimple ring. Then J coincides with the Jacobson radical J' of R as ungraded ring; if R is a graded division ring, it is also a division ring as ungraded ring. For any graded R-module M, the radicals* $\mathrm{rad}\ M$ *of M as graded module and* $\mathrm{Rad}\ M$ *of M as ungraded R-module coincide. Similarly, the socles of M as graded or ungraded R-module coincide. Finally, M has a unique maximal (resp., unique simple) submodule as graded module iff it has a unique maximal (resp., unique simple) submodule as ungraded module.*

Proof. It is well known that J' contains any nilpotent ideal of R, and that if R is ungraded Artinian, then J' is the largest (under inclusion) nilpotent ideal of R. Since R is Artinian as graded ring, one has similarly that J is the largest nilpotent graded ideal of R. Hence $J \subseteq J'$, so $\mathrm{Rad}\ R/J = J'/J = 0$ since R/J is semisimple.

Now one has $J'M \subseteq \mathrm{Rad}\ M$ and $JM \subseteq \mathrm{rad}\ M$ in general. Since M/JM is a graded R/J-module, it is semisimple as graded or ungraded R-module and thus $0 = \mathrm{rad}\ (M/JM) = (\mathrm{rad}\ M)/JM = 0$ and $0 = \mathrm{Rad}\ (M/J'M) = (\mathrm{Rad}\ M)/J'M$. Therefore $\mathrm{rad}\ M = JM = J'M = \mathrm{Rad}\ M$.

A graded (resp., ungraded) R-module is semisimple iff it is annihilated by J. Hence the socle of a graded R-module M as either graded or ungraded R-module is $\{m \in M | Jm = 0\}$.

Simple graded R-modules are simple as ungraded R-modules, so the length of a composition series of a graded R-module N as graded module is the same as the length of a composition series of N as ungraded module. The final claim follows by applying this observation to the head $M/\text{rad}\, M$ (resp., socle) of M as (graded or ungraded) R-module. □

Remarks. The ring R/J is a trivially graded semisimple ring if R is (unital and) basic semiperfect.

C.20. **Proof of Lemma 1.39.** Let J denote the radical of A. For any i, $e_iJ_{1_G}e_i$ is the radical of the local ring $e_iA_{1_G}e_i \cong \text{End}(M_i)^{\text{op}}$. We now show that $e_iA_ge_j \subseteq J$ if $i \neq j$ or $g \neq 1$. Let $f \in e_iA_ge_j$. If $f \notin J$, there exists $f' \in e_jA_{g^{-1}}e_i$ so $e_i - ff'$ is not left invertible in $e_iA_1e_i$. Since $e_iA_1e_i$ is local, this means $ff'k = e_i$ for some $k \in e_iA_1e_i$. This implies that the homomorphism $f\colon M_i\langle g\rangle \to M_j$ has a left inverse. Similarly, f has a right inverse and hence an inverse. Thus, $M_i\langle g\rangle \cong M_j$, so $i = j$ and $g = 1$

Hence we have the decomposition $A/J = \oplus_i e_i(A/J)_1e_i$ with $e_i(A/J)_1e_i \cong \text{End}(M_i)^{\text{op}}/\text{Rad End}(M_i)^{\text{op}}$ a division ring, and A/J is basic semiperfect as claimed. The claims 1.39(a)–(c) follow.

Now for 1.39(d), consider the full and faithful functor (cf C.9) given by $\alpha := \hom(\mathbf{M}, ?)\text{add}\,\mathbf{M} \to A\text{-mod}$. The family $\alpha(\mathbf{M})$ is a Krull-Schmidt family in A-mod and idempotents split in A-mod. By the Krull-Schmidt theorem [2], any object in $\text{add}\,\alpha(\mathbf{M})$ is a direct sum with uniquely determined multiplicities of translates of objects $\alpha(M_i)$. Since any such object is clearly in the strict image of α, 1.39(d) follows.

C.21. Suppose that A is a Noetherian graded local ring with graded Jacobson radical J and that $k := A/J$ is a trivially graded division ring. Then

C.21.1. A f.g. graded A-module M is projective if and only if $\text{tor}_1^A(k, M) = 0$.

To see this, choose an exact sequence $0 \to L \to P \to M \to 0$ so $f\colon P \to M$ is a projective cover of M i.e. so f induces an isomorphism $k \otimes_A P \cong k \otimes_A M$. If $\text{tor}_1^A(k, M) = 0$, then application of $k\otimes_A ?$ to the short exact sequence above gives another short exact sequence, so $k \otimes_A L = 0$ i.e. $L/JL = 0$ and thus $L = 0$ by Nakayama's lemma.

Now suppose in addition that A is commutative, so k is a field above, and that A is Noetherian even as ungraded ring.

C.21.2. A f.g. graded A-module P is projective iff its localization P_J at the maximal ideal J of A is a projective A_J-module.

For $\text{tor}_1^{A_J}(k_J, N_J) \cong \text{tor}_1^A(k, N)_J$ is zero if and only if

$$\text{tor}_1^A(k, N)_J/J\,\text{tor}_1^A(k, N)_J \cong \text{tor}_1^A(k, M)/J\,\text{tor}_1^A(k, M)$$

is zero, by Nakayama's lemma. The last module is zero iff $\text{tor}_1^A(k, M) = 0$, by the graded version of Nakayama's lemma.

C.22. **Semisimple rings.** In this final subsection, we record some facts which are useful in extending some of the results of this paper to the more general situation mentioned in Remark 1.39. The (standard) proofs will not be given (but see [26] and [41] for many of them if G is trivial).

Let us say that a graded, diagonalizable ring A is semisimple if A is completely reducible as left A-module, or equivalently, every M in A-mod is completely reducible. Let us say that A is simple if it is non-zero, semisimple and has no non-trivial proper two-sided ideals (or equivalently, any two simple left A-modules are isomorphic up to degree shift). Finally, say that A is semiperfect if A/J is a semisimple ring as defined above and every homogeneous idempotent of A/J lifts to a homogeneous idempotent of A.

Given G-graded L-diagonalizable rings I_p (with idempotents $\{e_i^p\}_{i\in L}$) such that for fixed $i \in L$, $e_i^p \in I_p$ is non-zero for only finitely many p, the direct sum $\oplus_p I_p$ is naturally a G-graded L-diagonalizable ring with respect to the idempotents $e_i := \sum e_i^p$, and the I_p are two-sided ideals of $\oplus_p I_p$.

Proposition. (a) *Any G-graded L-diagonalizable semisimple ring A is a direct sum $\oplus_p I_p$ as above of G-graded, L-diagonalizable simple rings I_p. Regarded as two-sided ideals of A, the I_p are uniquely determined.*

(b) *If A is a G-graded, L-diagonalizable simple ring, then there is G-graded division ring D and a family $\mathbf{M} = \{M_i\}_{i\in L}$ of f.g. graded D-modules (unique up to isomorphism and simultaneous degree shift) such that $A \cong \operatorname{end}(\mathbf{M})^{\mathrm{op}}$ as G-graded, L-diagonalizable ring.*

(c) *If A is a graded semiperfect ring with graded Jacobson radical J, then a map $P \to M$ with M a f.g. module in* A-mod *and P projective in* A-mod *is a projective cover of M iff it induces an isomorphism $P/JP \to M/JM$; in particular, M has a projective cover.*

(d) *If $\mathbf{M} = \{M_i\}_i$ is a family of objects in an additive category over G such that the endomorphism rings* $\operatorname{end}(M_i)$ *are graded local (or even unital graded semiperfect), then* $\operatorname{end}(\mathbf{M})^{\mathrm{op}}$ *is a graded semiperfect ring.*

References

[1] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p. *Asterisque*, 220:1–321, 1994.

[2] H. Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York. Amsterdam., 1968.

[3] A. Beilinson, V. Ginsburg, and W. Soergel. Koszul duality patterns in representation theory. *Jour. of the Amer. Math. Soc.*, 9:473–527, 1996.

[4] A. A. Beilinson and I. N. Bernstein. Localisation de $\mathfrak{g}$ -modules. *C. R. Acad. Sci. Paris, Sér I Math.*, 292:15–18, 1981.

[5] I. N. Bernstein and S. I. Gelfand. Tensor products of finite and infinite dimensional representations of semisimple Lie algebras. *Comp. Math.*, 41:245–285, 1980.

[6] N. Bourbaki. *Groupes ét algebres de Lie, Ch. 4–6.* Hermann, Paris, 1968.

[7] J. L. Brylinski and M. Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, 64:387–410, 1981.

[8] E. Cline, B. Parshall, and L. Scott. Derived categories and Morita theory. *J. of Alg*, 104:397–409, 1986.

[9] E. Cline, B. Parshall, and L. Scott. Derived categories, quasi-hereditary algebras and algebraic groups. *Carleton U. Math Lecture Notes*, 3:1–104, 1988.

[10] E. Cline, B. Parshall, and L. Scott. Finite dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[11] E. Cline, B. Parshall, and L. Scott. Integral and graded quasi-hereditary algebras. *J. of Alg.*, 131:126–160, 1990.

[12] C. W. Curtis and I. Reiner. *Methods of Representation Theory with Applications to Finite Groups and Orders*, volume I. John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1981.

[13] V. Deodhar, O. Gabber, and V. Kac. Structure of some categories of representations with highest weight of infinite-dimensional Lie algebras. *Adv. in Math.*, 45:92–116, 1982.

[14] V. Dlab and C. Ringel. A construction for quasi-hereditary algebras. *Comp. Math.*, 70:155–175, 1989.

[15] V. Dlab and C. Ringel. Quasi-hereditary algebras. *Illinois J. Math.*, 33:280–291, 1989.

[16] P. Dräxler, I. Reiten, S. Smalø, Ø. Solberg, and with an appendix by B. Keller. Exact categories and vector space categories. *Trans. of the Amer. Math. Soc.*, 351:647–682, 1999.

[17] M. J. Dyer. Modules for the dual nil Hecke ring. Preprint.

[18] M. J. Dyer. Kazhdan-Lusztig-Stanley polynomials and quadratic algebras I. Unpublished manuscript, 1991.

[19] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals II: twisted Bruhat orders. In *Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989)*, volume 139 of *Contemp. Math.*, pages 141–165, Providence, RI, 1992. Amer. Math. Soc.

[20] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Comp. Math.*, 89:91–115, 1993.

[21] M. J. Dyer. Algebras associated to Bruhat intervals and polyhedral cones. In *Finite-dimensional Algebras and Related Topics*, volume 424 of *Nato ASI Series C*, pages 95–121, Dordrecht, Boston, London, 1994. Kluwer Academic Publishers.

[22] M. J. Dyer. Bruhat intervals, polyhedral cones and Kazhdan-Lusztig-Stanley polynomials. *Math. Z.*, 215:223–236, 1994.

[23] M. J. Dyer. Representation theories from Coxeter groups. In *Representations of Groups*, volume 16 of *Can. Math. Soc. Conf. Proc*, pages 105–139, Providence, RI, 1995. Amer. Math. Soc.

[24] O. Gabber and A. Joseph. Towards the Kazhdan-Lusztig conjecture. *Ann. Scient. Éc. Norm. Sup.*, 14:261–302, 1981.

[25] P. Gabriel. Des catégories abéliennes. *Bull. Soc. Math. France*, 90:323–448, 1962.

[26] P. Gabriel and A. Roiter. *Algebra VIII: Representations of Finite-Dimensional Algebras.* Springer-Verlag, Berlin. Heidelberg. New York., 1992.

[27] R. Godement. *Topologie Algébrique et Théorie des Faisceaux.* Hermann, Paris, 1958.

[28] J. Graham and G. Lehrer. Cellular algebras. *Invent. Math.*, 123:1–34, 1996.

[29] Z. Hadad. *Infinite-dimensional flag varieties.* PhD thesis, M. I. T., 1984.

[30] A. Heller. Homological algebra in abelian categories. *Ann. of Math.*, 68:484–525, 1958.

[31] J. E. Humphreys. *Reflection Groups and Coxeter Groups.* Number 29 in Cambridge Studies in Advanced Mathematics. Camb. Univ. Press, Cambridge, 1990.

[32] R. Irving. BGG algebras and the BGG reciprocity principle. *J. of Alg.*, 135:363–380, 1990.

[33] R. Irving. Shuffled verma modules and principal series modules over complex semisimple Lie algebras. *J. London Math. Soc.*, 48:263–277, 1993.

[34] V. G. Kac. *Infinite Dimensional Lie Algebras.* Cambridge University Press, Cambridge, third edition, 1990.

[35] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.

[36] D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality. *Proc. Symp. Pure Math. of Amer. Math. Soc.*, 36:185–203, 1980.

[37] B. Keller. Chain complexes and stable categories. *Manus. Math.*, 67:379–417, 1990.

[38] S. MacLane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York. Berlin, 1971.

[39] P. McMullen. On simple polytopes. *Invent. Math.*, 113:419–444, 1993.

[40] B. Mitchell. *Theory of Categories*, volume XVII of *Pure and Applied Mathematics.* Academic Press, New York. London, 1965.

[41] B. Mitchell. Rings with several objects. *Adv. in Math.*, pages 1–161, 1972.

[42] C. Nastasescu and F. van Oystaeyen. *Graded Ring Theory*, volume 28 of *North-Holland mathematical library.* North-Holland Pub. Co., Amsterdam. New York., 1982.

[43] A. Neeman. The derived category of an exact category. *J. of Alg*, 135:388–394, 1990.

[44] B. Parshall. Finite-dimensional algebras and algebraic groups. In *Classical Groups and Related Topics*, volume 82 of *Contemp. Math.*, pages 97–114, Providence, RI, 1989. Amer. Math. Soc.

[45] D. Quillen. Higher algebraic K-theory I. In *Algebraic K-theory I. (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume 341 of *Lect. Notes in Math.*, pages 85–147. Springer, Berlin, 1973.

[46] C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208, 1991.

[47] A. Rocha-Charidi and N. R. Wallach. Projective modules over graded Lie algebras I. *Math. Z.*, 180:151–177, 1982.

[48] s. König. Quasi-hereditary orders. *Manuscripta Math.*, 68:417–433, 1990.

[49] M. Saito. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.*, 24:849–995, 1988.

[50] W. Soergel. The combinatorics of Harish-Chandra bimodules. *J. Reine Angew. Math*, 429:49–74, 1992.

[51] R. Stanley. Generalized H-vectors, intersection cohomology of toric varieties and related results. In *Commutative Algebra and Combinatorics*, volume 11 of *Adv. Stud. in Pure. Math.*, pages 187–213. North-Holland, Amsterdam and New York, 1987.

[52] R.W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift: A collection of articles written in Honor of the 60th Birthday of Alexander Grothendieck. Volume III*, volume 88 of *Progress in Mathematics*, pages 247–435. Birkhäuser, Boston, 1990.

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