The Four Equation New Keynesian Model *

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Abstract

This paper develops a New Keynesian model featuring financial intermediation, short and long term bonds, credit shocks, and scope for unconventional monetary policy. The log-linearized model reduces to four key equations – a Phillips curve, an IS equation, and policy rules for the short term nominal interest rate and the central bank's long bond portfolio (QE). The four equation model collapses to the standard three equation New Keynesian model under a simple parameter restriction. Credit shocks and QE appear in both the IS and Phillips curves. Optimal monetary policy entails adjusting the short term interest rate to offset natural rate shocks, but using QE to offset credit market disruptions. The ability of the central bank to engage in QE significantly mitigates the costs of a binding zero lower bound.

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1 Introduction

The textbook three equation New Keynesian (NK) model (see, e.g., Woodford 2003 or Galí 2008) has enormous influence in both policy circles and among academic researchers due to its elegance and tractability. The model boils down to a forward-looking IS equation characterizing aggregate demand, a Phillips curve describing aggregate supply, and a rule for the central bank's principal policy tool, the short term interest rate. The model has yielded several important insights, including the potential desirability of inflation targeting, the gains from policy commitment over discretion, and the importance of having the policy rate track the "natural" or "neutral" rate of interest.

In spite of its myriad uses, the textbook model has proven inadequate for examining a range of issues that have come to the fore in policy circles over the last decade. As it abstracts from the financial sector, the model is unable to address the consequences of financial market disruption of the sort that rocked the global economy in 2007-2009. It is also incapable of directly speaking to the potential benefits and costs of quantitative easing (QE) type policies. QE policies were among the first and most prominent of several unconventional policy interventions deployed to fight the global financial crisis once policy rates were lowered to zero. There is now a nascent literature incorporating QE into medium-scale DSGE models (e.g. Gertler and Karadi 2011, 2013; Carlstrom, Fuerst and Paustian 2017; or Sims and Wu 2019b). While this work has proven useful and generated several important insights, these quantitative frameworks lack the simplicity and transparency of the textbook three equation model.

Our paper bridges the gap between the complicated quantitative DSGE models that have been developed to study QE with the elegance and tractability of the textbook three equation model. Our model incorporates financial intermediaries, short and long term bonds, credit market shocks, and scope for central bank bond holdings to be economically relevant. The linearized version of our model reduces to four, rather than three, key equations. The IS and Phillips curves are similar to the three equation benchmark. The innovation is that credit shocks and central bank long bond holdings appear additively in *both* the IS and Phillips curves. This differs from many *ad-hoc* treatments of financial disturbances, which often simply include residuals in the IS equation meant to proxy for credit spreads (see, e.g., Smets and Wouters 2007). The model is closed with a rule for the short term policy rate (as in the benchmark three equation model) and a rule for the central bank's long bond portfolio.

We study optimal monetary policy in the context of our four equation model. Reflecting central banks' dual mandate, we focus on an objective function that minimizes a weighted sum of volatilities of inflation and the output gap. Because credit shocks appear in the Phillips curve, the so-called "Divine Coincidence" (Blanchard and Galí 2007) does not hold, and it is not possible to achieve the global minimum of the loss function with just one policy instrument. Optimal policy entails adjusting the short term interest rate to track fluctuations in the natural rate of interest (as in the benchmark three equation model), but adjusting the long bond portfolio to offset the effects of credit market disturbances. Our model therefore has an implication that differs from the conventional wisdom among policymakers that adjustment of short term interest rates is sufficient to meet a dual mandate of price and output stability – in general, quantitative easing policies ought to be used all the time to counter credit market shocks, not only when policy rates are constrained by the zero lower bound (ZLB).

We also explore the implications of the ZLB for policy. A couple of interesting results emerge. First, credit market shocks need not have differential effects at the ZLB in comparison to normal times. Adjusting the long bond portfolio in exactly the same way as it would absent a ZLB constraint, the central bank is able to stabilize both inflation and the output gap in response to credit shocks at the ZLB. Second, QE policies can serve as an effective (albeit imperfect) substitute for conventional policy actions in response to natural rate shocks. Without QE available, output and inflation react suboptimally to natural rate shocks when the short term policy rate is constrained, the more so the longer the anticipated period of the ZLB. A central bank can partially offset these non-optimal responses by adjusting its long bond portfolio. We derive an analytical expression for the optimal QE rule at the ZLB as a function of the relative welfare weight on the output gap in the loss function. Though it is not possible to completely stabilize both inflation and the gap, a central bank engaging in QE operations can significantly reduce the costs of the ZLB.

Our model has important implications for central banks facing a dual mandate to stabilize both inflation and real economic activity due to the failure of the Divine Coincidence. First, how much QE is desired at the ZLB depends critically on how much weight the central bank puts on inflation vs. output fluctuations. The more weight the central bank puts on the output gap, the less QE is required in response to a shock to the neutral rate of interest. Second, suppose the central bank were not allowed to respond to a credit shock with bond purchases. This was the operating framework of modern central banks prior to the Great Recession. The direction for the optimal short rate response to a credit shock depends on whether the central bank cares more about inflation or output stabilization. For a positive credit shock, a central bank focusing solely on inflation would increase the short rate; whereas if the central bank only cares about the output gap, it would instead cut the short rate. Alternatively, if a central bank can use bond purchases all the time as a policy instrument, there need not be any conflict between the two aspects of the dual mandate.

In the background of our four equation linear model, there are a number of different agents. The production side of the economy is identical to the standard three equation model. There are two types of households, which we refer to as the "parent" and the "child." The representative parent consumes, supplies labor, and has an equity share in production firms and financial intermediaries. It saves through one period nominal bonds. Each period, it makes an equity transfer to financial intermediaries, provides a lump sum transfer to the child, and receives dividends. The representative child does not supply labor and has no equity interest in firms or intermediaries. It is less patient than the parent. The child may not buy or sell short term debt, but may issue long term nominal bonds. It finances its consumption as well as the coupon payments on outstanding debt via the lump sum transfer it receives from the parent.

Debt markets are segmented such that only financial intermediaries can simultaneously access both the short term savings of the parent and the long term bonds issued by the child. Market segmentation is crucial for QE policies to work. New financial intermediaries are born each period and exist for only one period. They each receive a fixed amount of startup net worth at birth and return accumulated net worth to the parent upon exiting. Because of this setup, there is effectively a representative intermediary. In addition to startup net worth, the representative intermediary finances its operations with short term bonds. On the asset side of the balance sheet, it holds long term bonds issued by the child and interest-bearing reserves issued by the central bank. The intermediary is subject to a risk-weighted leverage constraint. Long bonds receive a risk-weight of one, while reserves have a risk-weight of zero. Risk-weighted assets cannot exceed an exogenous multiple of net worth. We refer to stochastic fluctuations in the leverage multiple as credit shocks. The model is calibrated such that the risk-weighted leverage constraint always binds so that the return on long term bonds is higher than that on short term bonds in expectation. The structure of financial intermediaries can be considered as a special case of Gertler and Karadi (2011, 2013) and Sims and Wu (2019b).

Unconventional monetary policy allows the central bank to also hold long term bonds issued by the child, and to in effect serve as an additional financial intermediary. It finances these holdings by creating interest-bearing reserves. It sets the interest rate on reserves, or the policy rate, according to some policy rule. In equilibrium, the interest rate on reserves equals the interest rate on short term bonds. Quantitative easing policies have effects isomorphic to positive credit shocks – when the central bank buys long term bonds, it eases the constraint facing the intermediary, leading to an expansion in the supply of credit and a reduction in long-short interest rate spreads.

Linearizing the model about the steady state, many of these details drop out, leaving a four equation system. In addition to the two policy rules, the linearized IS curve expresses the current output gap as a function of the expected future output gap and the spread between the real short term interest rate and the natural rate of interest, which is identical to the textbook three equation model. What is new in our model is a term related to credit shocks and the central bank's long bond portfolio. The Phillips curve relates current inflation to the current output gap, expected future inflation, and a new term capturing credit shocks as well as the central bank's long bond portfolio. Under the parameter restriction that all households are parents, both the IS and Phillips curves reduce to their standard expressions in the benchmark three equation model. Importantly, credit shocks and the central bank's long bond portfolio appear in *both* the IS and Phillips curves. This means that such shocks have both "demand" and "supply" effects, and also means that credit shocks generate a sort of endogenous "cost-push" term.

Our paper relates to a large literature on monetary policy in the New Keynesian model more generally and to that on unconventional policy actions in particular. Clarida et al.'s (1999) seminal work concerns monetary policy design in the canonical three equation New Keynesian model. Eggertsson and Woodford (2003) and Adam and Billi (2006, 2007) make early contributions on the consequences of a binding ZLB for optimal policy. Gertler and Karadi (2011, 2013), Carlstrom, Fuerst and Paustian (2017), Sims and Wu (2019b), and Mau (2019) all represent attempts to model large scale asset purchases in a quantitative DSGE framework. Distinct from this strand of the literature, one important contribution of our paper is to incorporate the financial frictions giving rise to effective QE policies in these papers into the tractable small-scale New Keynesian model of Clarida, Galí and Gertler (1999) that is so popular among academics and policymakers alike. In that sense, our paper is similar to Wu and Zhang (2017), who propose replacing the policy rate in the three equation New Keynesian model with the shadow Federal Funds rate at the ZLB, meant as a summary statistic for unconventional interventions. We differ in that our model features four equations instead of three, explicitly modeling QE as a separate policy instrument.

The implications of our model relate to an empirical literature suggesting that unconventional policy actions have been successful antidotes to the ZLB – see, for example, Swanson and Williams (2014), Wu and Xia (2016), Wu and Zhang (2017, 2019), Garín, Lester and Sims (2019), Debortoli, Galí and Gambetti (2016), Mouabbi and Sahuc (2017), Swanson (2018a,b), and Sims and Wu (2019a). Different from this literature, we emphasize why and how the substitutability between conventional and unconventional policy tools is not perfect.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 discusses optimal central bank policy. Section 4 offers concluding thoughts.

2 Model

This section presents our model. We first present the four equation linearized model in Subsection 2.1, on which we base our subsequent analysis. The full non-linear model is derived from first principles in Subsection 2.2. Subsection 2.3 studies positive properties of a calibrated version of the model before turning to normative issues in Section 3. Details are available in Appendixes A - C.

2.1 The Four Equation Model

The principal equations of our linearized model are an IS curve:

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1} - r_t^f\right) - z\Big[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\Big], \quad (2.1)$$

and a Phillips Curve:

$$\pi_t = \gamma\zeta x_t - \frac{z\gamma\sigma}{1-z}\Big[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\Big] + \beta\, \mathbb{E}_t\, \pi_{t+1}. \quad (2.2)$$

Lowercase variables with a t subscript denote log deviations about the non-stochastic steady state. π_t is inflation and $x_t = y_t - y_t^f$ denotes the output gap, where y_t^f is the equilibrium level of output consistent with price flexibility and no credit shocks. Similarly, r_t^f denotes the natural rate of interest – i.e. the real interest rate consistent with output equaling potential. It follows an exogenous process. θ_t captures credit conditions in the financial market; positive values correspond to more favorable conditions. This is described further in Subsection 2.2. We take it to be exogenous and henceforth refer to it as a credit shock. qe_t denotes the real market value of the central bank's long term bond portfolio. r_t^s is the short term nominal interest.

Letters without t subscripts are parameters or steady state values. σ, β, and γ are standard parameters – σ measures the inverse intertemporal elasticity of substitution, β is a subjective discount factor, and γ is the elasticity of inflation with respect to real marginal cost.[1] $\bar{b}^{FI}$ and $\bar{b}^{CB}$ are parameters measuring the steady state long-term bond holdings of financial intermediaries and the central bank, respectively, relative to total outstanding long term bonds. These coefficients sum to one, i.e. $\bar{b}^{FI} + \bar{b}^{CB} = 1$.

As described in Subsection 2.2, there are two kinds of households in our model. We will refer to these types of households as "parent" and "child," respectively. The parent is the standard household in a textbook New Keynesian model – it consumes, borrows or saves via one-period bonds, supplies labor, and owns firms. The child does not supply labor nor does it have an equity interest in production firms. It is less patient than the parent and finances its consumption by issuing long term bonds. It pays the servicing cost of these long term bonds with a transfer from the parent each period. The parameter $z \in [0, 1)$ represents the share of children in the total population. ζ is the elasticity of real marginal cost with respect to the output gap; it is conceptually similar to the corresponding parameter in the standard three equation model, but augmented to account for two types of households.[2] Our model collapses to the standard three equation NK model when $z = 0$. In this case, credit shocks and the central bank's long bond portfolio are irrelevant for the equilibrium dynamics

[1] In particular, $\gamma = \frac{(1-\phi)(1-\phi\beta)}{\phi}$ is the standard expression in the three equation model, where $\phi \in [0, 1)$ measures the probability of non-price adjustment.

[2] In particular, $\zeta = \frac{\chi(1-z)+\sigma}{1-z}$, where χ is the inverse Frisch labor supply elasticity for the parent.

of output and inflation. In addition, ζ reduces to the same expression as in the standard model.

Our four equation New Keyneian model consists of (2.1)-(2.2), together with policy rules for the short term interest rate r_t^s and central bank's long bond portfolio qe_t. Simple rule-based policies are specified in Subsection 2.3 for positive analyses, whereas we discuss optimal policies in Section 3.

QE vs. Conventional Monetary Policy Let us highlight an important difference between a QE shock and a conventional monetary policy shock concerning the impact on inflation. In our model, a QE shock is less inflationary than a conventional monetary policy rate cut. This finding is in-line with the results in the richer model of Sims and Wu (2019b), and empirically consistent with the lack of inflationary pressures from the expansive QE operations in the US and other parts of the world in the wake of the Great Recession. Economically, this finding emerges because the qe_t term enters in *both* the IS, (2.1), and Phillips Curves, (2.2). In particular, qe_t enters with a positive sign in the IS relationship, and hence serves as a positive demand shock, but with a negative sign in the Phillips Curve, and hence acts as a sort of endogenous "cost-push" shock. Both of these channels make QE expansionary for output, but have competing effects on inflation.

2.2 Derivation of the Four Equation Model

In this subsection, we present, from first principles, the economic environment giving rise to the linearized four equation model laid out in Subsection 2.1. The economy is populated by the following agents: two types of households (parent and child), a representative financial intermediary, production firms, and a central bank. We discuss the problems of each below.

2.2.1 Parent

A representative parent receives utility from consumption, C_t and disutility from labor, L_t. It discounts future utility flows by $\beta \in (0,1)$. Its lifetime utility is:

$$V_t = \max \ \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[\frac{C_{t+j}^{1-\sigma} - 1}{1-\sigma} - \psi \frac{L_{t+j}^{1+\chi}}{1+\chi} \right]. \tag{2.3}$$

$\sigma > 0$ is the inverse elasticity of intertemporal substitution, $\chi \geq 0$ is the inverse Frisch elasticity, and $\psi > 0$ is a scaling parameter.

The nominal price of consumption is P_t. The parent earns nominal income from labor, with a wage of W_t, receives dividends from ownership in firms and financial intermediaries, D_t and D_t^{FI}, respectively, and receives a lump sum transfer from the central bank, T_t. It can save via one period nominal bonds, S_t, which pay gross nominal interest rate R_t^s. In addition, it makes a time-varying transfer, X_t^b, to the child each period, as well as a fixed transfer, X^{FI}, to financial intermediaries.

$$P_t C_t + S_t \leq W_t L_t + R_{t-1}^s S_{t-1} + P_t D_t + P_t D_t^{FI} + P_t T_t - P_t X_t^b - P_t X^{FI}. \tag{2.4}$$

The objective is to pick a sequence of consumption, labor, and one period bonds to maximize (2.3) subject to the sequence of (2.4). The optimality conditions are standard:

$$\psi L_t^{\chi} = C_t^{-\sigma} w_t, \tag{2.5}$$

$$\Lambda_{t-1,t} = \beta \left(\frac{C_t}{C_{t-1}} \right)^{-\sigma}, \tag{2.6}$$

$$1 = R_t^s \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}. \tag{2.7}$$

In (2.5), $w_t = W_t/P_t$ is the real wage; and in (2.7), $\Pi_t = P_t/P_{t-1}$ is gross inflation. $\Lambda_{t-1,t}$ is the parent's stochastic discount factor.

2.2.2 Child

The child gets utility from consumption, $C_{b,t}$, and does not supply labor. Its flow utility function is the same as the parent, but it discounts future utility flows by $\beta_b < \beta$; i.e. it is less patient than the parent. Its lifetime utility is:

$$V_{b,t} = \mathbb{E}_t \sum_{j=0}^{\infty} \beta_b^j \left[\frac{C_{b,t+j}^{1-\sigma} - 1}{1-\sigma} \right]. \tag{2.8}$$

The child can borrow/save through long term bonds, the new issuance of which is denoted by CB_t. These bonds are structured as perpetuities with decaying coupon payments, as in Woodford (2001). Coupon payments decay at rate $\kappa \in [0,1]$. Issuing one unit of bonds in period t obligates the issuer to a coupon payment of 1 dollar in $t+1$, κ dollars in $t+2$, κ^2 dollars in $t+3$, and so on. The total coupon liability due in $t+1$ from past issuances is therefore:

$$B_t = CB_t + \kappa CB_{t-1} + \kappa^2 CB_{t-2} + \dots. \tag{2.9}$$

The attractive feature of these decaying coupon bonds is that one only needs to keep track of the total outstanding bonds, B_t, rather than individual issues. In particular:

$$CB_t = B_t - \kappa B_{t-1}. \tag{2.10}$$

New issuances in period t trade at market price Q_t dollars. Because of the structure of coupon payments, the prices of bonds issued at previous dates are proportional to the price of new issues; i.e. bonds issued in $t-j$ trade at $\kappa^j Q_t$ in t. The total value of the bond portfolio can therefore conveniently be written as $Q_t B_t$.

The nominal value of consumption plus coupon payments on outstanding debt cannot exceed the value of new bond issuances plus the nominal value of the transfer from the parent. The flow budget constraint facing the child is therefore:

$$P_t C_{b,t} + B_{t-1} \leq Q_t \left(B_t - \kappa B_{t-1}\right) + P_t X_t^b. \tag{2.11}$$

Define the gross return on the long bond as:

$$R_t^b = \frac{1 + \kappa Q_t}{Q_{t-1}}. \tag{2.12}$$

The optimality condition for the child is an Euler equation for long term bonds, where $\Lambda_{b,t-1,t}$ denotes its stochastic discount factor:

$$\Lambda_{b,t-1,t} = \beta_b \left(\frac{C_{b,t}}{C_{b,t-1}}\right)^{-\sigma}, \tag{2.13}$$

$$1 = \mathbb{E}_t \, \Lambda_{b,t,t+1} R_{t+1}^b \Pi_{t+1}^{-1}. \tag{2.14}$$

2.2.3 Financial Intermediaries

A representative financial intermediary (FI) is born each period and exits the industry in the subsequent period. It receives an exogenous and fixed amount of real net worth from the parent household, X^{FI}. It also attracts deposits, S_t^{FI}, from the parent household. It can hold newly issued long bonds issued by the child, CB_t^{FI}, or reserves on account with the central bank, RE_t^{FI}. The FI is structured as a special case of intermediaries in Sims and Wu (2019b) and Gertler and Karadi (2011, 2013), with intermediaries exiting after each period with probability one. Because the probability of exit after each period is unity, we can think of there being a (newly born) representative FI each period.

The balance sheet condition of the FI is:

$$Q_t CB_t^{FI} + RE_t^{FI} = S_t^{FI} + P_t X^{FI}. \tag{2.15}$$

The FI pays interest, R_t^s, on short term debt, earns interest, R_t^{re}, on reserves, and earns a return on long term bonds carried from t into $t+1$, R_{t+1}^b.

Upon exiting after period t, the FI therefore returns a dividend to the parent household that satisfies:

$$D_{t+1}^{FI} = \left(R_{t+1}^{b} - R_{t}^{s}\right) Q_t CB_t^{FI} + \left(R_t^{re} - R_t^{s}\right) RE_t^{FI} + R_t^{s} P_t X^{FI} \tag{2.16}$$

The FI may freely choose CB_t^{FI} and RE_t^{FI}. Since it exits after one period, it does not take into how its choice of CB_t^{FI} will affect the stock of bonds over which future FIs will act as custodians.

The FI is subject to a risk-weighted leverage constraint. We assume that this constraint applies to both newly purchased as well the outstanding stock of long bonds (in addition to on-balance sheet assets and liabilities as shown in (2.15), the FI acts as a custodian of the existing stock of long term bonds.) The value of the bonds over which the FI serves as custodian is $\kappa Q_t B_{t-1}^{FI}$. Long term bonds receive a risk weight of unity, while reserves on account with the central bank have a risk weight of zero. The leverage constraint is:

$$Q_t \left(CB_t^{FI} + \kappa B_{t-1}^{FI}\right) \leq \Theta_t P_t X^{FI}. \tag{2.17}$$

In other words, (2.17) says that the value of long bonds held by the FI cannot exceed a time-varying multiple, Θ_t, of its nominal net worth, $P_t X^{FI}$. We assume that Θ_t obeys a known stochastic process and refer to changes in Θ_t as credit shocks.

The objective of the FI is to maximize the expected one period ahead value of (2.16), discounted by the nominal stochastic discount factor of the parent household, i.e. $\Lambda_{t,t+1}\Pi_{t+1}^{-1}$, subject to (2.17). Letting Ω_t denote the multiplier on the constraint, the first order conditions are:

$$\mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1} \left(R_{t+1}^{b} - R_t^{s}\right) = \Omega_t, \tag{2.18}$$

$$\mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1} \left(R_t^{re} - R_t^{s}\right) = 0. \tag{2.19}$$

(2.19) says that the FI will hold an indeterminate amount of reserves so long as the return on reserves, R_t^{re}, equals the cost of funds, R_t^s. Absent a leverage constraint, the FI would buy newly issued long bonds up until the point at which the expected return on long bonds equals the cost of funds. The constraint being binding, i.e. $\Omega_t > 0$, generates excess returns.

2.2.4 Production

The production side of the economy is split into three sectors: final output, retail output, and wholesale output. There is a representative final good firm and representative wholesale producer. There are a continuum of retailers, indexed by $f \in [0,1]$.

The final output good, Y_t, is a CES aggregate of retail outputs, with $\varepsilon > 1$ the elasticity of substitution. This gives rise to a standard demand function for each variety of retail output and an aggregate price index:

$$Y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} Y_t, \tag{2.20}$$

$$P_t = \left[\int_0^1 P_t(f)^{1-\epsilon} df\right]^{\frac{1}{1-\epsilon}}. \tag{2.21}$$

Retailers purchase wholesale output at price $P_{m,t}$ and repackage it for sale at $P_t(f)$. $P_{m,t}$ has the interpretation as nominal marginal cost. Retailers are subject to a Calvo (1983) pricing friction – each period, there is a probability $1-\phi$ that a retailer may adjust its price, with $\phi \in [0,1]$. When given the opportunity to adjust, retailers pick a price to maximize the present discounted value of expected profits, where discounting is by the stochastic discount factor of the parent household. Optimization results in an optimal reset price, $P_{*,t}$, that is common across updating retailers. Letting $p_{m,t} = P_{m,t}/P_t$ denote real marginal cost, the optimal reset price satisfies:

$$P_{*,t} = \frac{\epsilon}{\epsilon - 1}\frac{X_{1,t}}{X_{2,t}}, \tag{2.22}$$

$$X_{1,t} = P_t^{\epsilon} p_{m,t} Y_t + \phi\, \mathbb{E}_t\, \Lambda_{t,t+1} X_{1,t+1}, \tag{2.23}$$

$$X_{2,t} = P_t^{\epsilon-1} Y_t + \phi\, \mathbb{E}_t\, \Lambda_{t,t+1} X_{2,t+1}. \tag{2.24}$$

The wholesale firm produces output, $Y_{m,t}$, according to a linear technology in labor:

$$Y_{m,t} = A_t L_t. \tag{2.25}$$

A_t is an exogenous productivity disturbance obeying a known stochastic process. Letting $w_t = W_t/P_t$ denote the real wage, the optimality condition is standard:

$$w_t = p_{m,t} A_t. \tag{2.26}$$

2.2.5 Central Bank

The central bank can hold a portfolio of long bonds, B_t^{cb}. It finances this portfolio via the creation of reserves, RE_t. Its balance sheet condition is:

$$Q_t B_t^{cb} = RE_t. \tag{2.27}$$

We will refer to the real value of the central bank's bond portfolio as $QE_t = Q_t b_t^{cb}$, where $b_t^{cb} = B_t^{cb}/P_t$, and shall assume that the central bank may freely choose this (equivalently, it can freely choose reserves). The central bank potentially earns an operating surplus that is returned to the parent household via a lump sum transfer.[3] This transfer satisfies:

[3] Alternatively, we could assume that the transfer is returned to the fiscal authority, who then adjusts the lump sum tax/transfer levied on households accordingly. Because we are not interested in describing fiscal policy, it is simpler to instead assume that the central bank provides rebates to the parent household directly.

$$P_tT_t = R_t^bQ_{t-1}B_{cb,t-1} - R_{t-1}^{re}RE_{t-1}. \tag{2.28}$$

2.2.6 Aggregation and Equilibrium

Market-clearing requires that $RE_t = RE_t^{FI}$ and $S_t = S_t^{FI}$ (i.e. the FI holds all reserves issued by the central bank and all one period bonds issued by the parent household), while $B_t = B_t^{FI} + B_t^{cb}$ (i.e. the total stock of long term bonds issued by the child must be held by the FI or the central bank). Some algebraic substitutions give rise to a standard aggregate resource constraint:

$$Y_t = C_t + C_{b,t}. \tag{2.29}$$

Aggregating across retailers gives rise to the aggregate production function, where v_t^p is a measure of price dispersion:

$$Y_tv_t^p = A_tL_t. \tag{2.30}$$

We assume that the transfer from parent to child, X_t^b, is time-varying in a way that represents a complete payoff of outstanding debt obligations each period:

$$P_tX_t^b = (1 + \kappa Q_t)B_{t-1}. \tag{2.31}$$

Neither the parent nor the child behaves as though it can influence the value of X_t^b. The particular assumption embodied in (2.31) implies that, even though the child solves a dynamic problem and has a forward-looking Euler equation, (2.14), its consumption is effectively static:

$$P_tC_{b,t} = Q_tB_t. \tag{2.32}$$

This assumption on the parent-child transfer allows us to eliminate a state variable and simplifies the system to four equations, although it is not crucial for the qualitative or quantitative properties of the model.

A_t and Θ_t obey conventional AR(1) processes in the log. We define potential output, Y_t^f, as the equilibrium level of output consistent with price flexibility (i.e. $\phi = 0$) and where the credit shock is constant, i.e. $\Theta_t = \Theta$. The natural rate of interest, R_t^f, is the gross real short term interest rate consistent with this level of output. $X_t = Y_t/Y_t^f$ is the gross output gap. The full set of equilibrium conditions are contained in Appendix A. The system can be greatly simplified, and the equilibrium conditions log-linearized about a zero inflation steady state can be reduced to the four equation system presented at the beginning of this section; i.e. (2.1)-(2.2) along with rules for the short term policy rate and the central bank's long bond portfolio. Details of the linearization may be found in Appendix B.

2.3 The Four vs. the Three Equation Model

Before turning to normative optimal policy analysis in Section 3, we first explore the positive properties of the linearized model as described above in Subsection 2.1.

For the purpose of studying positive properties of the model, we suppose that the short term rate follows a Taylor-type rule while the long bond portfolio obeys an exogenous process:

$$r_t^s = \rho_r r_{t-1}^s + (1-\rho_r)\Big[\phi_\pi \pi_t + \phi_x x_t\Big] + s_r \varepsilon_{r,t}, \tag{2.33}$$

$$qe_t = \rho_q qe_{t-1} + s_q \varepsilon_{q,t}. \tag{2.34}$$

r_t^f and θ_t, the natural rate of interest and credit shock, respectively, obey stationary AR(1) processes:

$$r_t^f = \rho_f r_{t-1}^f + s_f \varepsilon_{f,t}, \tag{2.35}$$

Table 1: Parameter Values of Linearized Model

Parameter	Value	Description (Target)
β	0.995	Discount factor
z	0.33	Consumption share of child
σ	1	Inverse elasticity of substitution
$\bar{b}^{FI}$	0.70	Weight on leverage in IS/PC curves
$\bar{b}^{cb}$	0.30	Weight on QE in IS/PC curves
γ	0.086	Elasticity of inflation w.r.t. marginal cost
ζ	2	Elasticity of gap w.r.t. marginal cost
ρ_r	0.8	Taylor rule smoothing
ϕ_π	1.5	Taylor rule inflation
ϕ_x	0	Taylor rule gap
ρ_f	0.8	AR natural rate
ρ_θ	0.8	AR leverage
ρ_q	0.8	AR QE

Note: this table lists the values of calibrated parameters of the linearized four equation model.

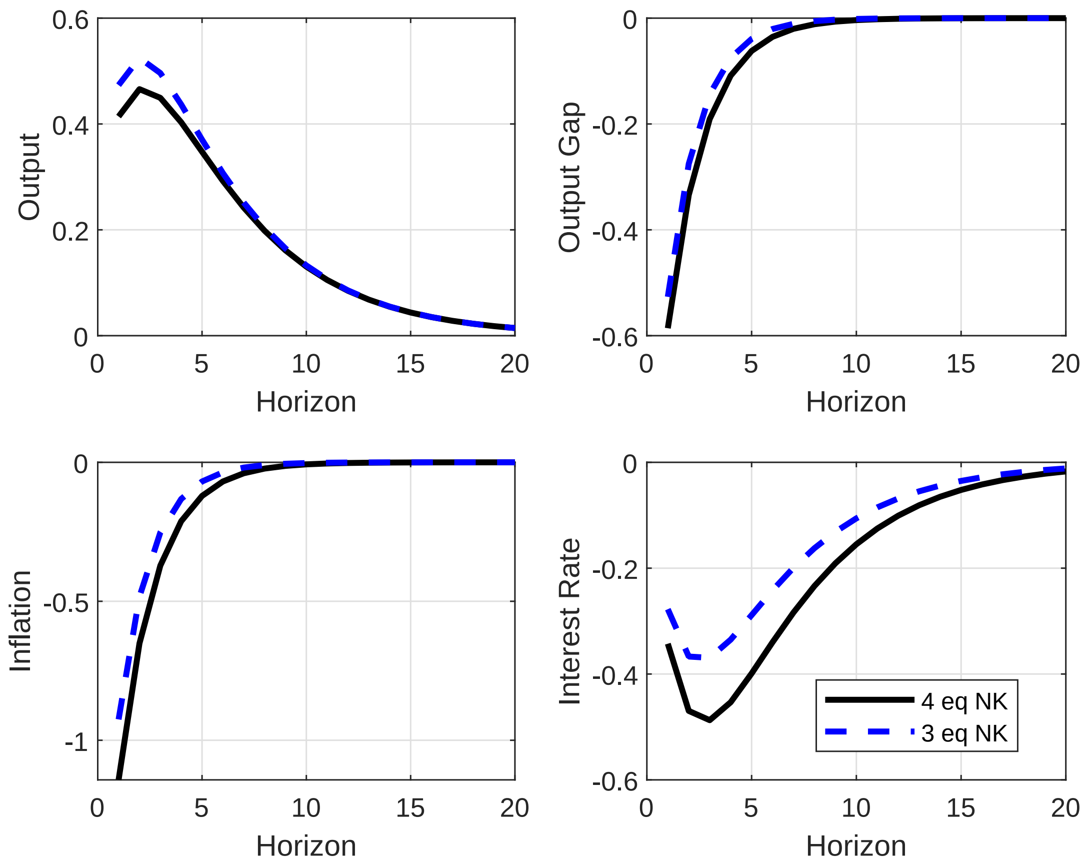
$$\theta_t = \rho_\theta \theta_{t-1} + s_\theta \varepsilon_{\theta,t}. \tag{2.36}$$

A full description and justification of the underlying parameter values of the non-linear model is provided in Appendix C. Here, we focus only on the parameter values necessary for solving the linearized model. These parameter values are listed in Table 1. The discount factor and elasticity of substitution take on standard values. The child-share of total consumption is set to one third. Given our calibrations of other steady state parameters (discussed further in Appendix C), we have $\bar{b}^{FI} = 0.7$ and $\bar{b}^{cb} = 0.3$. The elasticity of inflation with respect to real marginal cost is $\gamma = 0.086$ and the elasticity of the output gap with respect to real marginal cost is $\zeta = 2$, implying a slope of the Phillips Curve of 0.21. The parameters of the Taylor rule are standard. The autoregressive parameters in the exogenous processes are all set to 0.8.

Figure 1 displays impulse responses to a one percent positive shock to potential output.[4]

[4] As written, the linearized model presented in Subsection 2.1 writes the exogenous process in terms of the natural rate of interest. As shown in Appendix B, there is a mapping between the natural rate of interest and potential output. When comparing the four equation to the three equation model, the mapping between the natural rate of interest and potential output is not identical due to the presence of z in the four equation model. The comparison is more natural for an equal sized shock to potential output rather than the natural rate of interest.

Figure 1: **IRFs to Natural Rate Shock**

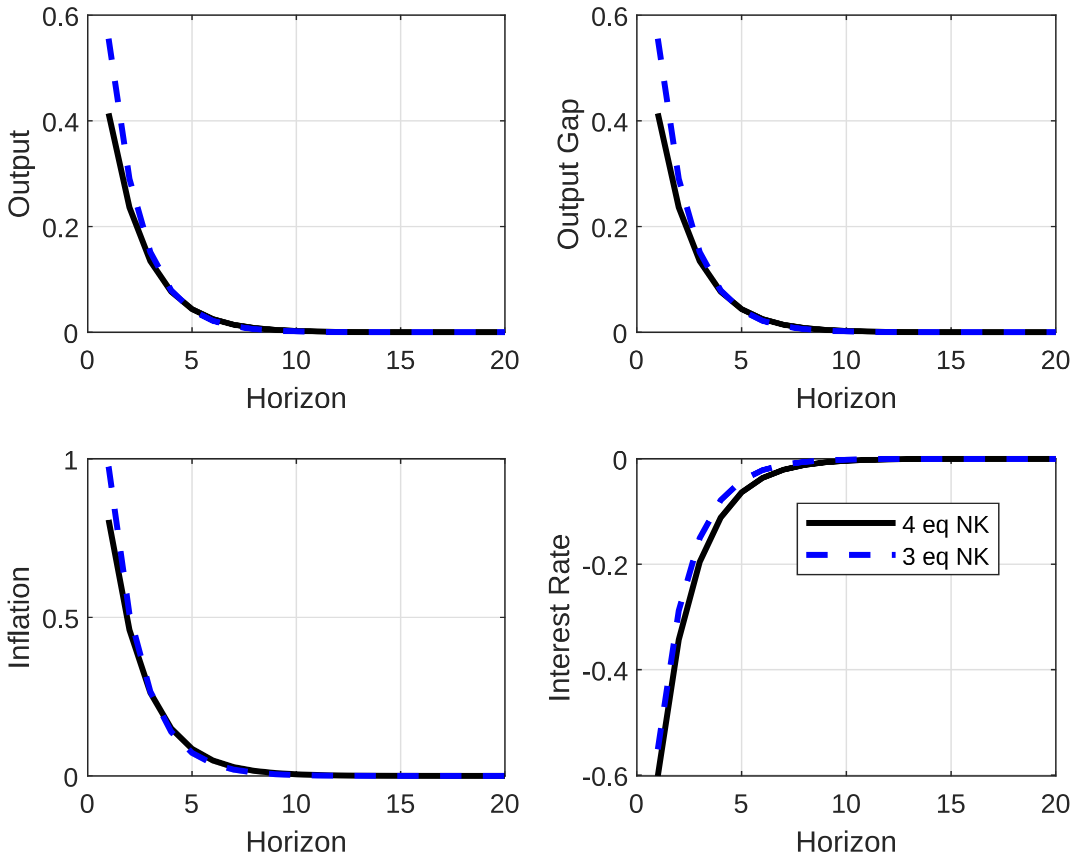


Notes: Black solid lines: IRFs to a one percentage point shock to the natural rate of output in the four equation model. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the baseline three equation NK model.

The solid black lines are responses in our baseline four equation model, whereas the dashed blue lines depict responses in the conventional three equation model (i.e. our model imposing $z = 0$). These responses are familiar and do not differ much in our model compared to the more standard three equation model. Output increases but by less than potential, resulting in a negative output gap. This puts downward pressure on inflation, which is met with policy accommodation with the short term interest rate declining. Relative to three equation model, output reacts slightly less on impact in our model, though this difference is not large.

Figure 2 plots impulse responses to a conventional monetary policy shock. The size and sign of the shock are chosen to generate the same impact response of output to the potential

Figure 2: **IRFs to Policy Shock**

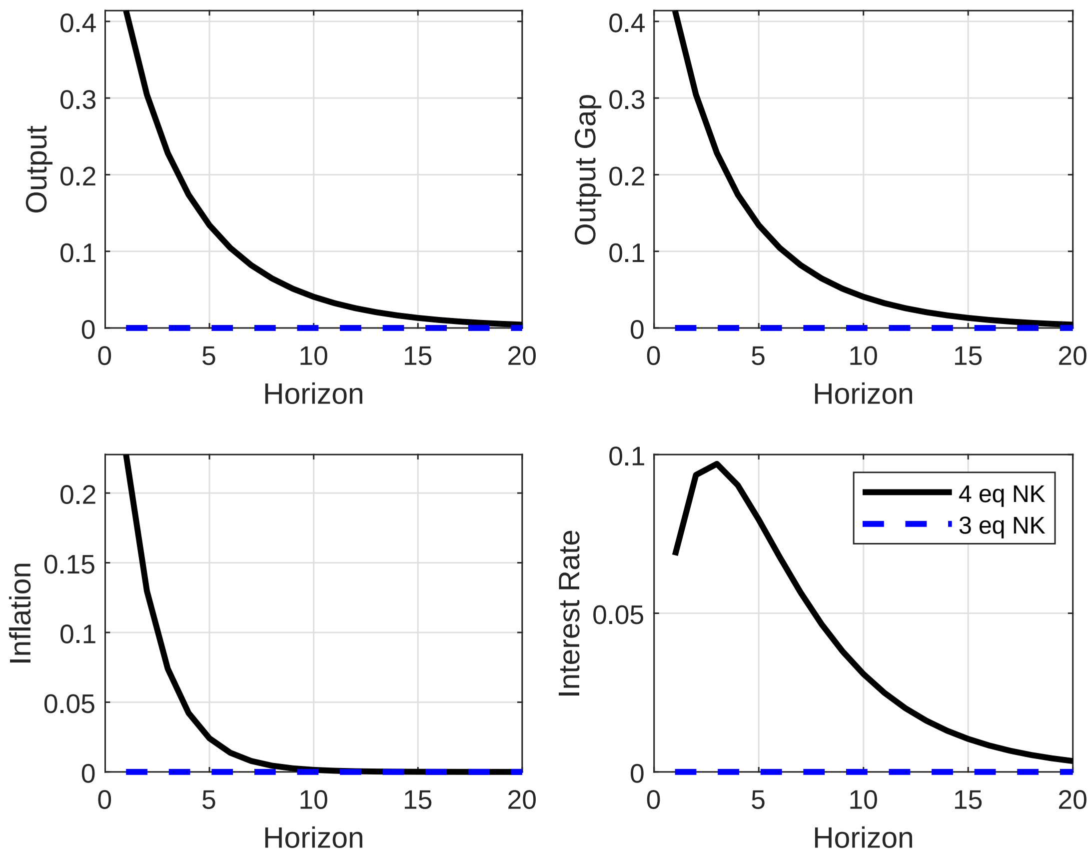


Notes: Black solid lines: IRFs to a conventional monetary policy shock. The size and sign of the shock are chosen to generate the same impact response of output as in Figure 1. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized policy shock in the baseline three equation NK model.

output shock in the four equation model. Output (and hence the output gap) rises on impact before reverting to its pre-shock value. Inflation rises and follows a similar dynamic path as output. As in the case of the potential output shock, there is little meaningful difference in the responses of variables in our four equation model relative to the baseline three equation model.

Figure 3 plots impulse responses to a leverage (θ_t) or QE (qe_t) shock. Because these differ only according to scale in the linear system (i.e. $\bar{b}^{FI} \neq \bar{b}^{cb}$), because we have assumed equal AR parameters (0.8), and because the shock sizes are normalized to produce the same

Figure 3: **IRFs to Leverage/QE Shock**



Notes: Black solid lines: IRFs to a leverage (θ_t) or QE (qe_t) shock. The size and sign of the shocks are chosen to generate the same impact response of output as in Figure 1. Because the QE and leverage shock only differ according to scale in the linearized model (i.e. $\bar{b}^{FI} \neq \bar{b}^{cb}$) and the AR parameters are the same, the normalized impulse responses are identical. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized policy shock in the baseline three equation NK model.

impact response of output, the IRFs of endogenous variables to a leverage or QE shock are identical. We therefore only show one set of impulse responses.

Unlike responses to the other shocks, in Figure 3, there is a meaningful difference between the four equation model and the three equation model. In the three equation model, both shocks are irrelevant for the dynamics of endogenous variables. In our four equation model, an increase in leverage (equivalently a central bank purchase of long bonds) is expansionary for output. In the current calibration, such an expansion also results in an increase in inflation and a resulting increase in the short term interest rate. That financial shocks have economic effects in-line with the traditional understanding of an aggregate demand shock and the fact that there is scope for QE policies represent a key advancement in our four equation model relative to the standard three equation model. These properties are critical for understanding the post-Crisis economy.

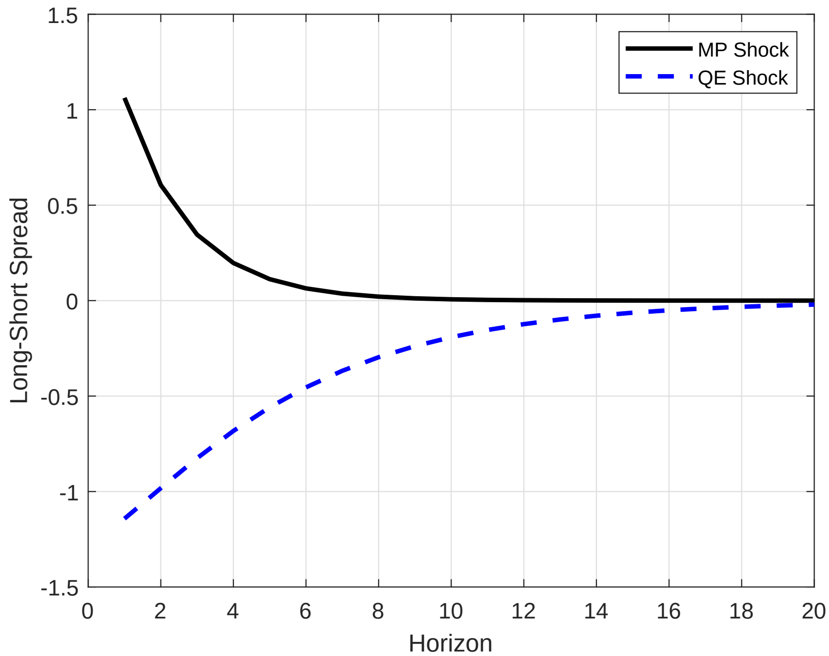
As noted above, an expansionary QE shock is less inflationary than a conventional monetary policy shock. Quantitatively comparing Figure 2 with Figure 3, one observes that a QE shock that increases output by the same amount as a conventional policy rate cut results in about one-third the response of inflation. Another important difference between a QE shock and a conventional policy shock concerns how each affects the yield curve. Though a long term interest rate does not appear in the baseline four equation model in Subsection 2.1, one is operating in the background and can be inferred from an alternative representation of the IS curve (which is derived in Appendix B):

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1}\right) - \frac{z}{\sigma}\left(\mathbb{E}_t\, r_{t+1}^b - r_t^s\right). \tag{2.37}$$

$\mathbb{E}_t\, r_{t+1}^b$ is the expected return on the long bond in the model. Hence, the last term in (2.37) can be interpreted as a credit spread.

The conventional expansionary monetary policy shock results in a steeping of the yield curve (i.e. an increase in the long rate relative to the short rate). In contrast, a stimulative QE shock results in a flattening of the yield curve. QE works by freeing up space on the FI's balance sheet to purchase long bonds, thereby pushing the price of these bonds higher and the yield lower. There is no direct effect on the short term rate except through the policy rule. As calibrated, the short rate actually rises modestly (due to the slightly inflationary nature of a QE shock under the current calibration). Impulse responses of the long-short spread to both a conventional policy shock and a QE shock are depicted in Figure 4.

Figure 4: **Response of Long-Short Spread to Monetary and QE Shocks**



Notes: This figures plots the responses of the annualized long short interest rate spread, i.e. $\mathbb{E}_t\, r^b_{t+1} - r^s_t$, inferred from (2.37), to a conventional monetary policy shock (solid black) and a QE shock (dashed blue). The shocks are normalized so as to generate the same impact increase in output as in Figure 2 and Figure 3.

3 Optimal Monetary Policy

In this section, we explore the design of optimal monetary policy in the context of our four equation NK model. Leverage shocks generate an endogenous cost-push term in the Phillips Curve, so they lead to a non-trivial tradeoff for a central bank wishing to solely implement policy via adjustment of the short term interest rate. As such, heretofore unconventional policies like quantitative easing ought to be used even when the short rate is unconstrained by the ZLB. Further, quantitative easing policies can be a useful (albeit imperfect) substitute for conventional policy when the short term rate is constrained by the ZLB.

Given policymakers' emphasis on the so-called dual mandate, we focus on a policy-relevant quadratic loss function in inflation and the output gap, which is also what much of the literature has used:

$$\mathbb{L} = \mu x_t^2 + \pi_t^2. \tag{3.1}$$

$\mu \geq 0$ is the relative weight attached to fluctuations in the output gap. An expression like (3.1) can be motivated as the micro-founded welfare criterion for a central bank in the standard three equation NK model under certain assumptions.[5,6]

In what follows, we first consider optimal policy when there are no constraints, and then study optimal policy when only one policy tool is available.

3.1 Unconstrained Optimal Policy

We begin by studying optimal monetary policy when both policy instruments are available. Because the credit shock appears in both the IS and Phillips curves, the so-called "Divine Coincidence" (Blanchard and Galí 2007) does not hold. This gives rise to Theorem 1:

Theorem 1 *It is not possible to completely stabilize both inflation and the output gap with the adjustment of a single policy instrument when both credit and natural rate shocks are present.*

Proof: See Appendix D.

Although the formal proof of Theorem 1 is more involved, the intuition is straightforward. In the benchmark three equation model with no credit shocks, setting $r_t^s = r_t^f$ would be consistent with $x_t = \mathbb{E}_t\, x_{t+1} = \mathbb{E}_t\, \pi_{t+1} = 0$ in the IS curve, (2.1), which would also be consistent with $\pi_t = x_t = \mathbb{E}_t\, \pi_{t+1} = 0$ in the Phillips Curve, (2.2). In other words, the global minimum of the loss function can be achieved by setting the short term interest rate equal to the natural rate, which would also be equivalent to implementing a strict inflation target of

[5]In particular, in the benchmark model (3.1) would be the micro-founded loss function when a Pigouvian tax is in place to undo the steady state distortion associated with monopolistic competition; see, e.g., Woodford (2003). The optimal weight on the output gap would satisfy $\mu = \frac{\gamma\zeta}{\epsilon}$, where $\gamma\zeta$ is the slope of the Phillips Curve and ϵ is the elasticity of substitution across varieties of retail goods. For conventional calibrations, this weight would be quite low.

[6] In our four equation model, a micro-founded loss function would be more complicated due to the two types of households, and would depend on arbitrary welfare weights on each. Rather than deriving such a loss function, we choose to focus on a policy-relevant loss function like (3.1) and consider a variety of different values of μ.

$\pi_t = 0$.[7] With credit shocks and $z \neq 0$, in contrast, setting $r_t^s = r_t^f$ would entail fluctuations in both π_t and x_t. This result obtains because credit shocks appear in the Phillips curve as well as the IS curve. Via similar logic, were the short term rate exogenously fixed, it would not be possible to endogenously adjust qe_t so as to implement $\pi_t = x_t = 0$ either.

Given the impossibility result of Theorem 1, the central bank should use both the short term rate and its long bond portfolio as policy instruments. Doing so, it is in principle possible to achieve the global minimum of the loss function with zero inflation and a zero output gap. The optimal policy is described in Proposition 1.

Proposition 1 *With both instruments available, optimal policy for any μ entails setting $r_t^s = r_t^f$ and $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$. This policy results in $\pi_t = x_t = 0$.*

The proof of Proposition 1 is simple. Setting $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$ causes the θ_t and qe_t terms to drop out from both the IS and Phillips curves. Then the model is isomorphic to the standard three equation model, and consequently setting $r_t^s = r_t^f$ stabilizes both inflation and the output gap in response to shocks to the natural rate of interest. The implication of Proposition 1 is that QE-type policies in principle ought to be used to offset credit market shocks all the time, not only when conventional policy is constrained by the ZLB. While somewhat counter to conventional wisdom, this implication is rather intuitive in our model – QE-type policies work similarly to exogenous credit market disturbances, and hence can be deployed to offset them, leaving room for the short term policy rate to counter the sticky price distortion as in the standard three equation model.

3.2 Optimal QE at the ZLB

Although QE type policies should always be used to offset credit market disturbances in our model, they only became popular when short term interest rates were pushed to the

[7] Note that we do not concern ourselves with issues of equilibrium determinacy. It is well-known that interest rate pegs are inconsistent with a determinate rational expectations equilibrium (e.g. Sargent and Wallace 1975). An interest rate rule with a sufficiently strong reaction to an endogenous variable, e.g. $r_t^s = r_t^f + \phi_\pi \pi_t$ with $\phi_\pi > 1$, would be consistent with $r_t^s = r_t^f$ and $\pi_t = x_t = 0$ being the unique equilibrium outcome.

ZLB in the wake of the Financial Crisis and ensuing Great Recession at the end of the first decade of the 2000s. In this section, we study how QE policies might be used to mitigate the consequences of a binding ZLB.

We approximate the effects of a binding ZLB in our linearized model following Eggertsson and Woodford (2003) and Christiano, Eichenbaum and Rebelo (2011). Suppose that a central bank has been following the jointly optimal policy described in Proposition 1. But then in period t, suppose the natural rate of interest falls below zero, so that $r_t^s = 0$. Suppose that it will stay there in each subsequent period with probability $\alpha \in [0, 1)$, where this probability is invariant over time. The expected duration of the ZLB is therefore $1/(1-\alpha)$. This means that interest rate policy can be characterized as follows:

$$r_t^s = 0 \tag{3.2}$$

$$\mathbb{E}_t\, r_{t+1}^s = \mathbb{E}_t\, r_{t+1}^f \text{ with probability } 1-\alpha \tag{3.3}$$

$$\mathbb{E}_t\, r_{t+1}^s = 0 \text{ with probability } \alpha \tag{3.4}$$

To solve for the equilibrium, we must in addition specify the path of the central bank's long bond portfolio.

3.2.1 QE Only Stabilizing Credit Shocks

As a first step, we suppose that the central bank adjusts its long bond portfolio to offset credit market disturbances regardless of whether the short term interest rate is stuck at the ZLB. This means that $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$ as in Proposition 1. We later study alternative policy rules in which the long bond portfolio also reacts to natural rate shocks when the short term interest rate is stuck at zero in Section 3.2.2.

We solve for the policy functions assuming the short rate is stuck at zero and QE policies are managed as described above. This gives rise to Lemma 1.

Lemma 1 *When the short rate is constrained for* $1/(1-\alpha)$ *periods in expectation and the* *central bank's long bond portfolio obeys* $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$*, the equilibrium dynamics of inflation and the output gap are not impacted by credit shocks and* $x_t = \omega_1 r_t^f$ *and* $\pi_t = \omega_2 r_t^f$*, where* ω_1 *and* ω_2 *are functions of underlying structural parameters.*

Proof: See Appendix D.

An important and novel implication of Lemma 1 is that the ZLB need not pose a problem for credit shocks – adjusting QE policies exactly as a central bank would absent a ZLB completely stabilizes the gap and inflation. The same is not true, however, for shocks to the natural rate of interest.

Focus on the parameter space where[8]

$$\omega_1 > 0,\ \omega_2 > 0. \tag{3.5}$$

Whereas absent a ZLB constraint the optimal policy would completely stabilize both inflation and the output gap, at the ZLB both inflation and the gap fall in response to a negative shock to the natural rate. The inability to lower the policy rate leaves policy too tight relative to what is optimal, resulting in a contraction in aggregate demand. These effects are more marked the larger is α (i.e. the longer is the expected duration of the ZLB).

3.2.2 QE: an Imperfect Substitute for Conventional Policy

The more interesting and policy-relevant question is whether, and to what extent, QE can be an effective substitute for conventional monetary policy during periods in which the short term interest rate is constrained by zero. This was the original motivation for the use of QE in countries like Japan and the US when policy rates moved to the ZLB.

[8] As noted in Carlstrom, Fuerst and Paustian (2014), a caveat here is that for sufficiently large α, the signs of ω_1 and ω_2 can flip from positive to negative (see, e.g., (D.14)-(D.15) in Appendix D). Where this perverse sign flip occurs depends on the values of other parameters, such as the slope of the Phillips Curve, $\gamma\zeta$. We restrict attention to values of α consistent with ω_1 and ω_2 being positive. An alternative experiment would be to make the duration of the ZLB deterministic rather than stochastic. There would be no sign flip at some sufficiently long duration, but the analytic expressions for ω_1 and ω_2 would be significantly more complicated.

Lemma 2 *When the short rate is constrained for* $1/(1-\alpha)$ *periods in expectation and the central bank's long bond portfolio obeys* $qe_t = \tau r_t^f - \frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$*, the equilibrium dynamics of inflation and the output gap are characterized by* $x_t = \widehat{\omega}_1 r_t^f$ *and* $\pi_t = \widehat{\omega}_2 r_t^f$*, where:*

$$\widehat{\omega}_1 = \omega_1 + d_1\tau \tag{3.6}$$

$$\widehat{\omega}_2 = \omega_2 + d_2\tau \tag{3.7}$$

ω_1 *and* ω_2 *are identical to the values in Lemma 1.*

Proof: See Appendix D.

In Lemma 2, in addition to reacting optimally to credit shocks, the central bank's long bond portfolio adjusts to changes in the natural rate of interest via the parameter τ. The resulting policy functions for the output gap and inflation are given in (3.6)-(3.7).

We focus on values of $\tau < 0$, which means the central bank provides positive stimulus in the face of a decline in the natural rate of interest. We also focus on the region of the parameter space where, in addition to ω_1 and ω_2 being positive, d_1 and d_2 are positive. Most standard calibrations of the underlying parameters place the economy in this region of the parameter space. Hence, larger (in absolute value) values of τ result in smaller declines in both the output gap and inflation in response to a natural rate shock (i.e. $\widehat{\omega}_1$ and $\widehat{\omega}_2$ are less positive). For sufficiently large values of τ, the signs of $\widehat{\omega}_1$ or $\widehat{\omega}_2$ could flip from positive to negative.

Next, for a central bank following a QE rule such as the one described in Lemma 2, we characterize the optimal value of the parameter τ in Proposition 2:

Proposition 2 *If the short rate is constrained for* $1/(1-\alpha)$ *periods in expectation, the central bank's long bond portfolio obeys* $qe_t = \tau r_t^f - \frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$*, and the central bank's objective is to minimize (3.1), then the optimal* τ *is:*

$$\tau^* = -\left(\frac{\mu d_1\omega_1 + d_2\omega_2}{\mu d_1^2 + d_2^2}\right) \tag{3.8}$$

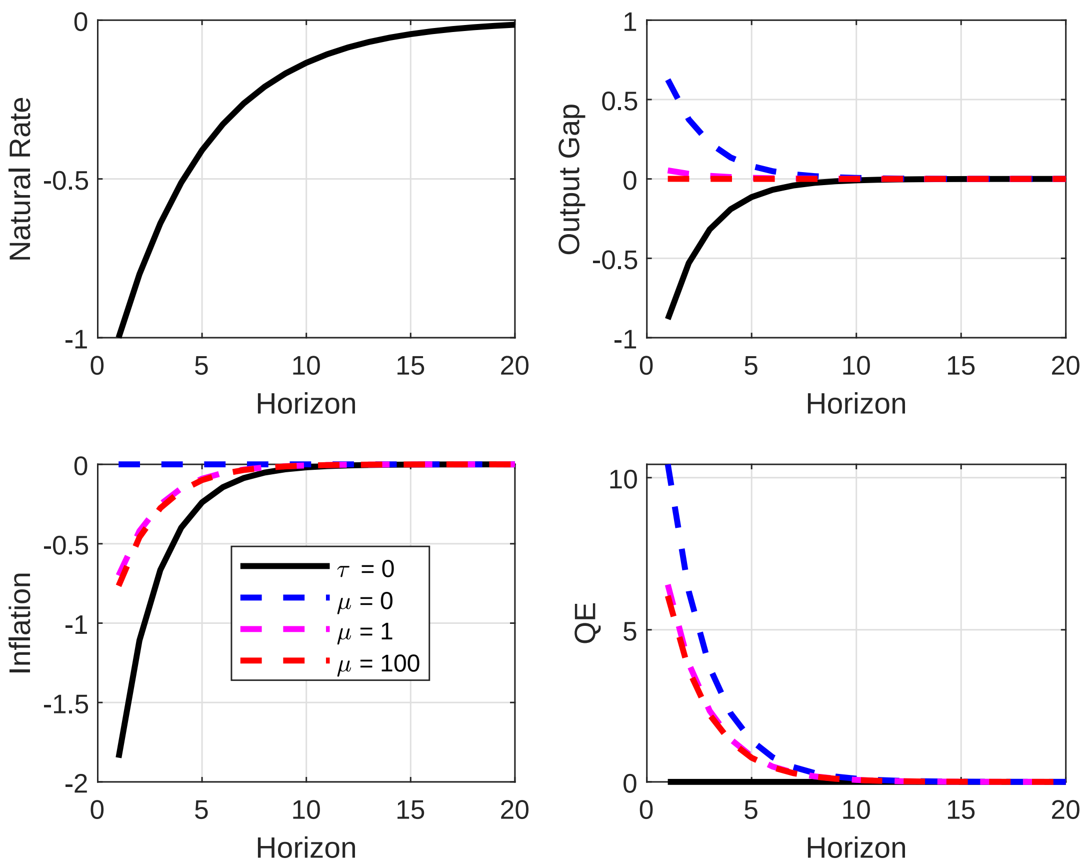
Proof: See Appendix D.

Under our maintained assumptions concerning the parameter space, all relevant parameters in (3.8) are positive, so that the optimal $\tau^* < 0$. Figure 5 plots responses of the output gap and inflation to a natural rate shock when τ is chosen optimally for different values of μ, the relative weight on fluctuations in the output gap. The solid black line shows responses when $\tau = 0$ for point of comparison. When the central bank places no weight on the output gap (i.e. $\mu = 0$), inflation is completely stabilized, the output gap increases quite markedly, and the central bank increases the size of its long bond portfolio by a sizeable amount. When virtually all weight is placed on the gap ($\mu = 100$), in contrast, inflation declines, the gap is completely stabilized, and the increase in the value of the long bond portfolio is much more modest. The case of equal weight on inflation and the gap (shown in pink) is quite close to the case of nearly all weight being on the gap in the loss function.

The results described in Figure 5 suggest that quantitative easing can be an effective, albeit imperfect, substitute for conventional policy in response to natural rate shocks at the ZLB. For example, in the case of equal relative weights ($\mu = 1$), the output gap essentially does not react to the natural rate shock and inflation falls by about two-thirds of a percent given optimal QE policy. In comparison, with no endogenous QE at the ZLB, the output gap would decline by nearly a full percentage point and inflation would fall by about three times as much. Endogenous QE therefore entails a sizeable welfare improvement over doing nothing at the ZLB. This will be true regardless of the value of μ.

We close this section by plotting the optimal τ as a function of μ. This is shown in Figure 6. The optimal τ is always negative, but is increasing in the relative weight on the output gap. That is, for a central bank concerned solely with stabilizing inflation, it is optimal to adjust the long bond portfolio quite strongly in response to natural rate shocks. For a central bank more concerned with gap stabilization, the optimal QE response remains sizeable but is nevertheless quite a bit smaller than for values of μ close to zero. The optimal values of τ are very similar for values of $\mu \geq 0.5$. To our knowledge, we are the

Figure 5: **IRFs to Natural Rate Shock at the ZLB, Endogenous QE, Optimal τ, Different μ**



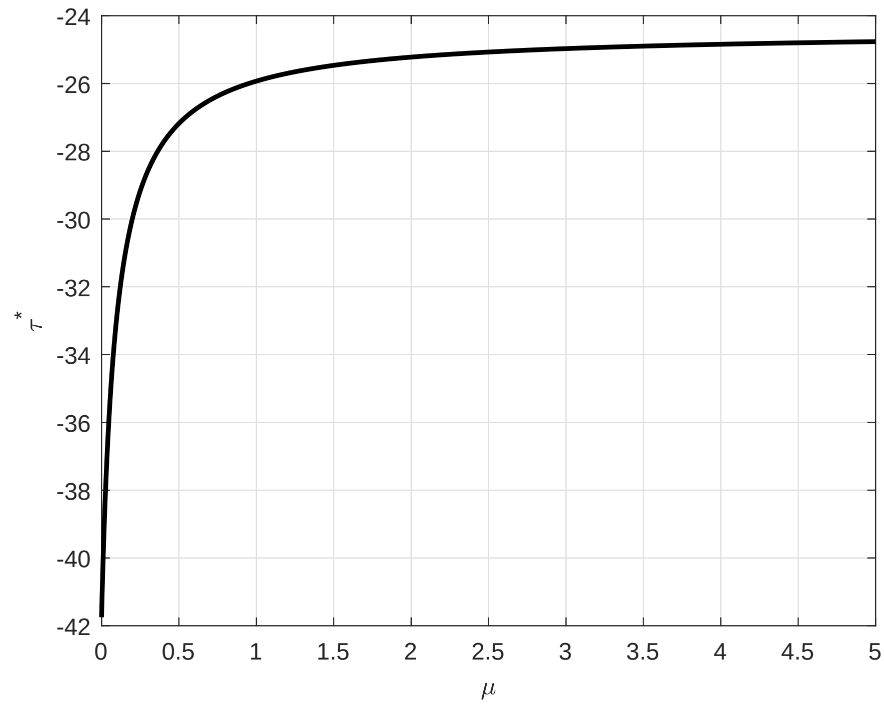
Notes: Black solid lines: IRFs to a one hundred basis point shock to the natural rate of interest in the four equation model when the short term interest rate is constrained by the ZLB for $1/(1-\alpha)$ periods in expectation, where $\alpha = 3/4$, and $\tau = 0$ so that there is no endogenous QE to the natural rate shock. The dashed lines plot responses with the optimally chosen τ for different welfare weights on the output gap, μ. The output gap is expressed in percentage points, while the responses of inflation and is in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the baseline three equation NK model.

first to discuss how QE should be implemented differently when central banks place different weights on their dual mandate.

3.3 Optimal Policy without QE

Next, consider an operating framework similar to the one prevailing in the US prior to the Great Recession in which the central bank uses the short term interest rate as its sole policy

Figure 6: **Optimal τ As a Function of μ**



Notes: This figure plots the optimal τ, i.e. (3.8), as a function of μ, the welfare weight on the output gap.

instrument. This subsection studies the optimal adjustment of the short term rate in this scenario.

Lemma 3 *Suppose $qe_t = 0$ and the central bank obeys the policy rule $r_t^s = r_t^f + \eta\theta_t$ for all t. Then, in equilibrium, the responses of the gap and inflation to credit shocks will be given by $x_t = \widehat{\varphi}_1\theta_t$ and $\pi_t = \widehat{\varphi}_2\theta_t$, where:*

$$\widehat{\varphi}_1 = \varphi_1 + a_1\eta \tag{3.9}$$

$$\widehat{\varphi}_2 = \varphi_2 + a_2\eta \tag{3.10}$$

Proof: See Appendix D.

A policy rule such as the one described in Lemma 3 completely stabilizes the output gap and inflation in response to natural rate shocks. This is not true conditional on credit shocks, where in general it is impossible to choose η such that $\widehat{\varphi}_1 = \widehat{\varphi}_2 = 0$. Suppose that the central bank wishes to choose η so as to minimize the welfare loss. The optimal η is given in Proposition 3:

Proposition 3 *Suppose $qe_t = 0$ and the central bank obeys the policy rule $r_t^s = r_t^f + \eta\theta_t$ for all t. If the central bank's objective is to minimize (3.1), then the optimal η is:*

$$\eta^* = -\left(\frac{\mu\varphi_1 a_1 + \varphi_2 a_2}{\mu a_1^2 + a_2^2}\right) \tag{3.11}$$

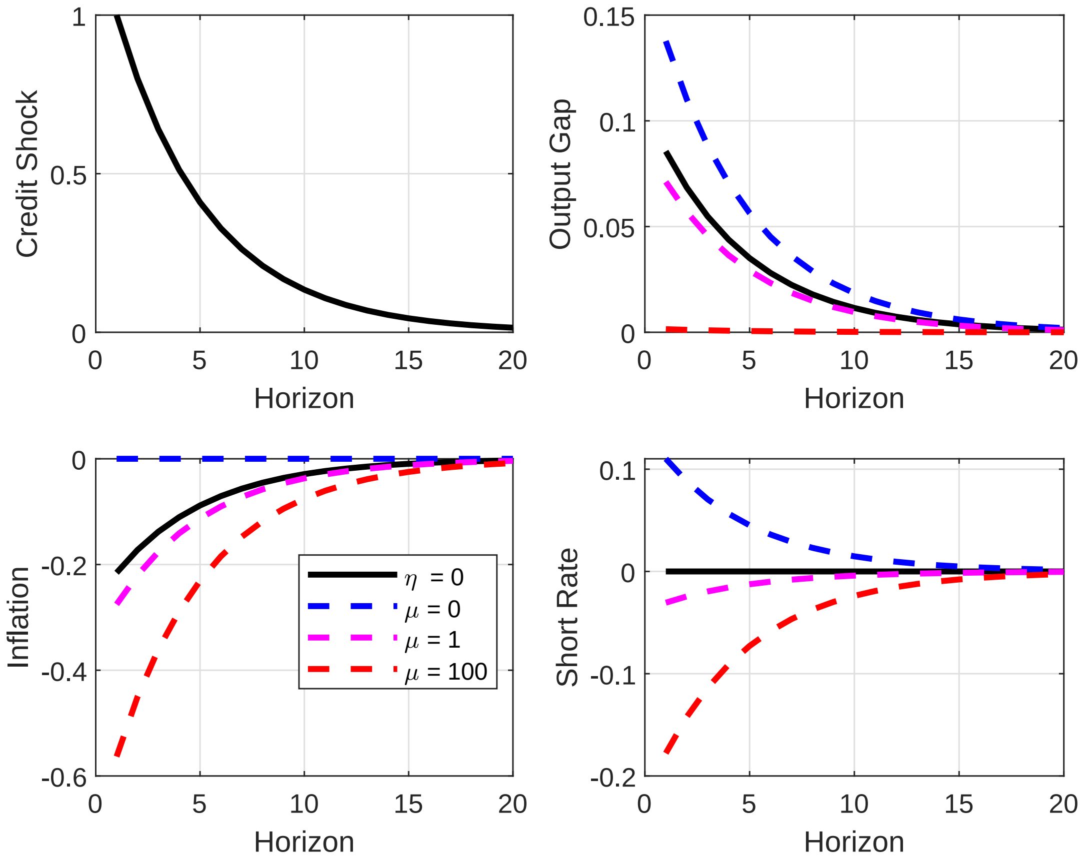
Proof: See Appendix D.

In general, the optimal value of η could be positive or negative, depending on the welfare weight on the gap, μ, as well as other parameters of the model. Figure 7 plots the responses of the output gap and inflation to a credit shock when η is chosen optimally as a function of different values of μ, taking our baseline calibration of other parameters. The solid black line shows responses for $\eta = 0$ for point of comparison.

When there is no weight placed on the output gap, (shown with the blue dashed lines corresponding to $\mu = 0$), the central bank raises the short term interest rate in response to a positive credit shock (i.e. $\eta^* > 0$). This completely stabilizes inflation but results in a sizeable increase in the output gap. In contrast, if the relative weight on the output gap is large ($\mu = 100$, shown in the red dashed lines), the central bank optimally cuts the policy rate in response to the credit shock (i.e. $\eta^* < 0$). This stabilizes the output gap but results in a significant decline in inflation. For equal weights on the output gap and inflation ($\mu = 1$, depicted via pink dashed lines), the optimal responses are not so different from the no response case – the policy rate decreases slightly, but the output gap rises and the inflation rate falls.

An interesting result from Figure 7 is that the *sign* of the optimal policy rate response to a credit shock depends on the relative weight placed on the output gap. A central bank mostly concerned with stabilizing output ought to cut the policy rate in the face of a positive

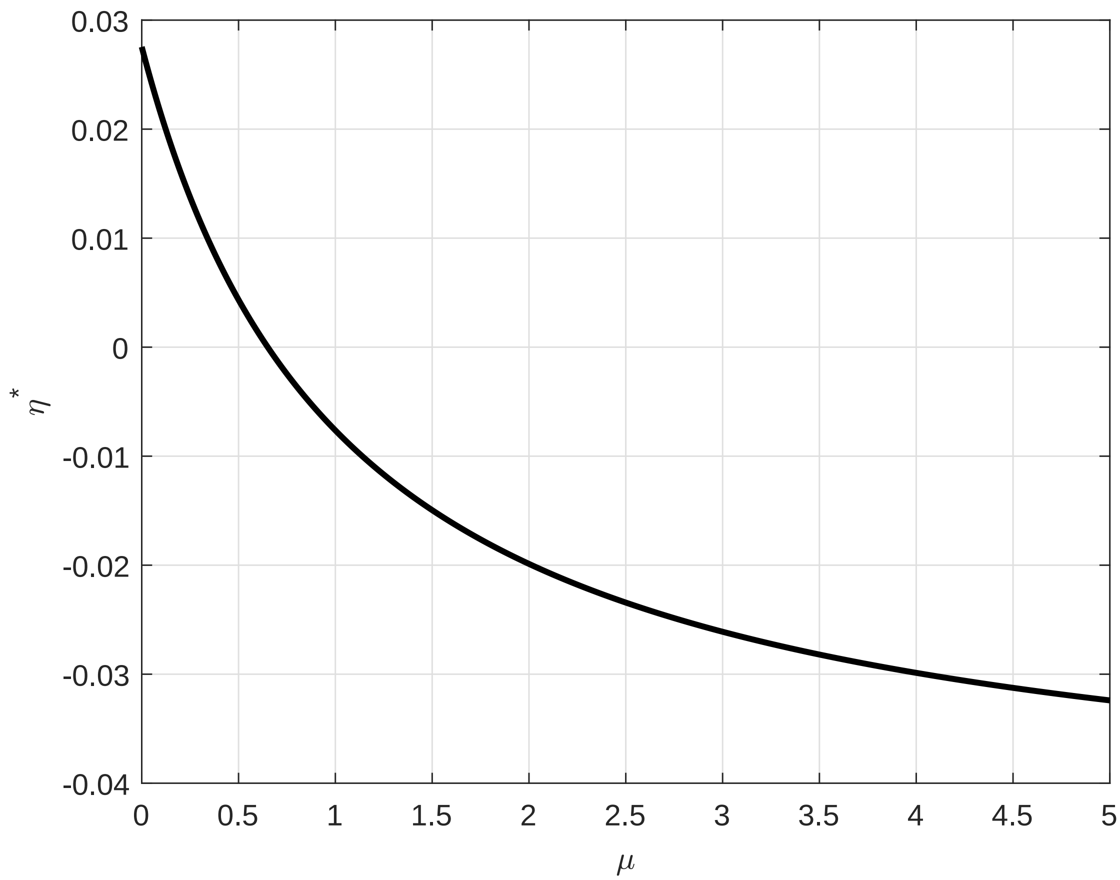
Figure 7: **IRFs to Credit Shock, Optimal η, Different μ**



Notes: IRFs to a one percentage point credit shock in the four equation model. $qe_t = 0$, while $r_t^s = r_t^f + \eta\theta_t$. The output gap is expressed in percentage points, while the responses of inflation and is in annualized percentage points. The black solid line corresponds to $\eta = 0$. The dashed lines choose η optimally given different welfare weights, μ.

credit shock, whereas it should raise the policy rate if it is mostly concerned with stabilizing inflation. Figure 8 plots the optimal value of η as a function of μ. Consistent with what is observed in Figure 7, the optimal η is positive when μ is very small and turns negative as μ gets bigger, crossing zero at around $\mu = 0.6$. For central banks facing a dual mandate, this tradeoff between stabilizing inflation or the output gap can be eliminated if they can deploy QE.

Figure 8: **Optimal η As a Function of μ**



Notes: This figure plots the optimal η, i.e., as a function of μ, the welfare weight on the output gap.

4 Conclusion

In this paper, we developed a four equation New Keynesian model with credit shocks, financial intermediation, short and long term debt, and a channel for central bank long bond holdings to be economically relevant. The model inherits the tractability and elegance of the benchmark three equation New Keynesian model. It mainly differs in that credit shocks appear as wedges in both the IS and Phillips curves. In addition to a rule for the short term policy rate, the fourth equation in the model is a rule for QE.

The model allows us to address the consequences of credit market disturbances as well as the effects of large scale asset purchases. We produce several analytical results concerning monetary policy design. The presence of credit market frictions breaks the Divine Coincidence, meaning it is not possible to completely stabilize inflation and the output gap with just one policy instrument. Optimal monetary policy entails adjusting the short term interest rate to match fluctuations in the natural rate of interest, but manipulating the central bank's long bond portfolio so as to neutralize credit shocks. When it is not possible to adjust the short term interest (for example, because of a binding ZLB), credit market shocks need not result in amplified fluctuations if the central bank adjusts its long bond portfolio as it would in normal times. In response to natural rate shocks, adjustment of the central bank's long bond portfolio can serve as a highly effective, albeit imperfect, substitute for conventional policy.

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A The Full Non-Linear Model

This appendix describes the full set of non-linear equilibrium conditions of the model.

The optimality condition for retail firms may be re-written in stationary terms by defining $p_{*,t} = P_{*,t}/P_t$, $x_{1,t} = X_{1,t}/P_t^{\epsilon}$, and $x_{2,t} = X_{2,t}/P_t^{\epsilon-1}$:

$$p_{*,t} = \frac{\epsilon}{\epsilon - 1}\frac{x_{1,t}}{x_{2,t}} \tag{A.1}$$

$$x_{1,t} = p_{m,t}Y_t + \phi\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon}x_{1,t+1} \tag{A.2}$$

$$x_{2,t} = Y_t + \phi\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_t^{\epsilon-1}x_{2,t+1} \tag{A.3}$$

The aggregate inflation rate evolves according to:

$$1 = (1-\phi)p_{*,t}^{1-\epsilon} + \phi\Pi_t^{\epsilon-1} \tag{A.4}$$

Price dispersion evolves according to:

$$v_t^p = (1-\phi)p_{*,t}^{-\epsilon} + \phi\Pi_t^{\epsilon}v_{t-1}^p \tag{A.5}$$

Define lowercase variables as real values of nominal bonds, i.e. $b_t = B_t/P_t$. The balance sheet condition of the FI may be written:

$$Q_t\left(b_t^{FI} - \kappa\Pi_t^{-1}b_{t-1}^{FI}\right) + re_t = s_t + X^{FI} \tag{A.6}$$

Note that $cb_t^{FI} = b_t^{FI} - \kappa\Pi_t^{-1}b_{t-1}^{FI}$. The leverage constraint of the FI may be written:

$$Q_t b_t^{FI} \geq \Theta_t X^{FI} \tag{A.7}$$

The central bank's balance sheet can be written:

$$Q_t b_t^{cb} = re_t \tag{A.8}$$

Similarly, the market-clearing condition for long term bonds in real terms is:

$$b_t = b_t^{FI} + b_t^{cb} \tag{A.9}$$

The auxiliary QE_t variable is just the real value of the central bank's long bond portfolio:

$$QE_t = Q_t b_t^{cb} \tag{A.10}$$

Under our assumption on the transfer from parent to child, the consumption of the child may be written:

$$C_{b,t} = Q_t b_t \tag{A.11}$$

A_t and Θ_t obey stationary AR(1) processes, where the non-stochastic steady state value of productivity is normalized to unity and Θ denotes the non-stochastic steady state value of leverage.

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{A.12}$$

$$\ln \Theta_t = (1-\rho_\theta)\ln \Theta + \rho_\theta \ln \Theta_{t-1} + s_\theta \varepsilon_{\theta,t} \tag{A.13}$$

To close the model, it is necessary to specify rules for the policy rate and the central bank's bond holdings. For the analysis in Subsection 2.3, we assume that the policy rate is set according to a Taylor rule and that the central bank's bond holdings obey an exogenous AR(1) process:

$$\ln R_t^s = (1-\rho_r)\ln R^s + (1-\rho_r)\left[\phi_\pi(\ln \Pi_t - \ln \Pi) + \phi_x(\ln Y_t - \ln Y_t^f)\right] + s_r \varepsilon_{r,t} \tag{A.14}$$

$$\ln QE_t = (1-\rho_q)\ln QE + \rho_q \ln QE_{t-1} + s_q \varepsilon_{q,t} \tag{A.15}$$

In (A.14)-(A.15), R^s and QE denote the non-stochastic steady state values of the policy rate and the central bank's balance sheet, respectively.

Y_t^f can be found by solving the full system of equations assuming price flexibility ($\phi = 0$) where the credit shock is constant, i.e. $\Theta_t = \Theta$. Taking Y_t^f as given, the output gap is then:

$$\ln X_t = \ln Y_t - \ln Y_t^f \tag{A.16}$$

The optimality conditions for the parent household, (2.5)-(2.7); the definition of the return on the long bond, (2.12); the optimality conditions for the child, (2.13)-(2.14); the optimality conditions for the FI, (2.18)-(2.19); the balance sheet condition for the FI and the leverage constraint re-written in real terms, (A.6)-(A.7); the labor demand condition for the wholesale firm, (2.26); the optimality condition for optimal price-setting for retailers, re-written in stationary form, (A.1)-(A.3); the market-clearing condition and aggregate production function, (2.29)-(2.30); the central bank's policy rule for R_t^{re}, (A.14); the central bank's balance sheet and definition of the QE_t variable, (A.8) and (A.10); the central bank's QE rule, (A.15); the consumption of the child, (A.11); the bond market-clearing condition, (A.9); the evolution of inflation and price dispersion, (A.4)-(A.5); the definition of the output gap, (A.16); and the exogenous processes (A.12)-(A.3) constitute twenty-seven variables, $\{L_t, C_t, w_t, \Lambda_{t-1,t}, R_t^s, \Pi_t, \Lambda_{b,t-1,t}, R_t^b, Q_t, b_t^{FI}, \Theta_t, re_t, s_t, \Omega_t, R_t^{re}, p_{*,t}, x_{1,t}, x_{2,t}, p_{m,t}, Y_t, A_t, C_{b,t}, v_t^p, b_{cb,t}, b_t, QE_t, X_t\}$ in twenty-seven equations.

B Details of the Linearized Model

This Appendix provides details of the linearization of the non-linear model, the equilibrium conditions for which are given in Appendix A. Where possible, lowercase variables denote log deviations from steady state, e.g. $\theta_t = \ln \Theta_t - \ln \Theta$. Where the corresponding level variable is already lowercase, a "hat" is put atop the relevant variable to denote a log deviation from steady state, e.g. $\widehat{p}_{m,t} = \ln p_{m,t} - \ln p_m$. Variables without a time subscript denote non-stochastic steady state values. The model is linearized about a steady state with zero trend inflation (i.e. $\Pi = 1$) where the leverage constraint on intermediaries binds. The complete list of linearized equilibrium conditions are as follows:

$$\chi l_t = -\sigma c_t + \widehat{w}_t \tag{B.1}$$

$$\lambda_{t-1,t} = -\sigma(c_t - c_{t-1}) \tag{B.2}$$

$$0 = \mathbb{E}_t \lambda_{t,t+1} + r_t^s - \mathbb{E}_t \pi_{t+1} \tag{B.3}$$

$$\lambda_{b,t-1,t} = -\sigma(c_{b,t} - c_{b,t-1}) \tag{B.4}$$

$$r_t^b = \frac{\kappa}{R^b} q_t - q_{t-1} \tag{B.5}$$

$$0 = \mathbb{E}_t \lambda_{b,t,t+1} + \mathbb{E}_t r_{t+1}^b - \mathbb{E}_t \pi_{t+1} \tag{B.6}$$

$$q_t + \widehat{b}_t^{FI} = \theta_t \tag{B.7}$$

$$\left[Qb^{FI}(1-\kappa\Pi^{-1})\right] q_t + Qb^{FI}\widehat{b}_t^{FI} - \kappa\Pi^{-1}Qb^{FI}\widehat{b}_{t-1}^{FI} + \kappa\Pi^{-2}Qb^{FI}\pi_t + re \cdot \widehat{re}_t = s \cdot \widehat{s}_t \tag{B.8}$$

$$\mathbb{E}_t \lambda_{t,t+1} - \mathbb{E}_t \pi_{t+1} + \frac{R^b}{sp} \mathbb{E}_t r_{t+1}^b - \frac{R^s}{sp} r_t^s = \omega_t \tag{B.9}$$

$$r_t^{re} = r_t^s \tag{B.10}$$

$$\widehat{p}_{*,t} = \widehat{x}_{1,t} - \widehat{x}_{2,t} \tag{B.11}$$

$$\widehat{x}_{1,t} = (1-\phi\beta)\widehat{p}_{m,t} + (1-\phi\beta)y_t + \phi\beta\,\mathbb{E}_t \lambda_{t,t+1} + \epsilon\phi\beta\,\mathbb{E}_t \pi_{t+1} + \phi\beta\,\mathbb{E}_t \widehat{x}_{1,t+1} \tag{B.12}$$

$$\widehat{x}_{2,t} = (1-\phi\beta)y_t + \phi\beta\,\mathbb{E}_t \lambda_{t,t+1} + (\epsilon-1)\phi\beta\,\mathbb{E}_t \pi_{t+1} + \phi\beta\,\mathbb{E}_t \widehat{x}_{2,t+1} \tag{B.13}$$

$$\widehat{w}_t = \widehat{p}_{m,t} + a_t \tag{B.14}$$

$$(1-z)c_t + zc_{b,t} = y_t \tag{B.15}$$

$$\widehat{v}_t^p + y_t = a_t + l_t \tag{B.16}$$

$$\widehat{v}_t^p = 0 \tag{B.17}$$

$$\pi_t = \frac{1-\phi}{\phi}\widehat{p}_{*,t} \tag{B.18}$$

$$q_t + \widehat{b}_t^{cb} = \widehat{re}_t \tag{B.19}$$

$$\widehat{b}_t = \frac{b^{FI}}{b}\widehat{b}_t^{FI} + \frac{b^{cb}}{b}\widehat{b}_t^{cb} \tag{B.20}$$

$$c_{b,t} = q_t + b_t \tag{B.21}$$

$$qe_t = \rho_q qe_{t-1} + s_q\varepsilon_{q,t} \tag{B.22}$$

$$a_t = \rho_A a_{t-1} + s_A\varepsilon_{A,t} \tag{B.23}$$

$$\theta_t = \rho_\theta\theta_{t-1} + s_\theta\varepsilon_{\theta,t} \tag{B.24}$$

$$r_t^{re} = \rho_r r_{t-1}^{re} + (1-\rho_r)\left[\phi_\pi\pi_t + \phi_x x_t\right] + s_r\varepsilon_{r,t} \tag{B.25}$$

$$qe_t = \widehat{re}_t \tag{B.26}$$

$$x_t = y_t - y_t^f \tag{B.27}$$

This is twenty-seven equations in $\big\{l_t, c_t, \widehat{w}_t, \lambda_{t-1,t}, r_t^s, \pi_t, \lambda_{b,t-1,t}, r_t^b, q_t, \widehat{b}_t^{FI}, \theta_t, \widehat{re}_t, \widehat{s}_t, \omega_t, r_t^{re},$

$\widehat{p}_{*,t}, \widehat{x}_{1,t}, \widehat{x}_{2,t}, \widehat{p}_{m,t}, y_t, a_t, c_{b,t}, \widehat{v}_t^p, \widehat{b}_t^{cb}, \widehat{b}_t, qe_t, x_t\}$ in twenty-seven variables.

The model can be reduced to the equations presented in Subsection 2.1 as follows. First, (B.10) can be used to eliminate r_t^{re}, so that the policy rule may be written as (2.33) in terms of r_t^s. Second, (B.11)-(B.13) can be combined with (B.18), which yields the textbook New Keynesian Phillips Curve expressed as a function of marginal cost, where $\gamma = \frac{(1-\phi)(1-\phi\beta)}{\phi}$:

$$\pi_t = \gamma \widehat{p}_{m,t} + \beta \, \mathbb{E}_t \, \pi_{t+1} \tag{B.28}$$

Combining (B.1) with (B.14) and (B.16), making note of the fact that $\widehat{v}_t^p = 0$ around a zero inflation steady state, yields:

$$\widehat{p}_{m,t} = \chi y_t - (1+\chi)a_t + \sigma c_t \tag{B.29}$$

Making use of (B.15) allows us to write this as:

$$\widehat{p}_{m,t} = \frac{\chi(1-z)+\sigma}{1-z} y_t - (1+\chi)a_t - \frac{\sigma z}{1-z} c_{b,t} \tag{B.30}$$

Combining (B.19)-(B.21) with (B.26) allows us to write:

$$c_{b,t} = \frac{b^{FI}}{b}\left(q_t + \widehat{b}_t^{FI}\right) + \frac{b^{cb}}{b} qe_t \tag{B.31}$$

Defining $\bar{b}^{FI} = b^{FI}/b$ and $\bar{b}^{cb} = b^{cb}/b$ (i.e. the fraction of total bonds held by financial intermediaries and the central bank, respectively, in steady state), and making use of the binding leverage constraint, (B.7), leaves:

$$c_{b,t} = \bar{b}^{FI}\theta_t + \bar{b}^{FI} qe_t \tag{B.32}$$

Plugging (B.32) into (B.30) then gives:

$$\widehat{p}_{m,t} = \frac{\chi(1-z)+\sigma}{1-z} y_t - (1+\chi)a_t - \frac{\sigma z}{1-z}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] \tag{B.33}$$

Define the hypothetical natural rate of output, y_t^f, as the level of output consistent with flexible prices and no credit market shocks. That is, y_t^f is the level of output consistent with $\widehat{p}_{m,t} = \theta_t = qe_t = 0$, or:

$$y_t^f = \frac{(1+\chi)(1-z)}{\chi(1-z)+\sigma} a_t \tag{B.34}$$

But then, using (B.27), we can write marginal cost as:

$$\widehat{p}_{m,t} = \frac{\chi(1-z)+\sigma}{1-z} x_t + \frac{\sigma z}{1-z}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] \tag{B.35}$$

Plugging (B.35) into (B.28), defining $\zeta = \frac{\chi(1-z)+\sigma}{1-z}$, yields (2.2).

To derive the IS equation, combine (B.2)-(B.4) with (B.6) and (B.16). Doing so yields:

$$y_t = \mathbb{E}_t \, y_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t \, \pi_{t+1}\right) - \frac{z}{\sigma}\left(\mathbb{E}_t \, r_{t+1}^b - \mathbb{E}_t \, \pi_{t+1}\right) \tag{B.36}$$

But from the Euler equation for the impatient household, along with the "full bailout" assumption embodied in (B.21), we can write:

$$\mathbb{E}_t\, r^b_{t+1} - \mathbb{E}_t\, \pi_{t+1} = \sigma\left[\mathbb{E}_t\, c_{b,t+1} - c_{b,t}\right] = \sigma\left[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\right] \quad \text{(B.37)}$$

Combining (B.37) with (B.36) yields:

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1-z}{\sigma}\left(r^s_t - \mathbb{E}_t\, \pi_{t+1}\right) - z\left[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\right] \quad \text{(B.38)}$$

Note that an alternative, and arguably more intuitive, way to write the IS expression is based on a simple algebraic manipulation of (B.36):

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1}{\sigma}\left(r^s_t - \mathbb{E}_t\, \pi_{t+1}\right) - \frac{z}{\sigma}\left(\mathbb{E}_t\, r^b_{t+1} - r^s_t\right) \quad \text{(B.39)}$$

(B.39) is the familiar IS/Euler equation, written in terms of output rather than the output gap, appended with a term equal to the long-short interest rate spread, i.e. $\mathbb{E}_t\, r^b_{t+1} - r^s_t$.

The natural rate of interest, r^f_t, is defined as the real rate consistent with the IS equation holding at the natural rate of output absent credit shocks. This implies that:

$$r^f_t = \frac{\sigma}{1-z}\left(\mathbb{E}_t\, y^f_{t+1} - y^f_t\right) \quad \text{(B.40)}$$

Adding and subtracting y^f_t and $\mathbb{E}_t\, y^f_{t+1}$ from both sides of (B.38) and re-arranging yields (2.1). Making use of (B.23), allows one to write an AR(1) process for r^f_t as in (2.35), where $\rho_f = \rho_A$ and $s_f = \frac{\sigma(\rho_A-1)(1+\chi)}{\chi(1-z)+\sigma}$.

$$r^f_t = \frac{\sigma(\rho_A - 1)}{1-z}\widehat{y}^f_t \quad \text{(B.41)}$$

Computing the dynamics of x_t, π_t, and r^s_t does not require keeping track of y_t, y^f_t,q_t, r^b_t, ω_t, $\widehat{s}_t$, $\widehat{b}^{FI}_t$, $\widehat{b}_t$, $\widehat{b}_{cb,t}$ or $c_{b,t}$. Given the solution for x_t, π_t, and r^s_t, the dynamics of these variables can be computed using the full system, (B.1)-(B.27).

C Model Calibration

The parameters of the model are calibrated as follows. The unit of time is a quarter. We assume a zero trend inflation rate, so $\Pi = 1$. This implies that steady state price dispersion is $v^p = 1$ and the steady state relative reset price is $p_* = 1$. We set $\epsilon = 11$, which implies a steady state price markup of ten percent. The discount factor of the parent is set to $\beta = 0.995$, which together with $\Pi = 1$ implies a steady state short term rate of 200 basis points at an annualized frequency (i.e. $R^s = 1.005$). We then target a steady state spread of the return on the long bond over the short term bond of 200 basis points at an annualized frequency, which implies $\beta_b = 0.99$ and $R^b = 1.01$. We set $\kappa = 1 - 40^{-1}$, implying a ten year duration of the long bond. Together with R^b, this implies a steady state value of Q.

The coefficient of relative risk aversion, σ, and the inverse Frisch elasticity, χ, are both set to 1. We target a steady state share of child consumption, $z = C_b/Y$, of one-third. We then pick ψ to normalize steady state labor input to unity. Together, these parameters imply a value of the steady state transfer from parent to child, X^b. We assume that the Calvo parameter is $\phi = 0.75$, implying a mean duration between price changes of one year. We assume that the size of the central bank's balance sheet is 10 percent of steady state output, i.e. $QE = 0.1 \times Y$. We pick a steady state target of the risk-weighted leverage ratio of $\Theta = 5$. This then implies a value of the steady state equity transfer from the parent to the FI, X^{FI}.

For the exercises in Subsection 2.3, we assume that the Taylor rule parameters are $\rho_r = 0.8$, $\phi_\pi = 1.5$, and $\phi_x = 0$. The autoregressive parameter of the QE process, ρ_q, is also set to 0.8. The autoregressive parameters for productivity and the leverage shock are also both set to 0.8. This implies, as shown below in Appendix B, that the AR parameter in the natural rate process is also 0.8.

Table C.1: Parameter Values of Full Model

Parameter	Value	Description (Target)
β	0.995	Discount factor, parent
σ	1	Inverse elasticity of substitution
χ	1	Inverse Frisch elasticity
ψ	1.36	Labor disutility scaling parameter (target $L = 1$)
β_b	0.99	Discount factor, child (target spread of 200 b.p. annualized)
Π	1	Steady state trend inflation
ϵ	11	Elasticity of substitution (target markup ten percent)
κ	$1 - 40^{-1}$	Coupon decay (target duration ten years)
ϕ	0.75	Calvo price
Θ	5	Steady state risk-weighted leverage
QE	0.1	Steady state central bank bond portfolio
z	0.33	Steady state child share of consumption
X^b	0.33	Steady state parent-child transfer
X^{FI}	0.046	Steady state parent-FI equity transfer
ρ_r	0.8	Taylor rule smoothing
ϕ_π	1.5	Taylor rule inflation
ϕ_x	0	Taylor rule gap
ρ_A	0.8	AR productivity
ρ_θ	0.8	AR leverage
ρ_q	0.8	AR QE

Note: this table lists the values of calibrated parameters for the exercises in Subsection 2.3.

D Proofs

D.1 Theorem 1

The theorem can be proved by contradiction. Suppose first we can achieve $x_t = \pi_t = 0$ with $qe_t = 0$ for all t. Then the Phillips curve may then be written as:

$$0 = -\frac{z\gamma\sigma}{1-z}\bar{b}^{FI}\theta_t, \tag{D.1}$$

which does not hold unless $\theta_t = 0$, which contradicts the assumption. Hence, there is a contradiction.

Second, suppose we can achieve $x_t = \pi_t = \mathbb{E}_t\, x_{t+1} = \mathbb{E}_t\, \pi_{t+1} = 0$ with $r_t^s = 0$ for all t. Then the Phillips Curve becomes

$$0 = \frac{z\gamma\sigma}{1-z}\Big[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\Big]. \tag{D.2}$$

This requires

$$qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t. \tag{D.3}$$

Note that (D.3) is identical to the QE rule given in Proposition 1. With this QE rule and the policy rate fixed, the IS curve becomes

$$0 = \frac{1-z}{\sigma} r_t^f - z\Big[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\Big]. \tag{D.4}$$

Since $\mathbb{E}_t\, \theta_{t+1} = \rho_\theta \theta_t$, this may be written:

$$qe_t - \mathbb{E}_t\, qe_{t+1} = \frac{\bar{b}^{FI}}{\bar{b}^{cb}}\left(\rho_\theta - 1\right)\theta_t - \frac{1-z}{\bar{b}^{cb} z\sigma} r_t^f. \tag{D.5}$$

Now guess that QE evolves according to:

$$qe_t = \alpha_1 \theta_t + \alpha_2 r_t^f, \tag{D.6}$$

which implies:

$$\mathbb{E}_t\, qe_{t+1} = \alpha_1 \rho_\theta \theta_t + \alpha_2 \rho_f r_t^f. \tag{D.7}$$

Now plug (D.6)-(D.7) into (D.5):

$$qe_t = \left(\alpha_1 \rho_\theta + \frac{\bar{b}^{FI}}{\bar{b}^{cb}}\left(\rho_\theta - 1\right)\right)\theta_t + \left(\alpha_2 \rho_f - \frac{1-z}{\bar{b}^{cb} z\sigma}\right) r_t^f. \tag{D.8}$$

Now solve for α_1 and α_2:

$$\alpha_1 = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}} \tag{D.9}$$

$$\alpha_2 = -\frac{1-z}{\bar{b}^{cb} z\sigma(1-\rho_f)}. \tag{D.10}$$

Therefore,

$$qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t - \frac{1-z}{\bar{b}^{cb} z\sigma(1-\rho_f)} r_t^f. \tag{D.11}$$

(D.3) and (D.11) are not the same unless $z = 1$ (which we have ruled out) or $r_t^f = 0$ (which contradicts the assumption). Hence, we have another contradiction. ■

D.2 Lemma 1

First, suppose that $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$. This means that the qe_t and θ_t terms drop out of both (2.1) and (2.2). After imposing the ZLB on the short rate:

$$x_t = \mathbb{E}_t\, x_{t+1} + \frac{1-z}{\sigma}\left(\mathbb{E}_t\, \pi_{t+1} + r_t^f\right) \tag{D.12}$$

$$\pi_t = \gamma\zeta x_t + \beta\, \mathbb{E}_t\, \pi_{t+1} \tag{D.13}$$

We then guess that $x_t = \omega_1 r_t^f$ and $\pi_t = \omega_2 r_t^f$ while the ZLB binds. After the ZLB lifts, $r_t^s = r_t^f$ and consequently $x_t = \pi_t = 0$. The ZLB lifts with probability $1-\alpha$ and remains in place with probability α. Making use of the guess, along with the fact that $x_t = \pi_t = 0$ once the ZLB lifts and $\mathbb{E}_t\, r_{t+1}^f = \rho_f r_t^f$, results in a system of two equations in two unknowns, which can be solved for as:

$$\omega_1 = \frac{(1-z)(1-\alpha\beta\rho_f)}{\sigma(1-\alpha\beta\rho_f)(1-\alpha\rho_f) - (1-z)\gamma\zeta\alpha\rho_f} \tag{D.14}$$

$$\omega_2 = \frac{(1-z)\gamma\zeta}{\sigma(1-\alpha\beta\rho_f)(1-\alpha\rho_f) - (1-z)\gamma\zeta\alpha\rho_f} \tag{D.15}$$

■

D.3 Lemma 2

Instead, suppose that the QE rule is:

$$qe_t = \tau r_t^f - \frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t \tag{D.16}$$

$\tau = 0$ is a special case of the QE rule in Lemma 1, hence ω_1 and ω_2 are identical to Lemma 1. With the rule in (D.16) imposing the ZLB, the key equations of the model are:

$$x_t = \mathbb{E}_t\, x_{t+1} + \frac{1-z}{\sigma}\left(\mathbb{E}_t\, \pi_{t+1} + r_t^f\right) - z\bar{b}^{cb}\tau\, \mathbb{E}_t\, r_{t+1}^f + z\bar{b}^{cb}\tau r_t^f \tag{D.17}$$

$$\pi_t = \gamma\zeta x_t + \beta\, \mathbb{E}_t\, \pi_{t+1} - \frac{\gamma\sigma z\bar{b}^{cb}\tau}{1-z} r_t^f \tag{D.18}$$

Guess that the policy functions are $x_t = \widehat{\omega}_1 r_t^f$ and $\pi_t = \widehat{\omega}_2 r_t^f$. One obtains the result in the text that these functions may be written as in (3.6)-(3.7). The expressions for d_1 and d_2 are:

$$d_1 = \frac{\sigma z\bar{b}^{cb}\big[(1-\alpha\beta\rho_f)(1-\alpha\rho_f) - \gamma\alpha\rho_f\big]}{\sigma(1-\alpha\beta\rho_f)(1-\alpha\rho_f) - (1-z)\gamma\zeta\alpha\rho_f} \tag{D.19}$$

$$d_2 = \frac{\sigma z\gamma\bar{b}^{cb}\Big[\zeta(1-z)\big[(1-\alpha\beta\rho_f)(1-\alpha\rho_f) - \gamma\alpha\rho_f\big] - \sigma(1-\alpha\beta\rho_f)(1-\alpha\rho_f) + (1-z)\gamma\zeta\alpha\rho_f\Big]}{(1-\alpha\beta\rho_f)(1-z)\big[\sigma(1-\alpha\beta\rho_f)(1-\alpha\rho_f) - (1-z)\gamma\zeta\alpha\rho_f\big]} \tag{D.20}$$

■

D.4 Proposition 2

Applying results in Lemma 2, the objective function (3.1) becomes

$$\mathbb{L} = (\mu\widehat{\omega}_1^2 + \widehat{\omega}_2^2)(r_t^f)^2. \tag{D.21}$$

Next,

$$\begin{aligned} \mu\hat{\omega}_1^2 + \hat{\omega}_2^2 &= \mu(\omega_1 + d_1\tau)^2 + (\omega_2 + d_2\tau)^2 \\ &= \mu\omega_1^2 + \omega_2^2 + 2(\mu\omega_1 d_1 + \omega_2 d_2)\tau + (\mu d_1^2 + d_2^2)\tau^2 \end{aligned}$$

Take the first order derivative

$$\frac{\partial(\mu\hat{\omega}_1^2 + \hat{\omega}_2^2)}{\partial\tau} = 2(\mu\omega_1 d_1 + \omega_2 d_2) + 2(\mu d_1^2 + d_2^2)\tau = 0 \tag{D.22}$$

The minimum is achieved at (3.8).■

D.5 Lemma 3

Suppose $qe_t = 0$ at all times. The IS and Phillips curves in this scenario may be written as:

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1} - r_t^f\right) - z\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) \tag{D.23}$$

$$\pi_t = \gamma\zeta x_t - \frac{z\gamma\sigma}{1-z}\bar{b}^{FI}\theta_t + \beta\, \mathbb{E}_t\, \pi_{t+1} \tag{D.24}$$

Given the linearity in the model, it will be possible and optimal to offset natural rate shocks via adjusting the short term rate one-for-one with the natural rate of interest. Suppose that the policy rule implemented by the central bank is therefore:

$$r_t^s = r_t^f + \eta\theta_t \tag{D.25}$$

With the policy rate adjusting one-to-one to fluctuations in the natural rate, neither the output gap nor inflation will react to natural rate shocks. Guess, therefore, that the policy functions mapping credit shocks into inflation and the output gap are given by: $x_t = \widehat{\varphi}_1\theta_t$ and $\pi_t = \widehat{\varphi}_2\theta_t$. The expressions for $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ are as in Lemma 3, where:

$$\varphi_1 = \frac{\sigma z \bar{b}^{FI}\big[(1-\beta\rho_\theta)(1-\rho_\theta) - \rho_\theta\gamma\big]}{\sigma(1-\beta\rho_\theta)(1-\rho_\theta) - (1-z)\rho_\theta\gamma\zeta} \tag{D.26}$$

$$a_1 = -\frac{(1-z)(1-\beta\rho_\theta)}{\sigma(1-\beta\rho_\theta)(1-\rho_\theta) - (1-z)\rho_\theta\gamma\zeta} \tag{D.27}$$

$$\varphi_2 = \frac{\sigma z\gamma\zeta\bar{b}^{FI}\big[(1-\beta\rho_\theta)(1-\rho_\theta) - \rho_\theta\gamma\big]}{(1-\beta\rho_\theta)\big[\sigma(1-\beta\rho_\theta)(1-\rho_\theta) - (1-z)\rho_\theta\gamma\zeta\big]} - \frac{\sigma z\gamma\bar{b}^{FI}}{(1-z)(1-\beta\rho_\theta)} \tag{D.28}$$

$$a_2 = -\frac{(1-z)\gamma\zeta}{\sigma(1-\beta\rho_\theta)(1-\rho_\theta) - (1-z)\rho_\theta\gamma\zeta} \tag{D.29}$$

■

D.6 Proposition 3

Given results in Lemma 3, the objective function (3.1) becomes:

$$\mathbb{L} = (\mu\widehat{\varphi}_1^2 + \widehat{\varphi}_2^2)\theta_t^2. \tag{D.30}$$

Next:

$$\begin{aligned}
\mu\widehat{\varphi}_1^2 + \widehat{\varphi}_2^2 &= \mu(\varphi_1 + a_1\eta)^2 + (\varphi_2 + a_2\eta)^2 \\
&= \mu\varphi_1^2 + \varphi_2^2 + 2(\mu\varphi_1 a_1 + \varphi_2 a_2)\eta + (\mu a_1^2 + a_2^2)\eta^2
\end{aligned}$$

Take the first order derivative

$$\frac{\partial(\mu\widehat{\varphi}_1^2 + \widehat{\varphi}_2^2)}{\partial\eta} = 2(\mu\varphi_1 a_1 + \varphi_2 a_2) + 2(\mu a_1^2 + a_2^2)\eta \tag{D.31}$$

The minimum is achieved at (3.11).■