# The Four Equation New Keynesian Model *

Eric Sims
*Notre Dame and NBER*

Jing Cynthia Wu
*Notre Dame and NBER*

Ji Zhang
*Tsinghua PBCSF*

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**Abstract**

This paper develops a New Keynesian model featuring financial intermediation, short- and long-term bonds, credit shocks, and scope for unconventional monetary policy. The log-linearized model reduces to four key equations – a Phillips curve, an IS equation, and policy rules for the short-term nominal interest rate and the central bank's long bond portfolio (QE). The four equation model collapses to the standard three equation New Keynesian model under a simple parameter restriction. Credit shocks and QE appear in both the IS and Phillips curves. In equilibrium, optimal monetary policy entails adjusting the short-term interest rate to offset natural rate shocks, but using QE to offset credit market disruptions. Such policies can be supported by implementable rules in which the short-term interest rate reacts strongly to inflation and the central bank's bond portfolio reacts strongly to the output gap. The ability of the central bank to engage in QE significantly mitigates the costs of a binding zero lower bound.

**Keywords:** quantitative easing, unconventional monetary policy, small-scale New Keynesian model, zero lower bound

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# 1 Introduction

The textbook three equation New Keynesian (NK) model (see, e.g., Woodford 2003 or Galí 2008) has enormous influence in both policy circles and among academic researchers due to its elegance and tractability. The model boils down to a forward-looking IS equation characterizing aggregate demand, a Phillips curve describing aggregate supply, and a rule for the central bank's principal policy tool, the short-term interest rate. The model has yielded several important insights, including the potential desirability of inflation targeting, the gains from policy commitment over discretion, and the importance of having the policy rate track the "natural" or "neutral" rate of interest.

In spite of its myriad uses, the textbook model has proven inadequate for examining a range of issues that have come to the fore in policy circles over the last decade. As it abstracts from the financial sector, the model is unable to address the consequences of financial market disruption of the sort that rocked the global economy in 2007-2009. It is also incapable of directly speaking to the potential benefits and costs of quantitative easing (QE) type policies. QE policies were among the first and most prominent of several unconventional interventions deployed to fight the global financial crisis once policy rates were lowered to zero. There is now a nascent literature incorporating QE into medium-scale DSGE models (e.g. Gertler and Karadi 2011, 2013; Carlstrom, Fuerst and Paustian 2017; or Sims and Wu 2021). While this work has proven useful and generated several important insights, these quantitative frameworks lack the simplicity and transparency of the textbook three equation model.

Our paper bridges the gap between the complicated quantitative DSGE models that have been developed to study QE with the elegance and tractability of the textbook three equation model. Our model incorporates financial intermediaries, short- and long-term bonds, credit market shocks, and scope for central bank bond holdings to be economically relevant. The linearized version of our model reduces to four, rather than three, key equations. The IS and Phillips curves are similar to the three equation benchmark. The innovation is that credit shocks and central bank long bond holdings appear additively in *both* the IS and

Phillips curves. This differs from many *ad-hoc* treatments of financial disturbances, which often simply include residuals in the IS equation meant to proxy for credit spreads (see, e.g., Smets and Wouters 2007). The model is closed with a rule for the short-term policy rate (as in the benchmark three equation model) and a rule for the central bank's long bond portfolio.

We study optimal monetary policy in the context of our four equation model. Reflecting central banks' dual mandate, we focus on an objective function that minimizes a weighted sum of volatilities of inflation and the output gap. Because credit shocks appear in the Phillips curve, the so-called "Divine Coincidence" (Blanchard and Galí 2007) does not hold, and it is not possible to achieve the global minimum of the loss function with just one policy instrument.

When the central bank can actively manage both the short-term interest rate and a bond portfolio, it can achieve complete stabilization of both inflation and the output gap. In equilibrium, optimal policy entails adjusting the short-term interest to track fluctuations in the natural rate of interest (as in the benchmark three equation model), but adjusting the long bond portfolio to offset the effects of credit market disturbances. Our model therefore has an implication that differs from the conventional wisdom among policymakers that adjustment of short-term interest rates is sufficient to meet a dual mandate of price and output stability – in general, QE policies ought to be used all the time to counter credit market shocks, not only when policy rates are constrained by the zero lower bound (ZLB).

We also explore the implications of the ZLB for policy. When the policy rate is unavailable, optimal balance sheet policy is characterized by a "lean against the wind" condition that nevertheless differs from the standard condition under discretion in the textbook three equation model. A couple of interesting results emerge. First, credit market shocks need not have differential effects at the ZLB in comparison to normal times. Adjusting the long bond portfolio in exactly the same way as it would absent a ZLB constraint, the central bank is able to stabilize both inflation and the output gap in response to credit shocks at the ZLB.

Second, QE policies can serve as an effective (albeit imperfect) substitute for conventional policy in response to natural rate shocks. Without QE available, output and inflation react suboptimally to natural rate shocks when the short-term policy rate is constrained, the more so the longer the anticipated duration of the ZLB. A central bank can partially offset these non-optimal responses by adjusting its long bond portfolio. We derive an analytical expression for the optimal equilibrium path for QE at the ZLB as a function of the relative welfare weight on the output gap in the loss function. Though it is not possible to completely stabilize both inflation and the gap, a central bank engaging in QE operations can significantly reduce the costs of the ZLB.

Our model has important implications for central banks facing a dual mandate to stabilize both inflation and real economic activity due to the failure of the Divine Coincidence. How much QE is desired at the ZLB depends critically on how much weight the central bank puts on inflation vs. output fluctuations. The more weight the central bank puts on the output gap, the less QE is required in response to a shock to the neutral rate of interest. Prior to the Great Recession, active management of a long-bond portfolio was not a major feature of most central banks' toolkits.[1] With only one policy instrument available, a "lean against the wind" condition for the policy rate holds; in fact, this condition is the same as in the textbook three equation model. The direction for the optimal short rate response to a credit shock depends on whether the central bank cares more about inflation or output stabilization. For a positive credit shock, a central bank focusing solely on inflation would increase the short rate; whereas if the central bank only cares about the output gap, it would instead cut the short rate. Alternatively, if a central bank can use bond purchases all the time as a policy instrument, there need not be any conflict between the two aspects of the dual mandate.

Our analysis of optimal policy highlighted above studies how a central bank's two instruments (the policy rate and long bond portfolio) ought to optimally adjust in equilibrium to

[1] Japan is one notable exception.

stabilize its two targets (inflation and the output gap). While instructive, targeting rules of this sort may not be easily implementable. We also therefore consider an extension with "simple and implementable" rules for both the policy rate and the bond portfolio (Schmitt-Grohe and Uribe 2007): both instruments follow a Taylor-type rules that react to deviations in the two target variables (inflation and the output gap). When the long bond portfolio does not react to endogenous variables, the restrictions on parameter values of the rule for the policy rate necessary for equilibrium determinacy are identical to the standard three equation model. When the long bond portfolio does react to endogenous variables, determinacy is more likely when it responds strongly to the output gap and not inflation. We then show that having the policy rate react strongly to inflation and the bond portfolio react strongly to the output gap mimics the optimal allocations while also delivering a determinant equilibrium. We further show that an implementable rule for the central bank's long bond portfolio significantly ameliorates the adverse consequences of a binding ZLB on the policy rate.

Though irrelevant in a standard model (Wallace 1981), there are several potential channels by which QE can transmit to the real economy that have been explored in the literature (see Bhattarai and Neely 2020 for a thorough survey). One is a signaling channel, wherein accumulating a large balance sheet in the present might commit a central bank to lower short-term policy rates in the future (e.g. Bauer and Rudebusch 2014 and Bhattarai et al. 2019). Another is based on exogenous participation constraints that build on the preferred habitat theory of the term structure (e.g. Vayanos and Vila 2009, Hamilton and Wu 2012, and Chen et al. 2012). A third assumes leverage constraints on intermediaries (e.g. Gertler and Karadi 2011, 2013). The key friction in our model is a leverage constraint that allows for a long-short interest rate spread. Relative to more involved papers based on a leverage constraint, such as Sims and Wu (2021), our model makes a number of simplifying assumptions that allow us to reduce the model down to four equations. At the expense of some realism, these simplifying assumptions afford a great deal of tractability, which allows us to make clear statements about optimal policy. More expansive models with leverage constraints nevertheless generate similar quantitative predictions as our four equation model.

Our paper relates to the literature on unconventional monetary policy in the New Keynesian model. Gertler and Karadi (2011, 2013), Carlstrom, Fuerst and Paustian (2017), Sims and Wu (2021, 2020b), and Mau (2019) all represent attempts to model large scale asset purchases in a quantitative DSGE framework. Distinct from this strand of the literature, one important contribution of our paper is to incorporate the financial frictions giving rise to effective QE policies in these papers into the tractable small-scale New Keynesian model of Clarida, Galí and Gertler (1999) that is popular among academics and policymakers alike. The framework we present here can be used to address a number of important policy questions in a way similar to how the three equation model is used. For example, Sims and Wu (2020a) use the four equation model to relate the Fed's QE policies in the wake of the Great Recession to the Wu and Xia (2016) shadow rate series.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 discusses optimal central bank policy. Section 4 considers optimal implementable rules for both the policy rate and the central bank's long bond portfolio. Section 5 offers concluding thoughts.

# 2 Model

This section presents our model. We first present the four equation linearized model in Subsection 2.1. The full non-linear model is derived from first principles in Subsection 2.2. Subsection 2.3 studies positive properties of a calibrated version of the model before turning to normative issues in Section 3. Details are available in Appendices A - F.

## 2.1 The Four Equation Model

The principal equations of our linearized model are an IS curve:

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\,\pi_{t+1} - r_t^*\right) - z\Big[\bar{b}^{FI}\left(\mathbb{E}_t\,\theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\Big], \quad (2.1)$$

and a Phillips curve:

$$\pi_t = \gamma\zeta x_t - \frac{z\gamma\sigma}{1-z}\Big[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\Big] + \beta\,\mathbb{E}_t\,\pi_{t+1}. \quad (2.2)$$

Lowercase variables with a $t$ subscript denote log deviations about the non-stochastic steady state. $\pi_t$ is inflation and $x_t = y_t - y_t^*$ denotes the output gap, where $y_t^*$ is the equilibrium level of output consistent with price flexibility *and* no credit shocks.[2] We refer to this level of output as potential output. Similarly, $r_t^*$ denotes the natural rate of interest – i.e. the real interest rate consistent with output equaling potential. It follows an exogenous process. $\theta_t$ captures credit conditions in the financial market; positive values correspond to more favorable conditions. This variable is described further in Subsection 2.2. We take it to be exogenous and henceforth refer to it as a credit shock. $qe_t$ denotes the real market value of the central bank's long-term bond portfolio. $r_t^s$ is the short-term nominal interest.

Letters without $t$ subscripts are parameters or steady state values. $\sigma$, $\beta$, and $\gamma$ are standard parameters – $\sigma$ measures the inverse intertemporal elasticity of substitution, $\beta$ is a subjective discount factor, and $\gamma$ is the elasticity of inflation with respect to real marginal cost.[3] $\bar{b}^{FI}$ and $\bar{b}^{CB}$ are parameters measuring the steady state long-term bond holdings of financial intermediaries and the central bank, respectively, relative to total outstanding long-term bonds. These coefficients sum to one, i.e. $\bar{b}^{FI} + \bar{b}^{CB} = 1$.

[2] Traditionally in New Keynesian models, potential output is defined as the hypothetical level of output consistent with price flexibility and is denoted $y_t^f$. As described below, in our model both price stickiness and financial frictions distort the competitive equilibrium. It is therefore natural to define potential output as a concept wherein both frictions are neutralized rather than just price rigidity. See further details in Appendix D.

[3] In particular, $\gamma = \frac{(1-\phi)(1-\phi\beta)}{\phi}$ is the standard expression in the three equation model, where $\phi \in [0, 1)$ measures the probability of non-price adjustment.

As described in Subsection 2.2, there are two kinds of households in our model. We will refer to these types of households as "parent" and "child," respectively. The parent is the standard household in a textbook New Keynesian model – it consumes, borrows or saves via one-period bonds, supplies labor, and owns firms. The child does not supply labor nor does it have an equity interest in production firms. It is less patient than the parent and finances its consumption by issuing long-term bonds. It pays the servicing cost of these long-term bonds with a transfer from the parent each period. The parameter $z \in [0, 1)$ represents the share of children in the total population. $\zeta$ is the elasticity of real marginal cost with respect to the output gap; it is conceptually similar to the corresponding parameter in the standard three equation model, but augmented to account for two types of households.[4] Our model collapses to the standard three equation NK model when $z = 0$. In this case, credit shocks and the central bank's long bond portfolio are irrelevant for the equilibrium dynamics of output and inflation.

Our four equation New Keyneian model consists of (2.1)-(2.2), together with policy rules for the short-term interest rate $r_t^s$ and central bank's long bond portfolio $qe_t$. Simple rule-based policies are specified in Subsection 2.3 for positive analyses, whereas we discuss optimal policies in Section 3.

## 2.2 Derivation of the Four Equation Model

In this subsection, we present, from first principles, the economic environment giving rise to the linearized four equation model laid out in Subsection 2.1. The economy is populated by the following agents: two types of households (parent and child), a representative financial intermediary, production firms, and a central bank. We discuss the problems of each below.

Note that we make several simplifying assumptions in this section in order to get the system to reduce to just four equations. This is intentional and for tractability. Neverthe-

[4] In our model, $\zeta = \frac{\chi(1-z)+\sigma}{1-z}$, where $\chi$ is the inverse Frisch labor supply elasticity for the parent. If $z = 0$, $\zeta$ would be identical to the textbook three equation model.

less, the quantitative implications of our small-scale model are similar to more complicated models. For example, the dynamics of the child's consumption in our model are in-line with the behavior of investment in Sims and Wu (2021). In Appendix F, we show some quantitative results when we relax a few of the assumptions that allow the system to reduce to four equations.

### 2.2.1 Parent

A representative parent receives utility from consumption, $C_t$ and disutility from labor, $L_t$. It discounts future utility flows by $\beta \in (0,1)$. Its lifetime utility is:

$$V_t = \max \ \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[ \frac{C_{t+j}^{1-\sigma} - 1}{1-\sigma} - \psi \frac{L_{t+j}^{1+\chi}}{1+\chi} \right]. \tag{2.3}$$

$\sigma > 0$ is the inverse elasticity of intertemporal substitution, $\chi \geq 0$ is the inverse Frisch elasticity, and $\psi > 0$ is a scaling parameter.

The nominal price of consumption is $P_t$. The parent earns nominal income from labor, with a wage of $W_t$, receives dividends from ownership in firms and financial intermediaries, $D_t$ and $D_t^{FI}$, respectively, and receives a lump sum transfer from the fiscal authority, $T_t$. It can save via one-period nominal bonds, $S_t$, that pay gross nominal interest rate $R_t^s$. In addition, it makes a transfer, $X_t^b$, to the child each period, as well as a transfer, $X_t^{FI}$, to financial intermediaries; though time-varying, neither of these are choice variables. The parent's budget constraint is:

$$P_t C_t + S_t \leq W_t L_t + R_{t-1}^s S_{t-1} + P_t D_t + P_t D_t^{FI} + P_t T_t - P_t X_t^b - P_t X_t^{FI}. \tag{2.4}$$

The objective is to pick a sequence of consumption, labor, and one-period bonds to maximize (2.3) subject to the sequence of (2.4). The optimality conditions are standard:

$$\psi L_t^{\chi} = C_t^{-\sigma} w_t, \tag{2.5}$$

$$\Lambda_{t-1,t} = \beta \left( \frac{C_t}{C_{t-1}} \right)^{-\sigma}, \tag{2.6}$$

$$1 = R_t^s \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{-1}. \tag{2.7}$$

In (2.5), $w_t = W_t/P_t$ is the real wage; and in (2.7), $\Pi_t = P_t/P_{t-1}$ is gross inflation. $\Lambda_{t-1,t}$ is the parent's stochastic discount factor.

### 2.2.2 Child

The child gets utility from consumption, $C_{b,t}$, and does not supply labor. Its flow utility function is the same as the parent, but it discounts future utility flows by $\beta_b < \beta$; i.e. it is less patient than the parent. Its lifetime utility is:

$$V_{b,t} = \mathbb{E}_t \sum_{j=0}^{\infty} \beta_b^j \left[ \frac{C_{b,t+j}^{1-\sigma} - 1}{1 - \sigma} \right]. \tag{2.8}$$

The child can borrow/save through long-term bonds, the new issuance of which is denoted by $NB_t$. These bonds are structured as perpetuities with decaying coupon payments. Coupon payments decay at rate $\kappa \in [0, 1]$. Issuing one unit of bonds in period $t$ obligates the issuer to a coupon payment of one dollar in $t + 1$, $\kappa$ dollars in $t + 2$, $\kappa^2$ dollars in $t + 3$, and so on. The total coupon liability due in $t + 1$ from past issuances is therefore:

$$B_t = NB_t + \kappa NB_{t-1} + \kappa^2 NB_{t-2} + \ldots. \tag{2.9}$$

The attractive feature of these decaying coupon bonds is that one only needs to keep track of the total outstanding bonds, $B_t$, rather than individual issues. In particular:

$$NB_t = B_t - \kappa B_{t-1}. \tag{2.10}$$

New issuances in period $t$ trade at market price $Q_t$ dollars. Because of the structure of coupon payments, the prices of bonds issued at previous dates are proportional to the price of new issues; i.e. bonds issued in $t-j$ trade at $\kappa^j Q_t$ in $t$. The total value of the bond portfolio can therefore conveniently be written as $Q_t B_t$.

The nominal value of consumption plus coupon payments on outstanding debt cannot exceed the value of new bond issuances plus the nominal value of the transfer from the parent. The flow budget constraint facing the child is therefore:

$$P_t C_{b,t} + B_{t-1} \leq Q_t \left( B_t - \kappa B_{t-1} \right) + P_t X_t^b. \tag{2.11}$$

Define the gross return on the long bond as:

$$R_t^b = \frac{1 + \kappa Q_t}{Q_{t-1}}. \tag{2.12}$$

The optimality condition for the child is an Euler equation for long-term bonds, where $\Lambda_{b,t-1,t}$ denotes its stochastic discount factor:

$$\Lambda_{b,t-1,t} = \beta_b \left( \frac{C_{b,t}}{C_{b,t-1}} \right)^{-\sigma}, \tag{2.13}$$

$$1 = \mathbb{E}_t \, \Lambda_{b,t,t+1} R_{t+1}^b \Pi_{t+1}^{-1}. \tag{2.14}$$

### 2.2.3 Financial Intermediaries

A representative financial intermediary (FI) is born each period and exits the industry in the subsequent period. This is a special case of Gertler and Karadi (2011, 2013), and Sims and Wu (2021), who allow financial intermediaries to live for multiple periods and exit randomly. We make this simplifying assumption because it allows us to reduce the system into four equations. The FI receives an exogenous amount of net worth from the parent household, $P_t X_t^{FI}$. This equity transfer is comprised of two components – new real equity that is fixed at $\bar{X}^{FI}$, along with the stock of outstanding long bonds held by previous intermediaries, which are valued at $\kappa Q_t$:

$$P_t X_t^{FI} = P_t \bar{X}^{FI} + \kappa Q_t B_{t-1}^{FI} \tag{2.15}$$

The intermediary also attracts deposits, $S_t^{FI}$, from the parent household. It can hold long bonds issued by the child, $B_t^{FI}$, or reserves on account with the central bank, $RE_t^{FI}$. The FI is structured as a special case of intermediaries in Sims and Wu (2021) and Gertler and Karadi (2011, 2013), with intermediaries exiting after each period with probability one. Because the probability of exit after each period is unity, we can think of there being a (newly born) representative FI each period.

The balance sheet condition of the FI is:

$$Q_t B_t^{FI} + RE_t^{FI} = S_t^{FI} + P_t X_t^{FI}. \tag{2.16}$$

The FI pays interest, $R_t^s$, on short-term debt, earns interest, $R_t^{re}$, on reserves, and earns a return on long-term bonds carried from $t$ into $t+1$, $R_{t+1}^b$. Note that these are all nominal rates. Upon exiting after period $t$, the FI therefore returns a dividend to the parent household that satisfies:

$$P_{t+1} D_{t+1}^{FI} = \left(R_{t+1}^b - R_t^s\right) Q_t B_t^{FI} + \left(R_t^{re} - R_t^s\right) RE_t^{FI} + R_t^s P_t X_t^{FI} \tag{2.17}$$

The FI is subject to a risk-weighted leverage constraint. Long-term bonds receive a risk weight of unity, while reserves on account with the central bank have a risk weight of zero. The leverage constraint is:

$$Q_t B_t^{FI} \leq \Theta_t P_t \bar{X}^{FI}. \tag{2.18}$$

In other words, (2.18) says that the value of long bonds held by the FI cannot exceed a time-varying multiple, $\Theta_t$, of the new equity transferred from the parent, $P_t \bar{X}^{FI}$. We assume that $\Theta_t$ obeys a known stochastic process and refer to changes in $\Theta_t$ as credit shocks.

The objective of the FI is to maximize the expected one period ahead value of (2.17), discounted by the nominal stochastic discount factor of the parent household, i.e. $\Lambda_{t,t+1}\Pi_{t+1}^{-1}$, subject to (2.18). The intermediary can choose the quantity of long bonds and reserves that it holds. In doing so, it does not take into account that its choice of long bonds to hold today influences the total equity transfer future intermediaries will receive. In other words, even though the payouts are discounted because the household owner receives them in the future, the intermediary's problem is effectively static. Letting $\Omega_t$ denote the multiplier on the leverage constraint, the first order conditions are:

$$\mathbb{E}_t \, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^b - R_t^s\right) = \Omega_t, \tag{2.19}$$

$$\mathbb{E}_t \, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\left(R_t^{re} - R_t^s\right) = 0. \tag{2.20}$$

(2.20) says that the FI will hold an indeterminate amount of reserves so long as the return on reserves, $R_t^{re}$, equals the cost of funds, $R_t^s$. Absent a leverage constraint, the FI would buy long bonds up until the point at which the expected return on long bonds equals the cost of funds. The constraint being binding, i.e. $\Omega_t > 0$, generates excess returns.

### 2.2.4 Production

The production side of the economy is split into three sectors: final output, retail output, and wholesale output. There is a representative final good firm and representative wholesale producer. There are a continuum of retailers, indexed by $f \in [0, 1]$.

The final output good, $Y_t$, is a CES aggregate of retail outputs, with $\varepsilon > 1$ the elasticity of substitution. This gives rise to a standard demand function for each variety of retail output and an aggregate price index:

$$
\begin{aligned}
Y_t(f) &= \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} Y_t, && (2.21)\\
P_t &= \left[\int_0^1 P_t(f)^{1-\epsilon} df\right]^{\frac{1}{1-\epsilon}}. && (2.22)
\end{aligned}
$$

Retailers purchase wholesale output at price $P_{m,t}$ and repackage it for sale at $P_t(f)$. $P_{m,t}$ has the interpretation as nominal marginal cost. Retailers are subject to a Calvo (1983) pricing friction – each period, there is a probability $1-\phi$ that a retailer may adjust its price, with $\phi \in [0,1]$. When given the opportunity to adjust, retailers pick a price to maximize the present discounted value of expected profits, where discounting is by the stochastic discount factor of the parent household. Optimization results in an optimal reset price, $P_{*,t}$, that is common across updating retailers. Letting $p_{m,t} = P_{m,t}/P_t$ denote real marginal cost, the optimal reset price satisfies:

$$
\begin{aligned}
P_{*,t} &= \frac{\epsilon}{\epsilon-1}\frac{X_{1,t}}{X_{2,t}}, && (2.23)\\
X_{1,t} &= P_t^{\epsilon} p_{m,t} Y_t + \phi\, \mathbb{E}_t\, \Lambda_{t,t+1} X_{1,t+1}, && (2.24)\\
X_{2,t} &= P_t^{\epsilon-1} Y_t + \phi\, \mathbb{E}_t\, \Lambda_{t,t+1} X_{2,t+1}. && (2.25)
\end{aligned}
$$

The wholesale firm produces output, $Y_{m,t}$, according to a linear technology in labor:

$$
Y_{m,t} = A_t L_t. \tag{2.26}
$$

$A_t$ is an exogenous productivity disturbance obeying a known stochastic process. Letting $w_t = W_t/P_t$ denote the real wage, the optimality condition is standard:

$$
w_t = p_{m,t} A_t. \tag{2.27}
$$

### 2.2.5 Central Bank and Fiscal Authority

The central bank can hold a portfolio of long bonds, $B_t^{cb}$. It finances this portfolio via the creation of reserves, $RE_t$. Its balance sheet condition is:

$$Q_t B_t^{cb} = RE_t. \tag{2.28}$$

We will refer to the real value of the central bank's bond portfolio as $QE_t = Q_t b_t^{cb}$, where $b_t^{cb} = B_t^{cb}/P_t$, and shall assume that the central bank may freely choose this (equivalently, it can freely choose reserves). The central bank potentially earns an operating surplus and then remits it to the fiscal authority. The fiscal authority then returns this revenue to the parent household via a lump sum transfer. When the surplus is negative, the transfer becomes a lump sum tax. This transfer satisfies:

$$P_t T_t = R_t^b Q_{t-1} B_{t-1}^{cb} - R_{t-1}^{re} RE_{t-1}. \tag{2.29}$$

In our model, we abstract from government bonds. As shown in Sims and Wu (2021), the effect of purchasing government bonds via QE would be qualitatively the same as private bonds, but quantitatively smaller by a constant fraction.

### 2.2.6 Aggregation and Equilibrium

Market-clearing requires that $RE_t = RE_t^{FI}$ and $S_t = S_t^{FI}$ (i.e. the FI holds all reserves issued by the central bank and all one period bonds issued by the parent household), while $B_t = B_t^{FI} + B_t^{cb}$ (i.e. the total stock of long-term bonds issued by the child must be held by the FI or the central bank). Some algebraic substitutions give rise to a standard aggregate resource constraint:

$$Y_t = C_t + C_{b,t}. \tag{2.30}$$

Aggregating across retailers gives rise to the aggregate production function, where $v_t^p$ is a measure of price dispersion:

$$Y_t v_t^p = A_t L_t. \tag{2.31}$$

We assume that the transfer from parent to child, $X_t^b$, is time-varying in a way that represents a complete payoff of outstanding debt obligations each period:

$$P_t X_t^b = (1 + \kappa Q_t) B_{t-1}. \tag{2.32}$$

Neither the parent nor the child behaves as though it can influence the value of $X_t^b$. The particular assumption embodied in (2.32) implies that, even though the child solves a dynamic problem and has a forward-looking Euler equation, (2.14), its consumption is effectively static:

$$P_t C_{b,t} = Q_t B_t. \tag{2.33}$$

This assumption on the parent-child transfer allows us to eliminate a state variable and simplifies the system to four equations, although it is not crucial for the qualitative or quantitative properties of the model. We refer to this assumption as a "full bailout" because, each period, the parent pays off the child's debt. We show, in Appendix F.1, that dropping the full bailout assumption, and instead considering a fixed transfer each period between parent and child, does not fundamentally alter the behavior of the model in response to shocks.

$A_t$ and $\Theta_t$ obey conventional AR(1) processes in the log. We define potential output, $Y_t^*$, as the equilibrium level of output consistent with price flexibility (i.e. $\phi = 0$) and where the credit shock is constant, i.e. $\Theta_t = \Theta$. The natural rate of interest, $R_t^*$, is the gross real short-term interest rate consistent with this level of output. $X_t = Y_t / Y_t^*$ is the gross output gap. The full set of equilibrium conditions are contained in Appendix A. The system can be greatly simplified, and the equilibrium conditions log-linearized about a zero inflation steady state can be reduced to the four equation system presented at the beginning of this section; i.e. (2.1)-(2.2) along with rules for the short-term policy rate and the central bank's long bond portfolio. Details of the linearization may be found in Appendix B.

## 2.3 The Four vs. the Three Equation Model

Before turning to normative optimal policy analysis in Section 3, we first explore the positive properties of the linearized model as described above in Subsection 2.1.

For the purpose of studying positive properties of the model, we suppose that the short-term rate follows a Taylor-type rule while the long bond portfolio obeys an exogenous process:

$$r_t^s = \rho_r r_{t-1}^s + (1-\rho_r)\Big[\phi_\pi \pi_t + \phi_x x_t\Big] + s_r \varepsilon_{r,t}, \tag{2.34}$$

$$qe_t = \rho_q qe_{t-1} + s_q \varepsilon_{q,t}. \tag{2.35}$$

$r_t^*$ and $\theta_t$, the natural rate of interest and credit shock, respectively, obey stationary AR(1) processes:

$$r_t^* = \rho_f r_{t-1}^* + s_f \varepsilon_{f,t}, \tag{2.36}$$

$$\theta_t = \rho_\theta \theta_{t-1} + s_\theta \varepsilon_{\theta,t}. \tag{2.37}$$

When we assume that the central bank's long bond portfolio is exogenous, as in (2.35), and close the model with a conventional Taylor rule for the policy rate, as in (2.34), the requirements for a unique rational expectations equilibrium are the same as in the standard three equation model. We show this formally in Appendix C.

A full description and justification of the underlying parameter values of the non-linear model is provided in Appendix E. Here, we focus only on the parameter values necessary

Table 1: Parameter Values of Linearized Model

| Parameter | Value | Description (Target) |
|---|---|---|
| $\beta$ | 0.995 | Discount factor |
| $z$ | 0.33 | Consumption share of child |
| $\sigma$ | 1 | Inverse elasticity of substitution |
| $\bar{b}^{FI}$ | 0.70 | Weight on credit in IS/PC curves |
| $\bar{b}^{cb}$ | 0.30 | Weight on QE in IS/PC curves |
| $\gamma$ | 0.086 | Elasticity of inflation w.r.t. marginal cost |
| $\zeta$ | 2.49 | Elasticity of marginal cost w.r.t. gap |
| $\rho_r$ | 0.8 | Taylor rule smoothing |
| $\phi_\pi$ | 1.5 | Taylor rule inflation |
| $\phi_x$ | 0 | Taylor rule gap |
| $\rho_f$ | 0.8 | AR natural rate |
| $\rho_\theta$ | 0.8 | AR credit |
| $\rho_q$ | 0.8 | AR QE |

Note: this table lists the values of calibrated parameters of the linearized four equation model.

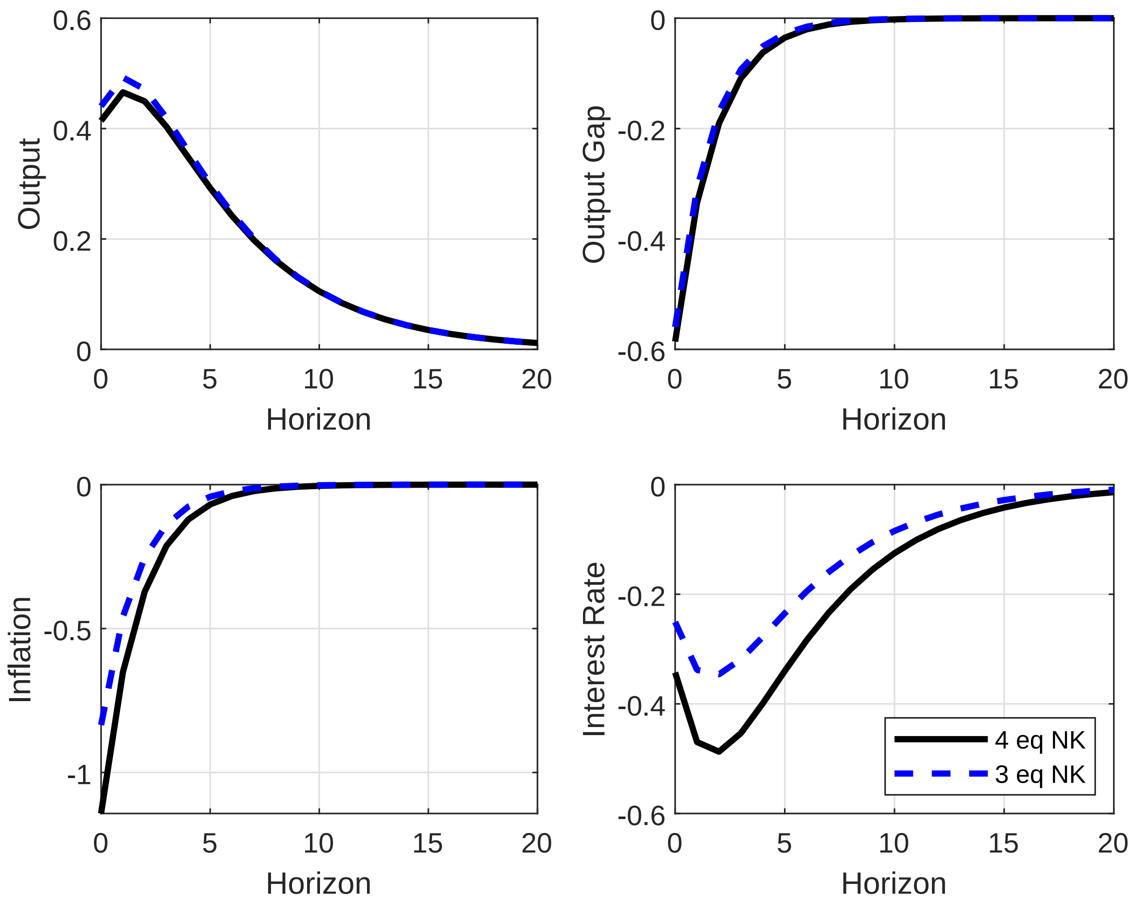
for solving the linearized model. These parameter values are listed in Table 1. The discount factor and elasticity of substitution take on standard values. The child-share of total consumption, $z$, is set to one-third. This is loosely calibrated to match the share of durable consumption and private investment in aggregate private non-government domestic expenditure.[5] The dynamics of the child's consumption in our model are roughly in-line with the behavior of investment in a larger model with physical capital accumulation (e.g. Sims and Wu 2021). In Appendix F.2, we present impulse responses with different values of $z$. Given our calibrations of other steady state parameters (discussed further in Appendix E), we have $\bar{b}^{FI} = 0.7$ and $\bar{b}^{cb} = 0.3$. The elasticity of inflation with respect to real marginal cost is $\gamma = 0.086$ and the elasticity of the output gap with respect to real marginal cost is $\zeta = 2.49$, implying a slope of the Phillips curve of 0.21. The parameters of the Taylor rule are standard. The autoregressive parameters in the exogenous processes are all set to 0.8.

Figure 1 displays impulse responses to a one percent positive shock to potential output.[6]

[5] In 2020Q3, the latest period for which we have data, these two categories composed 30 percent of non-government private domestic expenditure.

[6] As written, the linearized model presented in Subsection 2.1 writes the exogenous process in terms of the natural rate of interest. As shown in Appendix B, there is a mapping between the natural rate of interest and potential output. When comparing the four equation to the three equation model, the mapping between the natural rate of interest and potential output is not identical due to the presence of $z$ in the four equation model. The comparison is more natural for an equal sized shock to potential output rather than the natural

Figure 1: **IRFs to Shock to Potential Output**



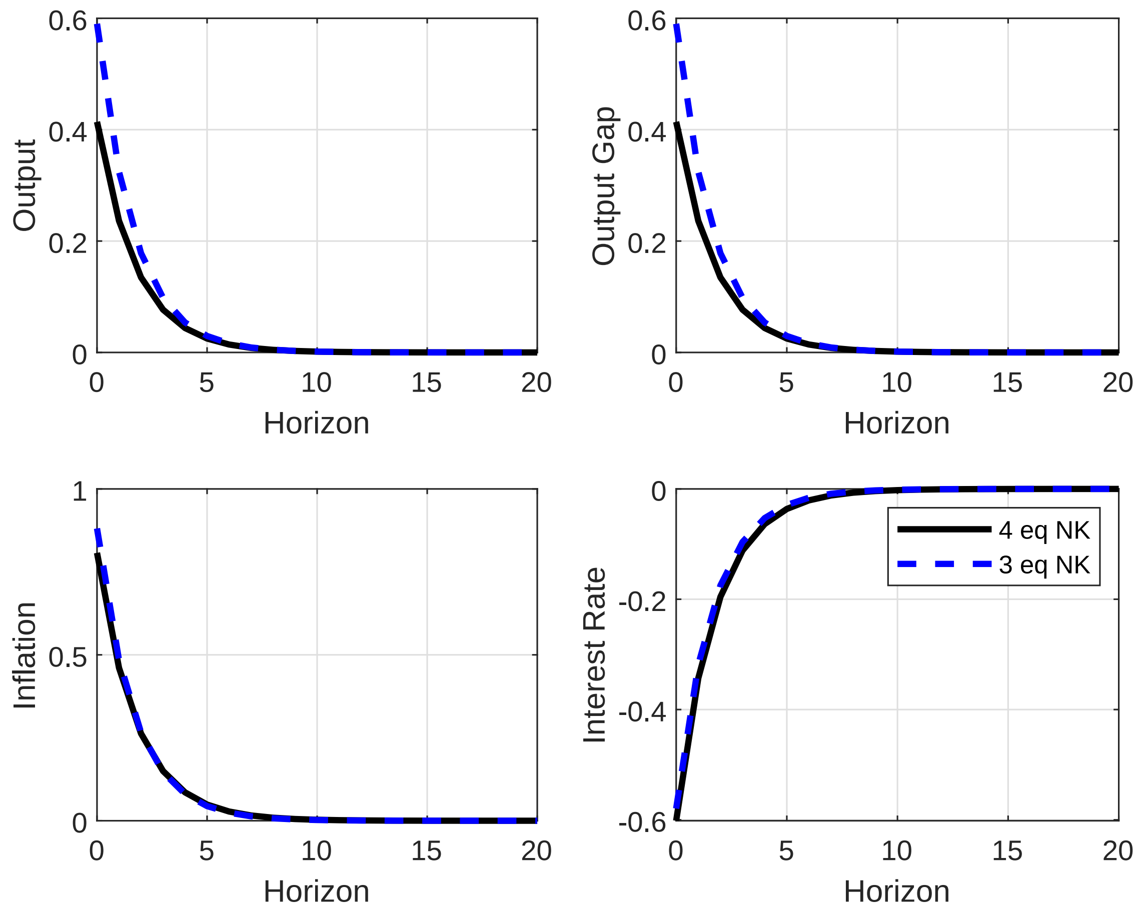

*Notes:* Black solid lines: IRFs to a one percentage point shock to potential output in the four equation model. Output and the output gap are expressed in percentage points, while the responses of inflation and the short-term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the baseline three equation NK model.

The solid black lines are responses in our baseline four equation model, whereas the dashed blue lines depict responses in the conventional three equation model (i.e. our model imposing $z = 0$). These responses are familiar and do not differ much in our model compared to the more standard three equation model. Output increases but by less than potential, resulting in a negative output gap. This puts downward pressure on inflation, which is met with policy accommodation with the short-term interest rate declining. Relative to three equation model, output reacts slightly less on impact in our model, though this difference is not large.

Figure 2 plots impulse responses to a conventional monetary policy shock. The size and

rate of interest.

Figure 2: **IRFs to Policy Shock**

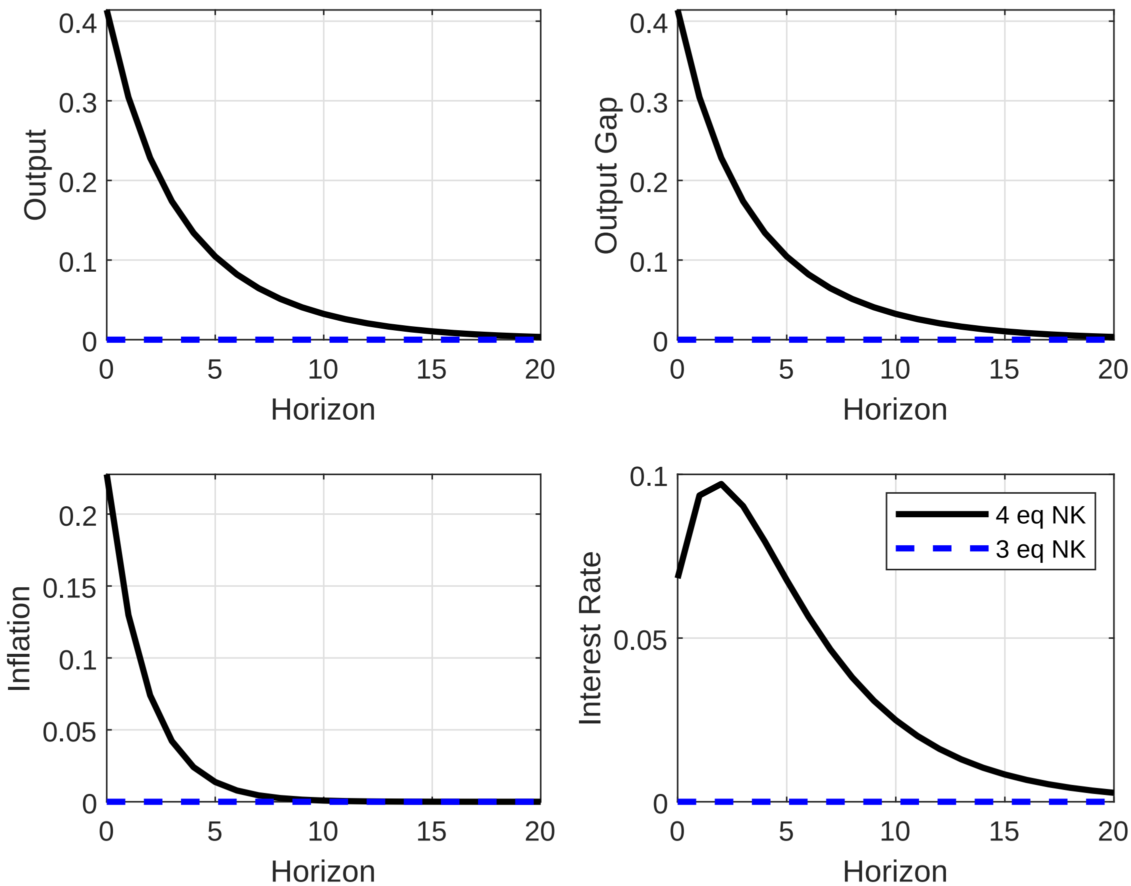



*Notes:* Black solid lines: IRFs to a conventional monetary policy shock. The size and sign of the shock are chosen to generate the same impact response of output as in Figure 1. Output and the output gap are expressed in percentage points, while the responses of inflation and the short-term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized policy shock in the baseline three equation NK model.

sign of the shock are chosen to generate the same impact response of output to the potential output shock in the four equation model. Output (and hence the output gap) rises on impact before reverting to its pre-shock value. Inflation rises and follows a similar dynamic path as output. As in the case of the potential output shock, there is little meaningful difference in the responses of variables in our four equation model relative to the baseline three equation model.

Figure 3 plots impulse responses to a credit ($\theta_t$) or QE ($qe_t$) shock. Because these differ only according to scale in the linear system (i.e. $\bar{b}^{FI} \neq \bar{b}^{cb}$), because we have assumed equal AR parameters (0.8), and because the shock sizes are normalized to produce the same impact

Figure 3: **IRFs to Credit/QE Shock**




*Notes:* Black solid lines: IRFs to a credit ($\theta_t$) or QE ($qe_t$) shock. The size and sign of the shocks are chosen to generate the same impact response of output as in Figure 1. Because the QE and credit shock only differ according to scale in the linearized model (i.e. $\bar{b}^{FI} \neq \bar{b}^{cb}$) and the AR parameters are the same, the normalized impulse responses are identical. Output and the output gap are expressed in percentage points, while the responses of inflation and the short-term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized credit/QE shock in the baseline three equation NK model.

response of output, the IRFs of endogenous variables to a credit or QE shock are identical. We therefore only show one set of impulse responses.

Unlike responses to the other shocks, in Figure 3, there is a meaningful difference between the four equation model and the three equation model. In the three equation model, both shocks are irrelevant for the dynamics of endogenous variables. In our four equation model, an increase in leverage (equivalently a central bank purchase of long bonds) is expansionary for output. In the current calibration, such an expansion also results in an increase in inflation and a resulting increase in the short-term interest rate. That financial shocks have economic effects in-line with the traditional understanding of an aggregate demand shock and the fact that there is scope for QE policies represent a key advancement in our four equation model relative to the standard three equation model. These properties are critical for understanding the post-Crisis economy.

## 2.4 Discussion

It is fairly standard in macro models to include reduced-form credit shocks as residuals in the IS equation (Smets and Wouters 2007). Our structural four equation model has this feature as well. But in our model, QE and credit shocks also appear as residuals in the Phillips Curve, which leads to a breakdown in the Divine Coincidence and results in potentially important policy tradeoffs.

Why do credit shocks appear as an endogenous cost-push wedge in the Phillips curve? Our model's Phillips curve written in terms of marginal cost is the same as in the standard three equation model:

$$\pi_t = \gamma \widehat{p}_{m,t} + \beta \, \mathbb{E}_t \, \pi_{t+1}, \tag{2.38}$$

where $\widehat{p}_{m,t}$ is real marginal cost linearized about the steady state, which is in turn equal to the log difference between the real wage and the marginal product of labor. See derivations in Appendix B.

Holding the aggregate level of output fixed, favorable credit conditions reallocate resources from the parent (the saver) to the child (the borrower). In our model, the parent supplies labor, similar to many other models of financial frictions where workers save and supply variable labor while entrepreneurs borrow and either do not supply labor or do so inelastically (e.g. Carlstrom and Fuerst 1997). The reallocation of resources when credit conditions are favorable therefore induces a negative wealth effect for the parent that puts downward pressure on the wage, and hence real marginal cost, for a given level of output.

This manifests itself as the endogenous cost-push term in the Phillips curve relation written in terms of the output gap. See further details in Appendix B.

The credit/QE shocks appearing as an endogenous cost-push wedge in the Phillips curve gives rise an important implication of our model. In our model, a QE shock is less inflationary than a conventional monetary policy rate cut. This finding is inline with the results in the richer model of Sims and Wu (2021, 2020b), and empirically consistent with the lack of inflationary pressures from the expansive QE operations in the US and other parts of the world in the wake of the Great Recession. The $qe_t$ term enters in both the IS, (2.1), and Phillips curves, (2.2). In particular, $qe_t$ enters with a positive sign in the IS relationship, and hence serves as a positive demand shock, but with a negative sign in the Phillips curve. Both of these channels make QE expansionary for output, but have competing effects on inflation. As parameterized, an expansionary QE shock in our model is nevertheless inflationary, albeit less so than a conventional monetary policy shock. There also exist parameterizations in which an expansionary QE shock can be deflationary.

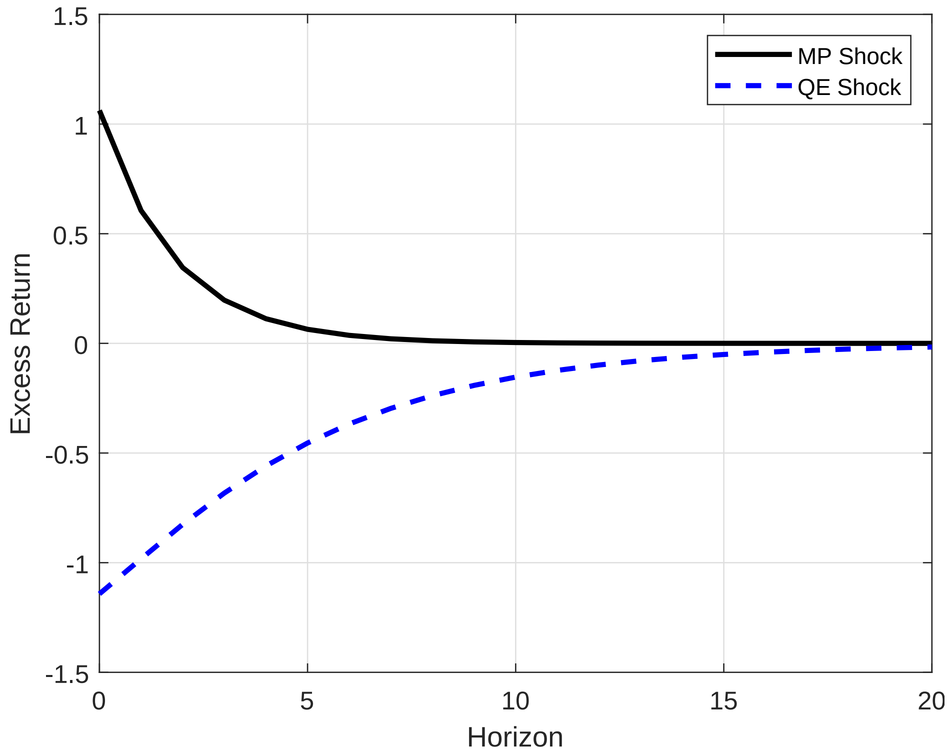
Another important difference between a QE shock and a conventional policy shock in our model concerns how each affects the yield curve. Though a long-term interest rate does not appear in the baseline four equation model in Subsection 2.1, one is operating in the background in an alternative representation of the IS curve (which is derived in Appendix B):

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1}\right) - \frac{z}{\sigma}\left(\mathbb{E}_t\, r_{t+1}^b - r_t^s\right). \tag{2.39}$$

$\mathbb{E}_t\, r_{t+1}^b$ is the expected return on the long bond in the model. Hence, the last term in (2.39) can be interpreted as an excess return of the long bond over the short-term rate.

A conventional expansionary monetary policy shock results in a steepening of the yield curve (i.e. an increase in the long rate relative to the short rate). In contrast, a stimulative QE shock results in a flattening of the yield curve. QE works by freeing up space on the FI's balance sheet to purchase long bonds, thereby pushing the price of these bonds higher and the yield lower. There is no direct effect on the short-term rate except through the policy

Figure 4: **Response of Excess Return of Long Bond to Monetary and QE Shocks**




*Notes:* This figures plots the responses of the annualized excess return, i.e. $\mathbb{E}_t\, r^b_{t+1} - r^s_t$, inferred from (2.39), to a conventional monetary policy shock (solid black) and a QE shock (dashed blue). The shocks are normalized so as to generate the same impact increase in output as in Figure 2 and Figure 3.

rule. As calibrated, the short rate actually rises modestly (due to the slightly inflationary nature of a QE shock under the current calibration). Impulse responses of the long-short spread to both a conventional policy shock and a QE shock are depicted in Figure 4.

# 3 Optimal Monetary Policy

In this section, we explore the design of optimal monetary policy in the context of our four equation NK model. Credit shocks generate an endogenous cost-push term in the Phillips curve, so they lead to a non-trivial tradeoff for a central bank wishing to solely implement policy via adjustment of the short-term interest rate. As such, heretofore unconventional policies like quantitative easing ought to be used even when the short rate is unconstrained by the ZLB. Further, quantitative easing policies can be a useful (albeit imperfect) substitute for conventional policy when the short-term rate is constrained by the ZLB.

Given policymakers' emphasis on the so-called dual mandate, we focus on a policy-relevant quadratic loss function in inflation and the output gap:

$$\mathbb{L} = \mu x_t^2 + \pi_t^2. \tag{3.1}$$

$\mu \geq 0$ is the relative weight attached to fluctuations in the output gap. An expression like (3.1) can be motivated as the micro-founded welfare criterion for a central bank in the standard three equation NK model under certain assumptions.[7,8] The central bank's welfare objective is a present discounted value of the period loss function given in (3.1). For the remainder of this section, we consider optimal policy under discretion, and so focus only on the period loss function.

## 3.1 Unconstrained Optimal Discretionary Policy

We begin by studying optimal monetary policy when both policy instruments are available. We start with an impossibility result spelled out formally in Theorem 1:

**Theorem 1** *It is not possible to completely stabilize both inflation and the output gap with the adjustment of a single policy instrument when both credit and natural rate shocks are present.*

**Proof**: See Appendix G.1.

[7] In particular, in the benchmark three equation model (3.1) would be the micro-founded loss function when a Pigouvian tax is in place to undo the steady state distortion associated with monopolistic competition; see, e.g., Woodford (2003). The optimal weight on the output gap would satisfy $\mu = \frac{\gamma\zeta}{\epsilon}$, where $\gamma\zeta$ is the slope of the Phillips curve and $\epsilon$ is the elasticity of substitution across varieties of retail goods. For conventional calibrations, this weight would be quite low.

[8] In our four equation model, a fully micro-founded loss function would be more complicated due to the two types of households, and would depend on arbitrary welfare weights on each. We instead choose to focus on a policy-relevant loss function like (3.1) and consider a variety of different values of $\mu$. One can motivate targeting $y_t^*$ as the appropriate output level in a version of a social planner's problem where the planner wishes to completely smooth the consumption of the child household. See Appendix D.

This result can be viewed as a straightforward application of Tinbergen (1952). But the result is particularly interesting and useful in our setting because the credit shock breaks the "Divine Coincidence" (Blanchard and Galí 2007), in which case one policy instrument is sufficient to hit both targets. In our model, it is not possible to simultaneously stabilize inflation and the output gap with only the short-term policy rate.

Given the impossibility result of Theorem 1, the central bank should use both the short-term rate and its long bond portfolio as policy instruments. Each period, the central bank minimizes its loss function in (3.1) with respect to the two policy instruments ($r_t^s$ and $qe_t$) subject to the IS and Phillips curves in (2.1) - (2.2). As shown formally in Appendix G.1, the optimal solution features $\pi_t = x_t = 0$; i.e., the central bank hits both of its targets. In equilibrium, simultaneously hitting both targets leads to the following optimal paths of the central bank's instruments:

**Proposition 1** *With both instruments available, the central bank achieves $\pi_t = x_t = 0$, and the equilibrium paths for the policy instruments are $r_t^s = r_t^*$ and $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$.*

The proof of Proposition 1 is simple. From the Phillips curve, inflation and the output gap always equaling zero implies that $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$. Then, from the IS equation, we must have $r_t^s = r_t^*$. The novel implication of Proposition 1 is that QE-type policies in principle ought to always be used as long as there are credit shocks, not only when the short-term policy rate is constrained by the ZLB.

## 3.2 Optimal QE at the ZLB

Although QE type policies should always be used to offset credit market disturbances in our model, they only became popular when short-term interest rates were pushed to the ZLB in the wake of the Financial Crisis and ensuing Great Recession. In this section, we study how QE policies might be used to mitigate the consequences of a binding ZLB.

We approximate the effects of a binding ZLB in our linearized model following Eggertsson and Woodford (2003) and Christiano, Eichenbaum and Rebelo (2011). Suppose that a central bank has been following the jointly optimal policy described in Proposition 1. But then in period $t$, suppose the natural rate of interest falls below zero, so that $r_t^s = 0$. Suppose that it will stay there in each subsequent period with probability $\alpha \in [0, 1)$, where this probability is invariant over time. The expected duration of the ZLB is therefore $1/(1-\alpha)$. This means that interest rate policy can be characterized as follows:

$$
\begin{aligned}
r_t^s &= 0 && (3.2) \\
\mathbb{E}_t \, r_{t+1}^s &= 0 \text{ with probability } \alpha && (3.3)
\end{aligned}
$$

Faced with a ZLB, the central bank will pick $qe_t$ to minimize its loss function, but is unable to pick the policy rate. The optimal choice of $qe_t$ leads to the following "lean against the wind condition":

$$
\pi_t = -\frac{\mu(1-z)}{\gamma\zeta(1-z) - \gamma\sigma} x_t. \tag{3.4}
$$

Proposition 2 describes the evolution of targets and instruments while the ZLB is binding.

**Proposition 2** *When the short rate is constrained by the ZLB, which will continue to bind in the next period with probability $\alpha$, with QE being the only viable policy instrument, the optimal targeting rule is characterized by (3.4). In equilibrium, the paths for inflation, output gap, and QE are*

$$
\begin{aligned}
\pi_t &= \omega_1 r_t^* && (3.5) \\
x_t &= \omega_1 \omega_2 r_t^* && (3.6) \\
qe_t &= \tau r_t^* - \frac{\bar{b}^{FI}}{\bar{b}^{cb}} \theta_t, && (3.7)
\end{aligned}
$$

*where*

$$
\begin{aligned}
\omega_1 &= \frac{\gamma(1-z)}{\gamma\sigma(1-\alpha\rho_f)\omega_2 - \alpha\gamma\rho_f(1-z) + (1-z)(1-\alpha\rho_f)(1-\gamma\zeta\omega_2-\alpha\beta\rho_f)} && (3.8)\\
\omega_2 &= -\frac{\gamma\zeta(1-z)-\gamma\sigma}{\mu(1-z)} && (3.9)\\
\tau &= -\frac{(1-\gamma\zeta\omega_2-\alpha\beta\rho_f)(1-z)}{z\gamma\sigma\bar{b}^{cb}}\omega_1 && (3.10)
\end{aligned}
$$

**Proof**: See Appendix G.2.

An important and novel implication of Proposition 2 is that the ZLB need not pose a problem for credit shocks – inflation and the gap can be completely stabilized when only QE is available, with the QE portfolio adjusting to credit shocks exactly as it would absent the ZLB.

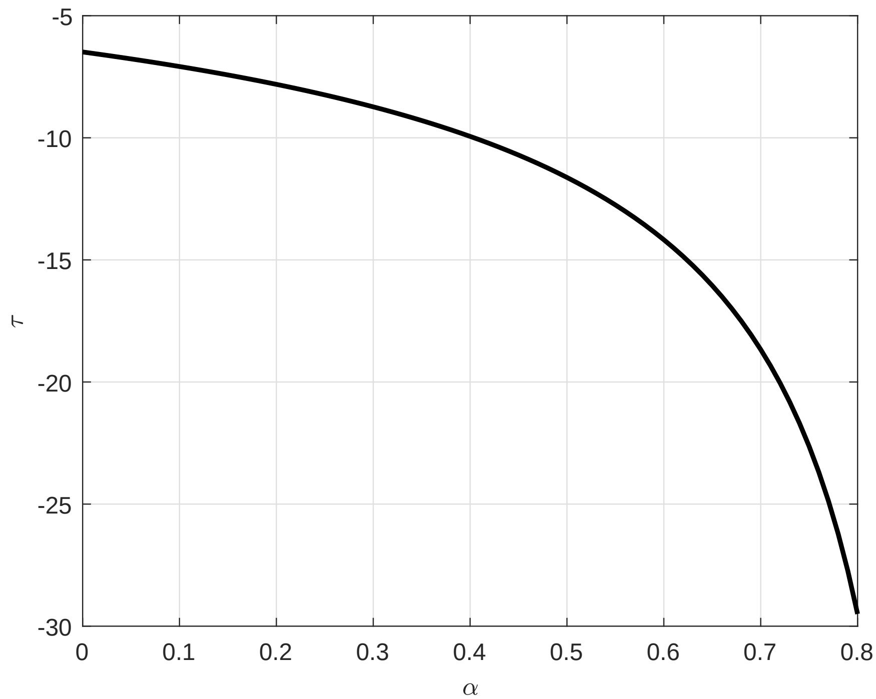
More importantly, we use the results from Proposition 2 to investigate the extent to which QE can serve as an effective substitute for conventional monetary policy during periods in which the short-term interest rate is constrained by zero. This was the original motivation for the use of QE in countries like Japan and the US when policy rates moved to the ZLB.

We restrict attention to values of $\alpha$ where $\omega_1 > 0$, so that reductions in the natural rate of interest cause inflation and the gap to decline when the ZLB binds.[9] In this region, $\tau < 0$, which means that the central bank provides positive stimulus in the form of bond purchases when the natural rate of interest declines. Figure 5 plots the equilibrium response of QE to a natural rate shock, $\tau$, as a function of $\alpha$. The figure is generated assuming $\mu = 1$. When the ZLB is expected to last longer, QE optimally reacts more aggressively to a natural rate shock.

Solid black lines in Figure 6 plot responses of the output gap and inflation to a negative

[9] As noted in Carlstrom, Fuerst and Paustian (2014), for sufficiently large $\alpha$, the sign of $\omega_1$ can flip from positive to negative. Where this perverse sign flip occurs depends on the values of other parameters, such as the slope of the Phillips curve, $\gamma\zeta$. We restrict attention to values of $\alpha$ consistent with $\omega_1$ being positive. For our parameterization, the sign flip occurs at an $\alpha$ consistent with an expected ZLB duration of 13 or more quarters. An alternative experiment would be to make the duration of the ZLB deterministic rather than stochastic. There would be no sign flip at some sufficiently long duration, but the analytic expression for $\omega_1$ would be significantly more complicated.

Figure 5: $\tau$ **As a Function of ZLB Duration,** $\alpha$

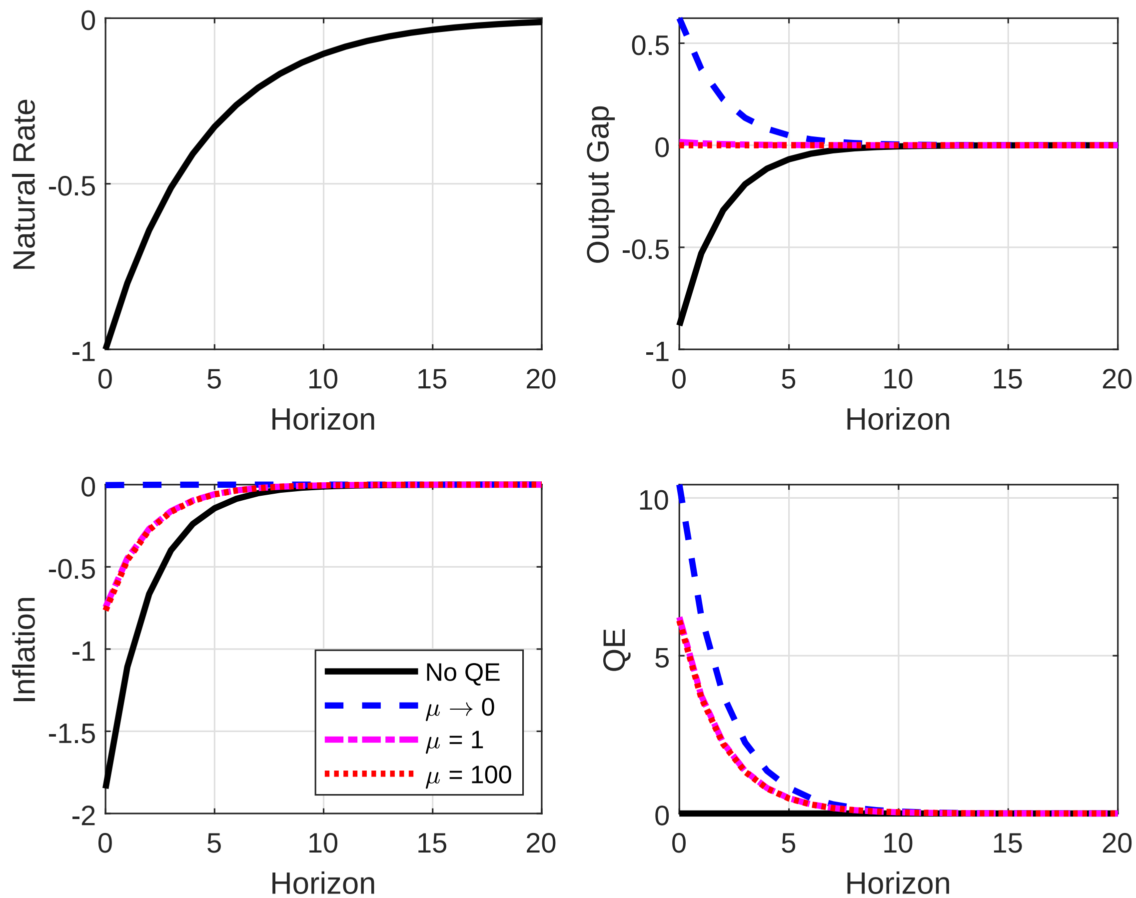



*Notes:* This figure plots the response of QE to a natural rate shock under optimal policy when the ZLB binds for different values of $\alpha$. The expected duration of the ZLB is $1/(1-\alpha)$, so higher values of $\alpha$ correspond to higher expected ZLB durations. The figure is generated assuming a value of $\mu = 1$.

natural rate shock when QE is unavailable. The ZLB binds for four quarters in expectation, with $\alpha = 3/4$. With neither QE nor the policy rate available, the output gap and inflation both decline significantly in response to a negative natural rate shock. A binding ZLB entails significant welfare losses.

The colored non-solid lines in Figure 6 plot responses when the ZLB binds but QE is optimally implemented. We consider different values of $\mu$, the relative weight on fluctuations in the output gap. When the central bank places no weight on the output gap (i.e. $\mu \to 0$), inflation is completely stabilized, the output gap increases (rather than decreases), and the central bank increases the size of its long bond portfolio by a sizeable amount. When virtually all weight is placed on the gap ($\mu = 100$), in contrast, inflation declines, although much less than the no-QE benchmark, the gap is completely stabilized, and the increase in the value

Figure 6: **IRFs to Natural Rate Shock at the ZLB, Optimal QE, Different $\mu$**

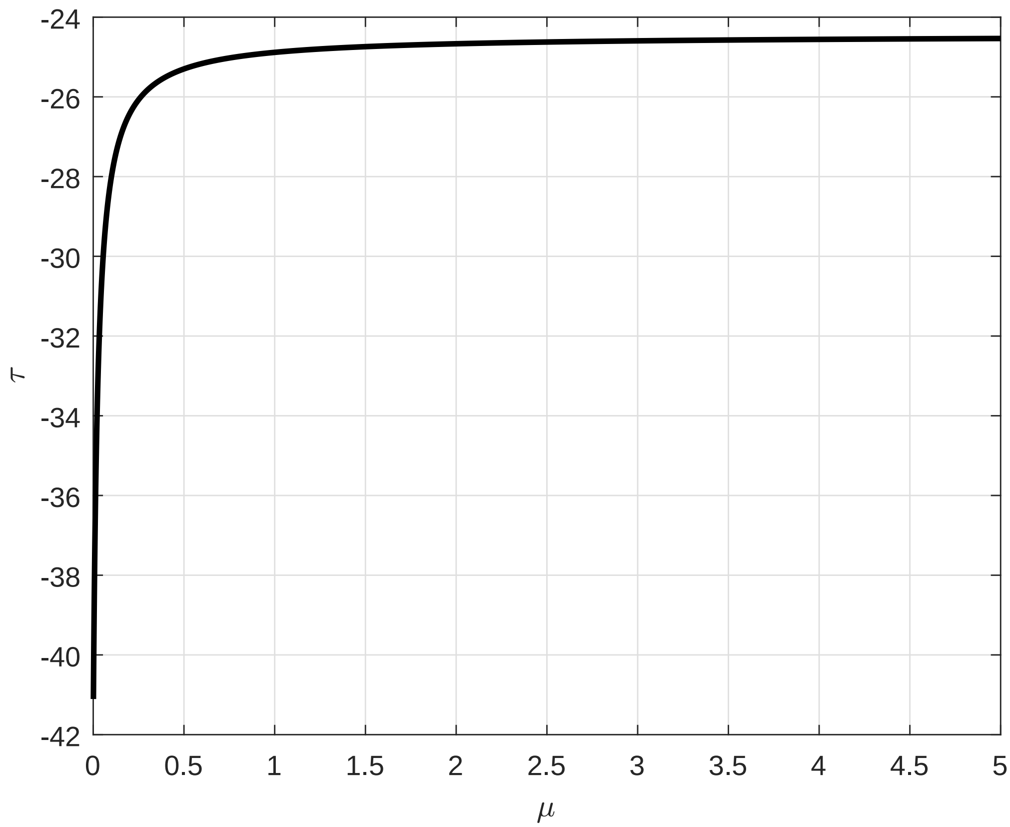



*Notes:* Black solid lines: IRFs to a one hundred basis point shock to the natural rate of interest in the four equation model when the short-term interest rate is constrained by the ZLB for $1/(1-\alpha)$ periods in expectation, where $\alpha = 3/4$, and there is no endogenous QE to the natural rate shock. The dashed lines plot responses with the optimally chosen $\tau$ for different welfare weights on the output gap, $\mu$. The output gap is expressed in percentage points, while the responses of inflation and the short-term interest rate are in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the baseline three equation NK model.

of the long bond portfolio is much more modest compared to the $\mu \to 0$ case. The case of equal weight on inflation and the gap is virtually indistinguishable from the case of nearly all weight being on the gap in the loss function.

The results described in Figure 6 suggest that quantitative easing can be an effective, albeit imperfect, substitute for conventional policy in response to natural rate shocks at the ZLB. For example, in the case of equal relative weights ($\mu = 1$), the output gap essentially does not react to the natural rate shock and inflation falls by about two-thirds of a percent given optimal QE policy. In comparison, with no endogenous QE at the ZLB, the output

Figure 7: $\tau$ **as a Function of** $\mu$




*Notes:* This figure plots the response of QE to a natural rate shock under optimal policy when the ZLB binds, with $\alpha = 3/4$, for different values of $\mu$, the welfare weight on the output gap.

gap would decline by nearly a full percentage point and inflation would fall by about three times as much. Endogenous QE therefore entails a sizeable welfare improvement over doing nothing at the ZLB. This will be true regardless of the value of $\mu$.

We close this subsection by plotting $\tau$ as a function of $\mu$, assuming a fixed value of $\alpha = 3/4$. $\tau$ measures the equilibrium response of QE to a natural rate shock under optimal discretionary policy. This is shown in Figure 7. $\tau$ is always negative, but is increasing in the relative weight on the output gap. That is, for a central bank concerned solely with stabilizing inflation, it is optimal to adjust the long bond portfolio quite strongly in response to natural rate shocks. For a central bank more concerned with gap stabilization, the optimal QE response remains sizeable but is nevertheless quite a bit smaller than for values of $\mu$ close to zero. The values of $\tau$ are very similar for values of $\mu \geq 0.5$. To our knowledge, we are the first to discuss how QE should be implemented differently when central banks place different weights on their dual mandate.

## 3.3 Optimal Policy without QE

Next, consider an operating framework similar to the one prevailing in the US prior to the Great Recession in which the central bank uses the short-term interest rate as its sole policy instrument. The ZLB does not bind. This subsection studies the optimal adjustment of the short-term rate in this scenario.

As shown in Appendix G.3, the optimal choice of the policy rate satisfies

$$\pi_t = -\frac{\mu}{\gamma\zeta} x_t. \tag{3.11}$$

Note that (3.11) is the same as the "lean against the wind condition" for the policy rate for optimal policy under discretion as in the canonical three equation model. With (3.11) characterizing optimal policy, the equilibrium paths of endogenous variables are given by:

**Proposition 3** *With the short rate being the only policy instrument, the optimal targeting rule is characterized by (3.11). In equilibrium, the paths for inflation, the output gap, and the policy rate are*

$$\pi_t = \varphi\theta_t \tag{3.12}$$

$$x_t = -\frac{\gamma\zeta}{\mu}\varphi\theta_t \tag{3.13}$$

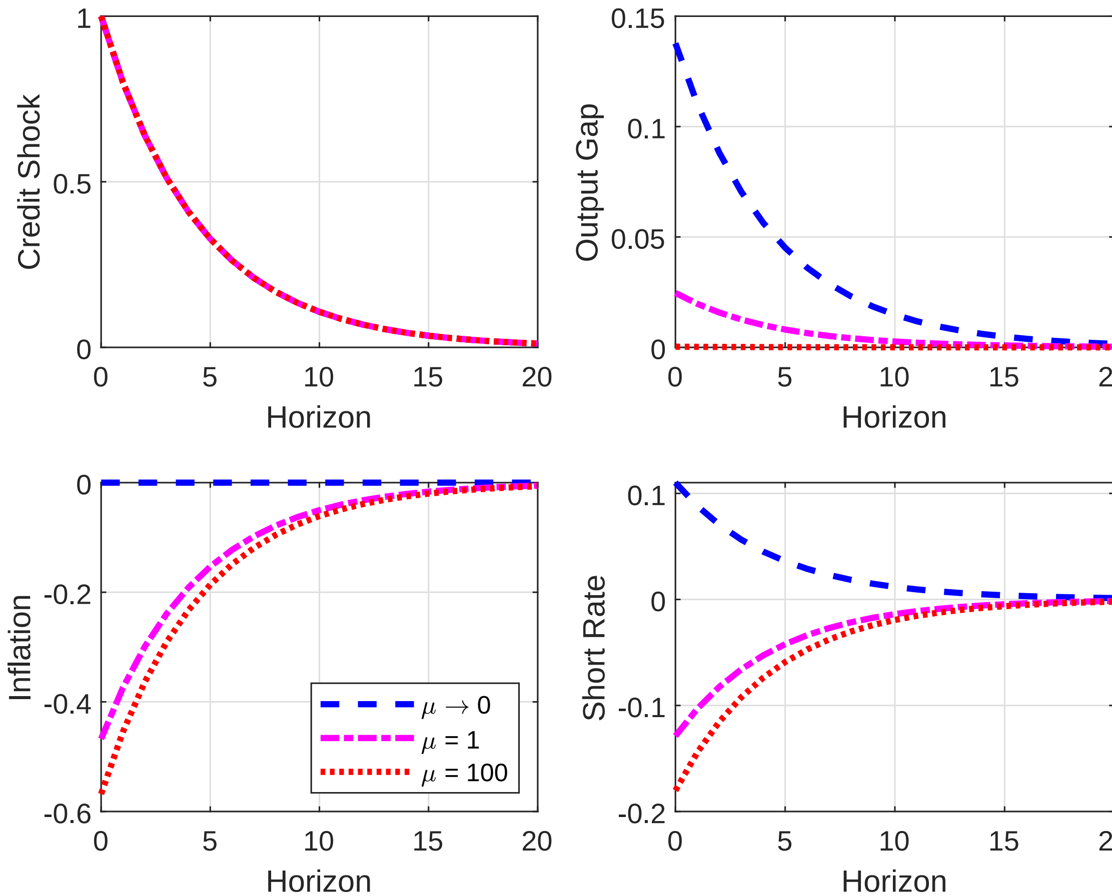
$$r_t^s = r_t^* + \eta\theta_t \tag{3.14}$$

*where*

$$\varphi = -\frac{\mu}{\gamma^2\zeta^2 + \mu(1-\beta\rho_\theta)} \frac{z\gamma\sigma}{1-z} \bar{b}^{FI} \tag{3.15}$$

$$\eta = \rho_\theta\varphi + \frac{\sigma(1-\rho_\theta)}{1-z}\frac{\gamma\zeta}{\mu}\varphi + \frac{(1-\rho_\theta)\sigma z \bar{b}^{FI}}{1-z}. \tag{3.16}$$

Figure 8: **IRFs to Credit Shock, Optimal Policy Rate, Different $\mu$**



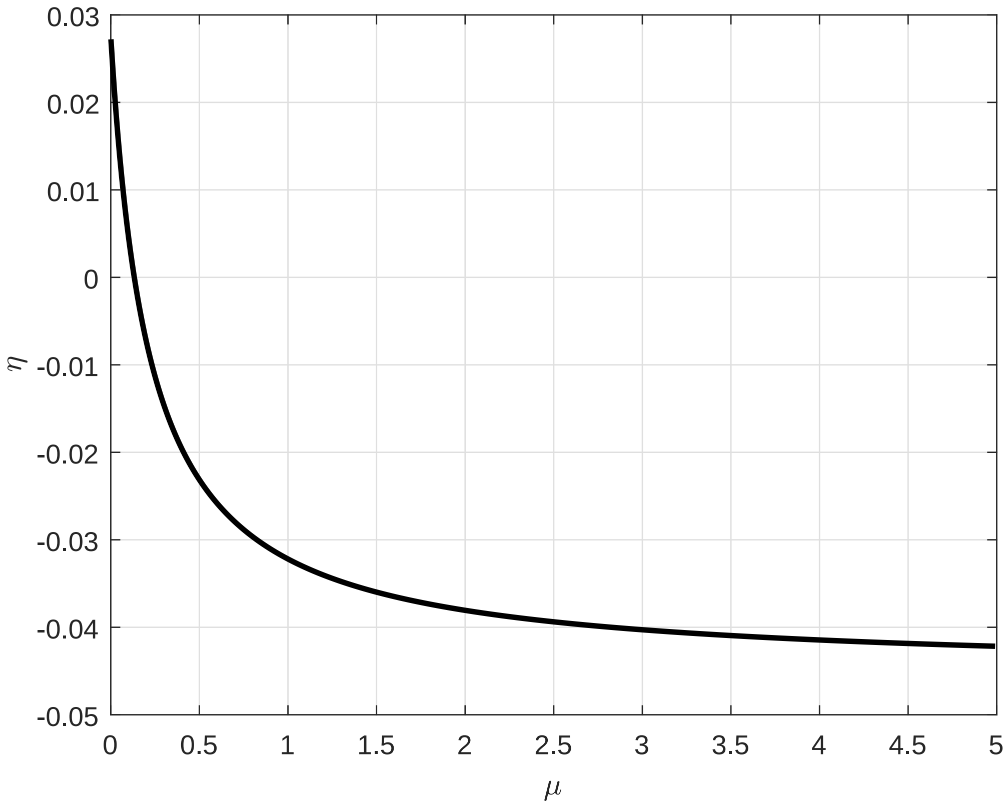

*Notes:* This figure plots IRFs to a one percentage point credit shock in the four equation model. $qe_t = 0$, and the interest rate is set according to the optimality condition described in Proposition 3. The output gap is expressed in percentage points, while the responses of inflation and the short-term interest rate are in annualized percentage points.

**Proof**: See Appendix G.3.

The optimal policy described in Proposition 3 completely stabilizes the output gap and inflation in response to natural rate shocks. This is not true conditional on credit shocks. Figure 8 plots the responses of the output gap and inflation to a credit shock for different values of $\mu$, taking our baseline calibration of other parameters.

When there is no weight placed on the output gap, (shown with the blue dashed lines corresponding to $\mu \to 0$), the central bank raises the short-term interest rate in response to a positive credit shock (i.e. $\eta > 0$). This completely stabilizes inflation but results in a sizeable increase in the output gap. In contrast, if the relative weight on the output gap is

Figure 9: $\eta$ **As a Function of** $\mu$




*Notes:* This figure plots $\eta$ as a function of $\mu$, the welfare weight on the output gap, for the case when $qe_t = 0$.

large ($\mu = 100$, shown in the red dashed lines), the central bank optimally cuts the policy rate in response to the credit shock (i.e. $\eta < 0$). This stabilizes the output gap but results in a significant decline in inflation. For equal weights on the output gap and inflation ($\mu = 1$, depicted via pink dashed lines), the policy rate decreases slightly, but the output gap rises and the inflation rate falls.

An interesting result from Figure 8 is that the *sign* of the optimal policy rate response to a credit shock depends on the relative weight placed on the output gap. A central bank mostly concerned with stabilizing output ought to cut the policy rate in the face of a positive credit shock, whereas it should raise the policy rate if it is mostly concerned with stabilizing inflation. Figure 9 plots $\eta$ as a function of $\mu$. Consistent with what is observed in Figure 8, $\eta$ is positive when $\mu$ is very small and turns negative as $\mu$ gets bigger, crossing zero at around $\mu = 1$. For central banks facing a dual mandate, this tradeoff between stabilizing inflation or the output gap can be eliminated if they can deploy QE.

## 4 Implementable Policy Rules

In Section 3, we explored optimal monetary policy under discretion. We derived first order conditions for a central bank facing a standard welfare function. These first order conditions are optimal targeting rules that imply paths of the short-term policy rate and the central bank's long bond portfolio. With both instruments available, it is possible to completely stabilize both inflation and the gap. In equilibrium, the policy rate moves one-for-one with the natural rate of interest and the long bond portfolio moves opposite credit shocks.

While optimal targeting rules have a long tradition in the monetary policy literature, there is also significant interest in the design of instrument rules where instruments react to fluctuations in endogenous variables. In this section, we therefore consider the optimal design of "simple and implementable" rules for both the short-term policy rate and the central bank's long bond holdings (Schmitt-Grohe and Uribe 2007). We assume that the short-term policy rate obeys a standard Taylor rule, (2.34). We further allow for the central bank's long bond portfolio to obey a similar Taylor-type rule that reacts to inflation and the output gap:

$$qe_t = \rho_q qe_{t-1} - (1 - \rho_q) [\lambda_\pi \pi_t + \lambda_x x_t] + s_q \varepsilon_{q,t}. \tag{4.1}$$

In postulating (4.1), which we refer to as a "QE rule," we assume that $\lambda_\pi \geq 0$ and $\lambda_x \geq 0$. The negative sign in front reflects the fact that, *a priori*, we think that the central bank would want to move its bond holdings opposite the direction of how it would adjust the policy rate in reaction to movements in both inflation and the output gap.

## 4.1 Determinacy

An important requirement for instrument rules is that they deliver a determinate rational expectations equilibrium. As shown in Appendix C, if there is no endogenous component to the QE rule (i.e. $\lambda_\pi = \lambda_x = 0$), then the restrictions necessary for determinacy on the coefficients of the Taylor rule for the policy rate are the same as in the standard three equation New Keynesian model. This will not necessarily be the case when the central bank's bond holdings react to inflation and the output gap. In this subsection, we consider how endogenous reactions in the QE rule impact equilibrium determinacy.

Let $\mathbf{z}_t = \left[\pi_t \; x_t \; r^s_{t-1} \; qe_{t-1}\right]'$ be the vector of linearized endogenous variables.[10] The system evolves according to:

$$\mathbb{E}_t \, \mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t. \tag{4.2}$$

With two predetermined states, a unique rational expectations equilibrium requires that there be exactly two unstable eigenvalues in $\mathbf{A}$.

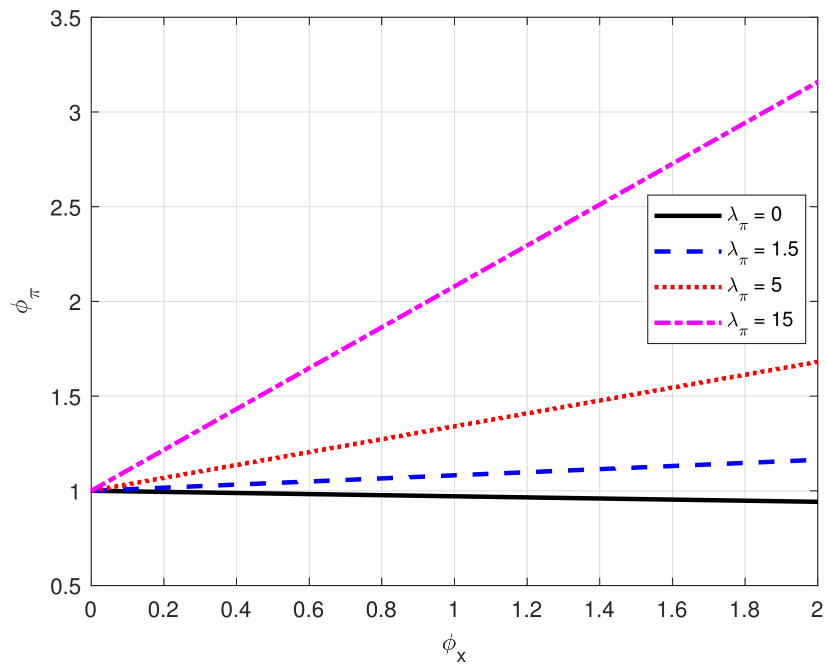
Because of the additional complexity of a fourth endogenous variable, we only numerically characterize the portion of the parameter space necessary for determinacy. We fix most parameter values at those listed in Table 1. We then consider different values of $\lambda_\pi$ and $\lambda_x$ and search for the minimum combination of $\phi_\pi$ and $\phi_x$ needed to generate determinacy, conditional on those values of $\lambda_\pi$ and $\lambda_x$.[11]

Consider first different values of $\lambda_\pi$, fixing $\lambda_x = 0$. We consider values of $\lambda_\pi$ of 0, 1.5, 5, and 15. Results are shown graphically in Figure 10. When $\lambda_\pi = \lambda_x = 0$, we have the familiar result (shown in the solid black line) that when $\phi_x = 0$, the central bank must respond at least one-to-one with inflation in the interest rate rule. As $\phi_x$ rises, the required value of $\phi_\pi$ falls, but determinacy is mostly governed by the response to inflation. When $\lambda_\pi > 0$, so long

[10] Please see Appendix C for more details.

[11] For these exercises, we set $\rho_r = \rho_q = 0.8$. Results are very similar for different values of these smoothing parameters.

Figure 10: **Policy Coefficients for Determinacy: Reacting to Inflation in the QE Rule**




*Notes:* This figure plots the minimum values of $\phi_\pi$ and $\phi_x$ (the reactions to inflation and the output gap, respectively, in the interest rate rule) necessary for equilibrium determinacy, conditional on different values of $\lambda_\pi$ (the reaction to inflation in the QE rule). $\lambda_x = 0$. The solid black line considers the case of $\lambda_\pi = 0$, the dashed blue line the case of $\lambda_\pi = 1.5$, the dotted red line the case of $\lambda_\pi = 5$, and the dash-dot magenta line the case of $\lambda_\pi = 15$. Values of $\phi_\pi$ above each line generate a unique rational expectations equilibrium.

as $\phi_x = 0$, it remains the case that the interest rate rule must react more than one-to-one to inflation for determinacy. There is an interaction effect between $\lambda_\pi$ and $\phi_x$, however. For modestly positive values of $\lambda_\pi$, as $\phi_x$ gets bigger, the requisite coefficient on inflation in the interest rate rule for equilibrium determinacy gets larger, instead of smaller. This effect is more noticeable the bigger is $\lambda_\pi$ and seems quantitatively relevant. In particular, suppose that $\phi_x = 1$. When $\lambda_\pi = \lambda_x = 0$, the requisite value of $\phi_\pi$ is slightly less than one. But when $\lambda_\pi = 5$, the required coefficient on inflation in the interest rate rule is about 1.3. When $\lambda_\pi = 15$, the necessary value of $\phi_\pi$ jumps to more than 2.

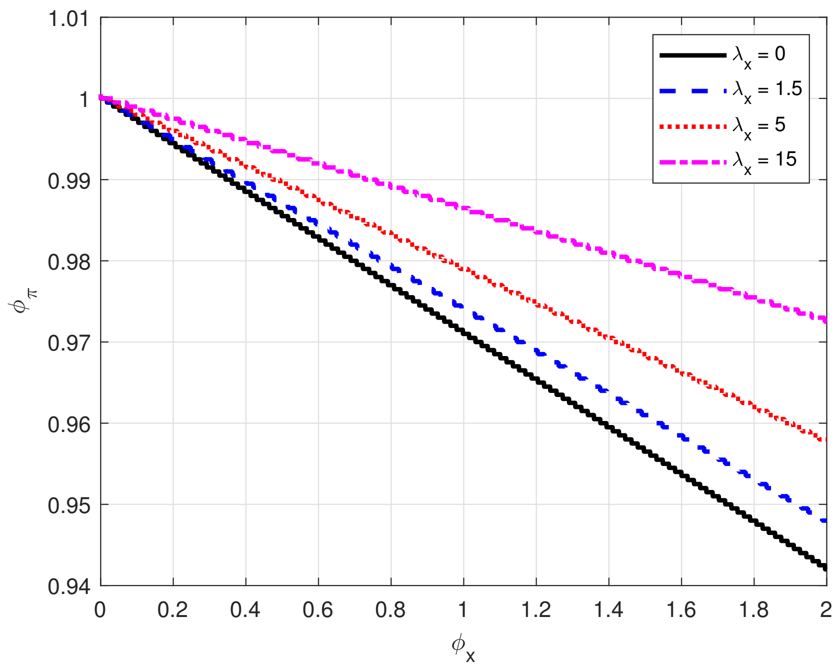
Our intuition for the above results is as follows. For a determinate equilibrium, the policy rate must react more than one-for-one to a permanent change in the inflation rate. If the Taylor rule for the policy rate does not react to the output gap, we have the standard Taylor principle condition that $\phi_\pi > 1$, regardless of whether the central bank's long bond portfolio reacts to inflation. When $\phi_x > 0$ and QE reacts to inflation, in contrast, the cutoff value of $\phi_\pi$ is bigger than one. When QE reacts negatively to inflation, there exists downward pressure on the output gap, other factors held constant. When the policy rate also reacts positively to the output gap, taken together an active QE rule reduces the overall sensitivity of the policy rate to inflation for a given $\phi_\pi$. The bigger the reaction of QE to inflation, and the larger the reaction of the policy rate to the gap, the more aggressive must be the direct response to inflation in the Taylor rule.

Consider next different values of $\lambda_x$, fixing $\lambda_\pi = 0$. We again consider values of $\lambda_x$ of 0, 1.5, 5, and 15. Results are depicted graphically in Figure 11. Here the determinacy results are more in line with the standard three equation model. In particular, responding more strongly to the output gap in the interest rate rule permits a smaller reaction to inflation for any value of $\lambda_x$. The required coefficient on inflation in the interest rate rule, $\phi_\pi$, is larger for each value of $\phi_x$ the bigger is the reaction to the gap in the QE rule, $\lambda_x$. But the differences in the necessary values of $\phi_\pi$ for each $\phi_x$ when $\lambda_x$ gets larger are quite small.

There are two noteworthy conclusions from these exercises. First, a QE rule that reacts aggressively to endogenous variables like inflation and the output gap does not make equilibrium determinacy more likely in our model. In fact, it makes it less likely – larger values of $\lambda_\pi$ or $\lambda_x$ reduce, rather than increase, the set of coefficients in the interest rate rule that yield a unique rational expectations equilibrium. Second, for the purposes of guaranteeing a determinate equilibrium, reacting to inflation in the QE rule seems more problematic than reacting to the output gap. QE responding to the output gap, but not inflation, hardly has any effect on the set of coefficients in the interest rate rule that result in determinacy. The QE rule reacting to inflation, in contrast, both introduces a tradeoff between reacting to the output gap and inflation in the interest rate rule, and significantly increases the required coefficient on inflation in that rule.[12] In practice, central banks have implemented

[12]To be clear, by "tradeoff" we mean that, conditional on reacting to inflation in the QE rule, the central bank must react *more* to inflation in the interest rate rule the more it reacts to the output gap. In contrast,

Figure 11: **Policy Coefficients for Determinacy: Reaction to the Output Gap in the QE Rule**




*Notes:* This figure plots the minimum values of $\phi_\pi$ and $\phi_x$ (the reactions to inflation and the output gap, respectively, in the interest rate rule) necessary for equilibrium determinacy, conditional on different values of $\lambda_x$ (the reaction to the output gap in the QE rule). $\lambda_\pi = 0$. The solid black line considers the case of $\lambda_x = 0$, the dashed blue line the case of $\lambda_x = 1.5$, the dotted red line the case of $\lambda_x = 5$, and the dash-dot magenta line the case of $\lambda_x = 15$. Values of $\phi_\pi$ above each line generate a unique rational expectations equilibrium.

QE during episodes of low interest rates and inflation, and have primarily used it to target real variables. In this case, indeterminacy is less likely to be an issue.

## 4.2 Optimal Implementable Rules

In this subsection, we consider optimal implementable policy rules, of the form (2.34) for the interest rate rule and (4.1) for the QE rule. There are six policy parameters – $\rho_r$, $\phi_\pi$, and $\phi_x$ for the interest rate rule, and $\rho_q$, $\lambda_\pi$, and $\lambda_x$ for the QE rule. The assumed objective function of a central bank, (3.1), features two targets (inflation and the output gap). Our model structure features two instruments (the short-term interest rate and the central bank's

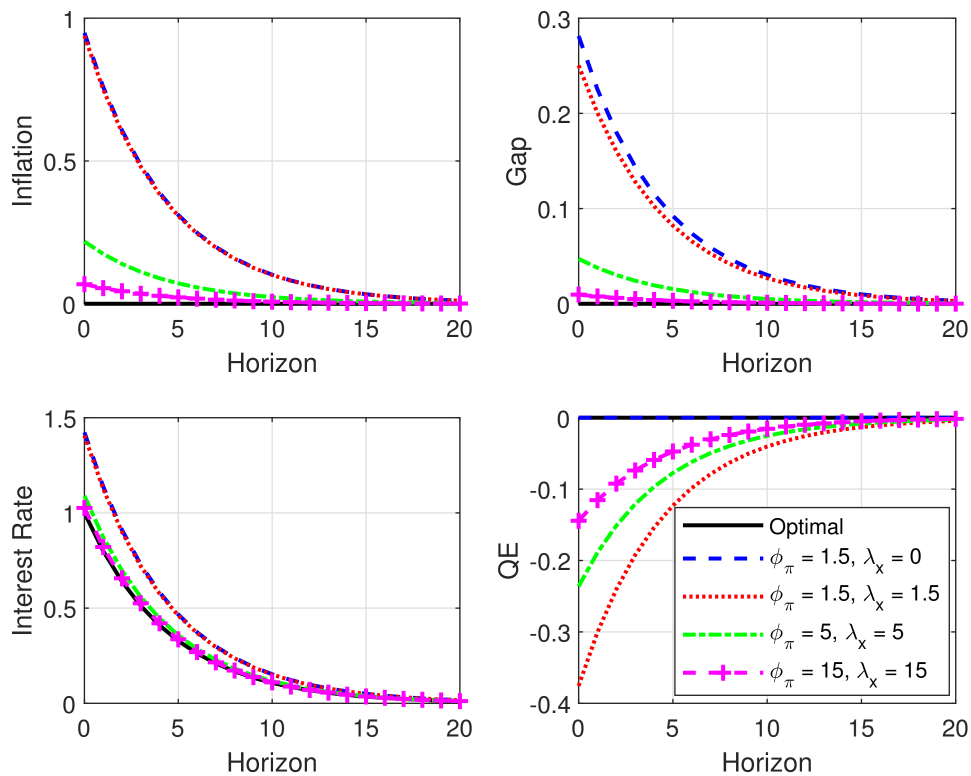
in the three equation model, no such tradeoff exists – responding more to the gap in the interest rate rule necessitates reacting less to inflation.

bond portfolio). With this many parameters and only two targets (inflation and the output gap), there may in principle be many configurations of these policy parameters that give rise to desirable outcomes. We focus on one particularly simple and transparent specification – the interest rate rule ought to react strongly to inflation, while the QE rule should react aggressively to the output gap. This both ensures equilibrium determinacy given our results above, and also seems to be the relevant case in practice.

For the purposes of the exercises which follow, we set both autoregressive parameters ($\rho_r$ and $\rho_q$) equal to zero. Results are qualitatively similar for positive values of the smoothing parameters. We set the coefficient on the output gap in the interest rate rule, $\phi_x$, and the coefficient on inflation in the QE rule, $\lambda_\pi$, equal to zero. We then show how both targets and instruments react to exogenous shocks for different values of the coefficient on inflation in the interest rate rule, $\phi_\pi$, and the coefficient on the output gap in the QE rule, $\lambda_x$.

Figure 12 shows impulse responses to a shock to the natural rate of interest. Solid black lines show the optimal responses discussed in Section 3. Under the optimal policy, inflation and the output gap are completely stabilized, with the interest rate reacting one-for-one with the natural rate and the central bank's bond portfolio unaffected. The dashed blue lines show the situation in which the interest rate rule reacts to inflation, with $\phi_\pi = 1.5$, but bond holdings are constant. Relative to the optimal outcome, the interest rate overreacts, with both inflation and the output gap increasing. The dotted red lines consider the case where $\phi_\pi = 1.5$ but the QE rule reacts to the output gap, with $\lambda_x = 1.5$. The central bank's bond holdings fall, with the output gap and inflation both increasing less than the case where $\phi_\pi = 1.5$ and $\lambda_x = 0$. The dashed dash-dot green lines consider the case where $\phi_\pi = \lambda_x = 5$. This represents a more noticeable improvement, with both inflation and the output gap reacting less to the shock. Magenta lines with plus markers consider the case where $\phi_\pi = \lambda_x = 15$. The output gap and inflation both increase, but only slightly. Furthermore, the paths of the interest rate and central bank bond holdings are closer to the optimal paths. As $\phi_\pi = \lambda_x \to \infty$, the responses of all variables – both targets and

Figure 12: **IRFs to Natural Rate Shock, Implementable Rules**



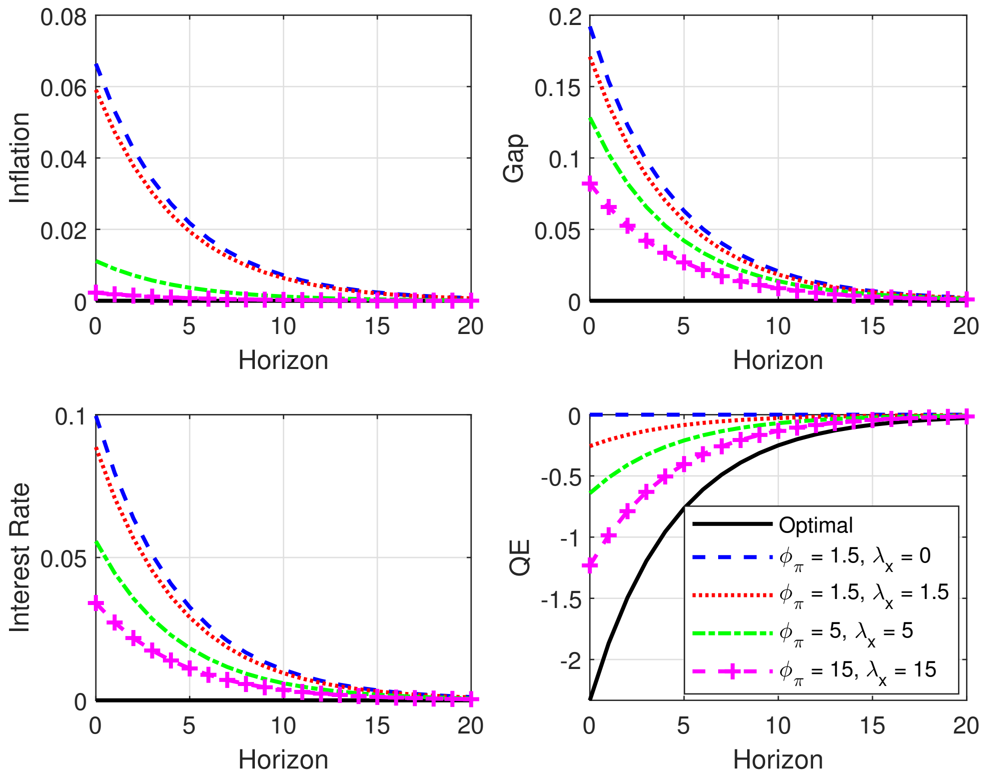

*Notes:* This figure shows impulse responses to a one unit shock to the natural rate of interest. Solid black lines show responses under the optimal path, where $r_t^s = r_t^*$ and $qe_t = 0$. Dashed lines consider parameter configurations of implementable rules, where we set $\rho_r = \rho_q = 0$, $\phi_x = 0$, and $\lambda_\pi = 0$.

instruments – approach their optimal paths.

Figure 13 is structured similarly, but considers responses to the credit shock, $\theta_t$. With the optimal policy, inflation and the output gap are constant, with the central bank's bond holdings falling and the interest rate being constant. When the central bank only reacts using the interest rate, with $\phi_\pi = 1.5$, and $\lambda_x = 0$, both inflation and the output gap increase, with the interest rate increasing as a result. As the central bank adjusts its bond portfolio more aggressively to the output gap, these movements are smaller. As in the case of the natural rate shock, as $\phi_\pi = \lambda_x \to \infty$, the responses of all variables approach their optimal paths.

In our four equation model, in equilibrium the optimal discretionary policy results in

Figure 13: **IRFs to Credit Shock, Implementable Rules**




*Notes:* This figure shows impulse responses to a one unit credit shock. Solid black lines show responses under the optimal path, where $r_t^s = 0$ and $qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t$. Dashed lines consider parameter configurations of implementable rules, where we set $\rho_r = \rho_q = 0$, $\phi_x = 0$, and $\lambda_\pi = 0$.

the interest rate moving one-for-one with changes in the exogenous natural rate of interest and the central bank's bond portfolio moving opposite the credit shock. A central bank can closely replicate these paths via Taylor-type instrument rules for both the interest rate and its bond portfolio. Doing so requires aggressively responding to inflation in the interest rate rule and reacting strongly to the output gap in the QE rule.

## 4.3 Implementable Rules and the ZLB

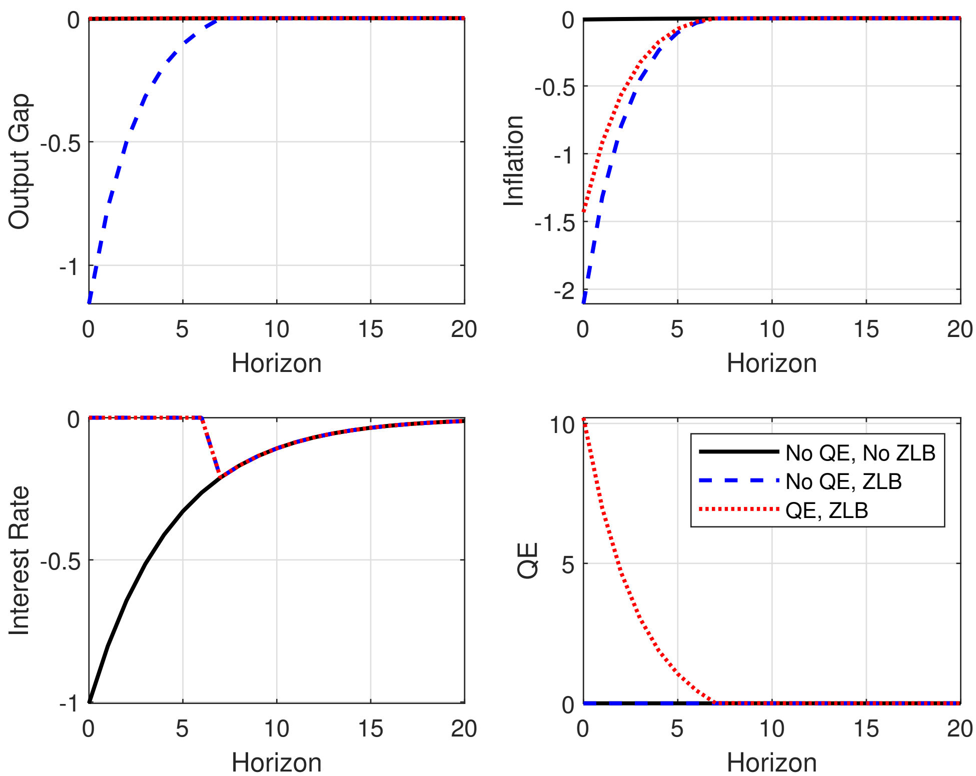
In practice, quantitative easing and other forms of unconventional monetary policy have been deployed primarily as antidotes to conventional policy paralysis at the ZLB. In Section 3, in the context of optimal targeting rules, we examined how QE could be deployed as a useful albeit imperfect substitute for conventional policy at the ZLB. In this subsection, we proceed similarly, but instead focus on an implementable rule for the central bank's long bond portfolio of the form (4.1).

We solve the linearized four equation model using a piecewise linear approximation subject to the constraint that the policy rate be non-negative.[13] As long as this constraint is not binding, the policy rate obeys (2.34). To implement a binding ZLB, we subject the economy to a sequence of natural rate shocks that force the non-negativity constraint to bind, in expectation for two years (eight quarters). To compute impulse responses, in the first period that the ZLB binds, we also subject the economy to a small shock to either the natural rate or the credit variable, where the shock is small enough so as to not change the expected length of time the ZLB is binding. We compare how the economy reacts to these shocks when QE is fixed versus when it obeys (4.1).

Figure 14 plots impulse responses to the natural rate shock under three scenarios. The solid black lines depict responses when there is no ZLB and the policy rate follows a simple Taylor rule with $\phi_\pi \to \infty$; QE is held fixed. As discussed above, a very large reaction to the inflation rate in the Taylor rule for the policy rate replicates the optimal allocations under discretion. The dashed blue lines depict responses when the ZLB binds for eight quarters and QE is still unavailable. During the period of the ZLB, $r_t^s = 0$; when the ZLB lifts, it follows the simple rule with a very large reaction to inflation. The binding ZLB results in both the output gap and inflation declining substantially while the ZLB binds; once the ZLB lifts, both return to zero. The red dotted lines consider the case where QE is not fixed; rather, it follows a rule in which it reacts very strongly to the output gap, with $\lambda_x \to \infty$. This results in completely stabilization of the gap, even during the ZLB. Inflation still falls, although not as markedly as when the ZLB binds and QE is unavailable. The central bank's long bond portfolio must react aggressively to the natural rate shock, as depicted in the lower right corner of the figure. Note that the responses in Figure 14 with implementable

[13] In particular, we follow Guerrieri and Iacoviello (2015).

Figure 14: **IRFs to Natural Rate Shock, ZLB, Implementable Rules**



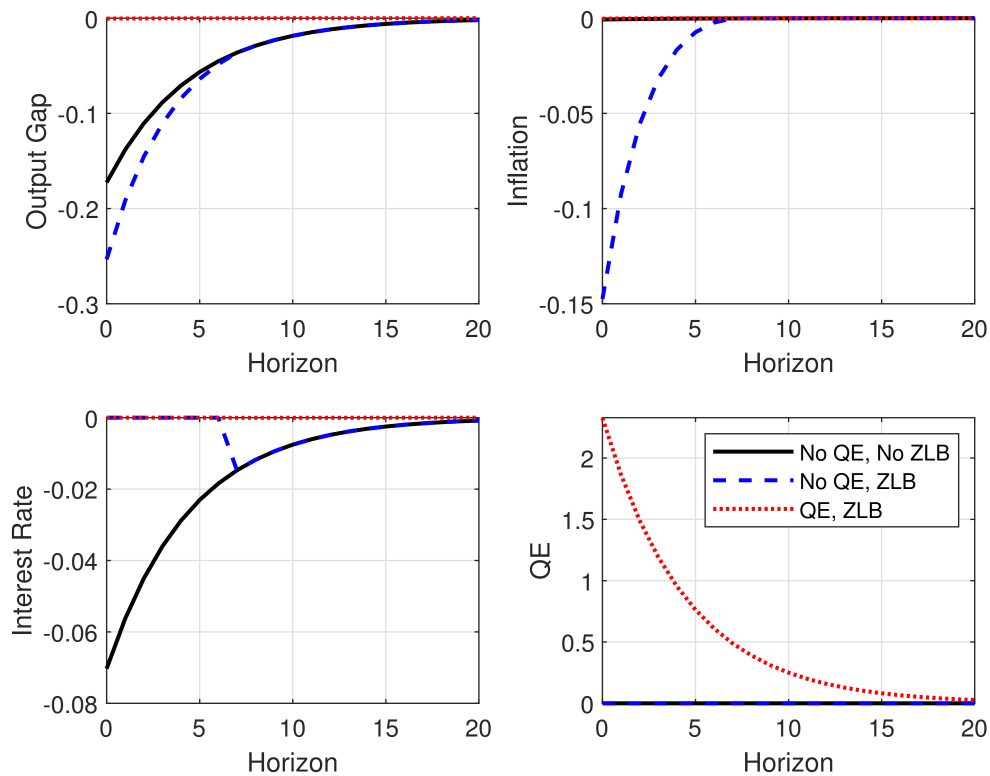

*Notes:* This figure shows impulse responses to a contractionary natural rate shock of one percentage point. Solid black lines show responses where the policy rate obeys a Taylor rule with $\phi_\pi \to \infty$ and $\rho_r = \phi_x = 0$, while QE is constant. Dashed blue lines depict responses when the ZLB binds for eight quarters, after which time the policy rate reverts to the simple rule; QE remains constant. Dotted red lines depict responses when the ZLB binds for eight quarters but QE follows a simple implementable rule with $\lambda_x \to \infty$, $\rho_q = 0$ and $\lambda_\pi = 0$.

rules are qualitatively similar to what is shown in Figure 6 for optimal targeting rules.[14]

Figure 15 is constructed similarly to Figure 14, but considers a contractionary credit shock. The black solid line shows responses with no ZLB where the policy rate reacts very strongly to inflation while the central bank's long bond portfolio is fixed. Inflation is stabilized, but the output gap falls. The dashed blue lines depict responses where QE is again assumed to be held constant, but the ZLB on the policy rate is in place for eight quarters. This results in inflation falling and the gap declining by more than it would absent

[14]Note in this exercise the ZLB lasts for eight periods with certainty, whereas in Section 3 the ZLB only lasted for four quarters in expectation. Hence, the scales in the two sets of figures are not directly comparable.

Figure 15: **IRFs to Credit Shock, ZLB, Implementable Rules**




*Notes:* This figure shows impulse responses to a contractionary credit shock of one percentage point. Solid black lines show responses where the policy rate obeys a Taylor rule with $\phi_\pi \to \infty$ and $\rho_r = \phi_x = 0$, while QE is constant. Dashed blue lines depict responses when the ZLB binds for eight quarters, after which time the policy rate reverts to the simple rule; QE remains constant. Dotted red lines depict responses when the ZLB binds for eight quarters but QE follows a simple implementable rule with $\lambda_x \to \infty$, $\rho_q = 0$ and $\lambda_\pi = 0$.

the ZLB. The red dashed lines depict responses when the ZLB on the policy rate binds for eight quarters, but QE follows a rule in which it reacts strongly to the output gap. This necessitates an increase in the central bank's long bond portfolio to offset the credit shock, as shown in the lower-right portion of the figure, but results in both inflation and the gap being completely stabilized. Similar to our result from Section 3, the ZLB on the policy rate poses no issues for the central bank with regard to credit shocks if QE is optimally implemented.

Our analysis in this section using implementable instrument rules reinforces our results from Section 3. In particular, QE can be used as an effective albeit imperfect substitute for the short-term policy rate conditional on natural rate shocks. Regardless of the relative weight the central bank attaches on the gap versus inflation, an instrument rule for QE that aggressively targets the output gap results in welfare improvements. Furthermore, a binding ZLB on the policy rate need not pose a problem for the central bank conditional on a credit shock – the same aggressive QE rule can completely stabilize both output and inflation.

# 5 Conclusion

In this paper, we developed a four equation New Keynesian model with credit shocks, financial intermediation, short- and long-term debt, and a channel for central bank long bond holdings to be economically relevant. The model inherits the tractability and elegance of the benchmark three equation New Keynesian model. It mainly differs in that credit shocks appear as wedges in both the IS and Phillips curves. In addition to a rule for the short-term policy rate, the fourth equation in the model is a rule for QE.

The model allows us to address the consequences of credit market disturbances as well as the effects of large scale asset purchases. We produce several analytical results concerning monetary policy design. The presence of credit market frictions breaks the Divine Coincidence, meaning it is not possible to completely stabilize inflation and the output gap with just one policy instrument. Optimal monetary policy entails adjusting the short-term interest rate to match fluctuations in the natural rate of interest, but manipulating the central bank's long bond portfolio so as to neutralize credit shocks. When it is not possible to adjust the short-term interest (for example, because of a binding ZLB), credit market shocks need not result in amplified fluctuations if the central bank adjusts its long bond portfolio as it would in normal times. In response to natural rate shocks, adjustment of the central bank's long bond portfolio can serve as a highly effective, albeit imperfect, substitute for conventional policy.

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# Appendix

The appendix contains the following sections to supplement the main body of the paper:

**A: The Full Non-Linear Model**: Here we show the full set of non-linear equilibrium conditions underlying our linearized four equation model.

**B: Details of the Linearized Model**: Here we show the full set of linearized equilibrium conditions of the model, and discuss how to reduce these down to the four equations presented in the text.

**C: Determinacy of Interest Rate Rules**: Here we derive conditions on the parameters of policy rules necessary for a unique rational expectations equilibrium.

**D: Potential, Flexible Price, and Efficient Output**: Here we derive expressions for potential, flexible price, and efficient output in our four equation model.

**E: Model Calibration**: Here we provide more detail on the baseline calibration of our four equation model.

**F: Robustness**: Here we compute impulse responses to shocks under a couple of different specifications of our four equation model.

**G: Optimal Monetary Policy**: Here we derive optimal equilibrium conditions for a central bank operating under discretion and provide formal proofs for some of the theorems and lemmas in the paper.

# A The Full Non-Linear Model

This appendix describes the full set of non-linear equilibrium conditions of the model.

The optimality condition for retail firms may be re-written in stationary terms by defining $p_{*,t} = P_{*,t}/P_t$, $x_{1,t} = X_{1,t}/P_t^{\epsilon}$, and $x_{2,t} = X_{2,t}/P_t^{\epsilon-1}$:

$$p_{*,t} = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} \tag{A.1}$$

$$x_{1,t} = p_{m,t} Y_t + \phi \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon} x_{1,t+1} \tag{A.2}$$

$$x_{2,t} = Y_t + \phi \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_t^{\epsilon-1} x_{2,t+1} \tag{A.3}$$

The aggregate inflation rate evolves according to:

$$1 = (1 - \phi) p_{*,t}^{1-\epsilon} + \phi \Pi_t^{\epsilon-1} \tag{A.4}$$

Price dispersion evolves according to:

$$v_t^p = (1 - \phi) p_{*,t}^{-\epsilon} + \phi \Pi_t^{\epsilon} v_{t-1}^p \tag{A.5}$$

Define lowercase variables as real values of nominal bonds, i.e. $b_t = B_t/P_t$. The balance sheet condition of the FI may be written:

$$Q_t b_t^{FI} + re_t = s_t + X_t^{FI} \tag{A.6}$$

The leverage constraint of the FI may be written:

$$Q_t b_t^{FI} \leq \Theta_t \bar{X}^{FI} \tag{A.7}$$

The central bank's balance sheet can be written:

$$Q_t b_t^{cb} = re_t \tag{A.8}$$

Similarly, the market-clearing condition for long term bonds in real terms is:

$$b_t = b_t^{FI} + b_t^{cb} \tag{A.9}$$

The auxiliary $QE_t$ variable is just the real value of the central bank's long bond portfolio:

$$QE_t = Q_t b_t^{cb} \tag{A.10}$$

Under our assumption on the transfer from parent to child, the consumption of the child may be written:

$$C_{b,t} = Q_t b_t \tag{A.11}$$

$A_t$ and $\Theta_t$ obey stationary AR(1) processes, where the non-stochastic steady state value of productivity is normalized to unity and $\Theta$ denotes the non-stochastic steady state value of leverage.

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{A.12}$$

$$\ln \Theta_t = (1-\rho_\theta)\ln\Theta + \rho_\theta \ln \Theta_{t-1} + s_\theta \varepsilon_{\theta,t} \tag{A.13}$$

To close the model, it is necessary to specify rules for the policy rate and the central bank's bond holdings. For the analysis in Subsection 2.3, we assume that the policy rate is set according to a Taylor rule and that the central bank's bond holdings obey an exogenous AR(1) process:

$$\ln R_t^s = (1-\rho_r)\ln R^s + (1-\rho_r)\left[\phi_\pi(\ln \Pi_t - \ln \Pi) + \phi_x(\ln Y_t - \ln Y_t^*)\right] + s_r \varepsilon_{r,t} \tag{A.14}$$

$$\ln QE_t = (1-\rho_q)\ln QE + \rho_q \ln QE_{t-1} + s_q \varepsilon_{q,t} \tag{A.15}$$

In (A.14)-(A.15), $R^s$ and $QE$ denote the non-stochastic steady state values of the policy rate and the central bank's balance sheet, respectively.

$Y_t^*$ can be found by solving the full system of equations assuming price flexibility ($\phi = 0$) where the credit shock is constant, i.e. $\Theta_t = \Theta$. Taking $Y_t^*$ as given, the output gap is then:

$$\ln X_t = \ln Y_t - \ln Y_t^* \tag{A.16}$$

The optimality conditions for the parent household, (2.5)-(2.7); the definition of the return on the long bond, (2.12); the optimality conditions for the child, (2.13)-(2.14); the optimality conditions for the FI, (2.19)-(2.20); the balance sheet condition for the FI and the leverage constraint re-written in real terms, (A.6)-(A.7); the labor demand condition for the wholesale firm, (2.27); the optimality condition for optimal price-setting for retailers, re-written in stationary form, (A.1)-(A.3); the market-clearing condition and aggregate production function, (2.30)-(2.31); the central bank's policy rule for $R_t^{re}$, (A.14); the central bank's balance sheet and definition of the $QE_t$ variable, (A.8) and (A.10); the central bank's QE rule, (A.15); the consumption of the child, (A.11); the bond market-clearing condition, (A.9); the evolution of inflation and price dispersion, (A.4)-(A.5); the definition of the output gap, (A.16); and the exogenous processes (A.12)-(A.3) constitute twenty-seven variables, $\{L_t, C_t, w_t, \Lambda_{t-1,t}, R_t^s, \Pi_t, \Lambda_{b,t-1,t}, R_t^b, Q_t, b_t^{FI}, \Theta_t, re_t, s_t, \Omega_t, R_t^{re}, p_{*,t}, x_{1,t}, x_{2,t}, p_{m,t}, Y_t, A_t, C_{b,t}, v_t^p, b_{cb,t}, b_t, QE_t, X_t\}$ in twenty-seven equations.

# B Details of the Linearized Model

This Appendix provides details of the linearization of the non-linear model, the equilibrium conditions for which are given in Appendix A. Where possible, lowercase variables denote log deviations from steady state, e.g. $\theta_t = \ln \Theta_t - \ln \Theta$. Where the corresponding level variable is already lowercase, a "hat" is put atop the relevant variable to denote a log deviation from steady state, e.g. $\widehat{p}_{m,t} = \ln p_{m,t} - \ln p_m$. Variables without a time subscript denote non-stochastic steady state values. The model is linearized about a steady state with zero trend inflation (i.e. $\Pi = 1$) where the leverage constraint on intermediaries binds. The complete list of linearized equilibrium conditions are as follows:

$$\chi l_t = -\sigma c_t + \widehat{w}_t \tag{B.1}$$

$$\lambda_{t-1,t} = -\sigma(c_t - c_{t-1}) \tag{B.2}$$

$$0 = \mathbb{E}_t \lambda_{t,t+1} + r_t^s - \mathbb{E}_t \pi_{t+1} \tag{B.3}$$

$$\lambda_{b,t-1,t} = -\sigma(c_{b,t} - c_{b,t-1}) \tag{B.4}$$

$$r_t^b = \frac{\kappa}{R^b} q_t - q_{t-1} \tag{B.5}$$

$$0 = \mathbb{E}_t \lambda_{b,t,t+1} + \mathbb{E}_t r_{t+1}^b - \mathbb{E}_t \pi_{t+1} \tag{B.6}$$

$$q_t + \widehat{b}_t^{FI} = \theta_t \tag{B.7}$$

$$\left[Qb^{FI}(1-\kappa)\right] q_t + Qb^{FI}\widehat{b}_t^{FI} - \kappa Qb^{FI}\widehat{b}_{t-1}^{FI} + \kappa Qb^{FI}\pi_t + re \cdot \widehat{re}_t = s \cdot \widehat{s}_t \tag{B.8}$$

$$\mathbb{E}_t \lambda_{t,t+1} - \mathbb{E}_t \pi_{t+1} + \frac{R^b}{sp} \mathbb{E}_t r_{t+1}^b - \frac{R^s}{sp} r_t^s = \omega_t \tag{B.9}$$

$$r_t^{re} = r_t^s \tag{B.10}$$

$$\widehat{p}_{*,t} = \widehat{x}_{1,t} - \widehat{x}_{2,t} \tag{B.11}$$

$$\widehat{x}_{1,t} = (1-\phi\beta)\widehat{p}_{m,t} + (1-\phi\beta)y_t + \phi\beta\, \mathbb{E}_t \lambda_{t,t+1} + \epsilon\phi\beta\, \mathbb{E}_t \pi_{t+1} + \phi\beta\, \mathbb{E}_t \widehat{x}_{1,t+1} \tag{B.12}$$

$$\widehat{x}_{2,t} = (1-\phi\beta)y_t + \phi\beta\, \mathbb{E}_t \lambda_{t,t+1} + (\epsilon-1)\phi\beta\, \mathbb{E}_t \pi_{t+1} + \phi\beta\, \mathbb{E}_t \widehat{x}_{2,t+1} \tag{B.13}$$

$$\widehat{w}_t = \widehat{p}_{m,t} + a_t \tag{B.14}$$

$$(1-z)c_t + zc_{b,t} = y_t \tag{B.15}$$

$$\widehat{v}_t^p + y_t = a_t + l_t \tag{B.16}$$

$$\widehat{v}_t^p = 0 \tag{B.17}$$

$$\pi_t = \frac{1-\phi}{\phi} \widehat{p}_{*,t} \tag{B.18}$$

$$q_t + \widehat{b}_t^{cb} = \widehat{re}_t \tag{B.19}$$

$$\widehat{b}_t = \frac{b^{FI}}{b} \widehat{b}_t^{FI} + \frac{b^{cb}}{b} \widehat{b}_t^{cb} \tag{B.20}$$

$$c_{b,t} = q_t + b_t \tag{B.21}$$

$$qe_t = \rho_q qe_{t-1} + s_q \varepsilon_{q,t} \tag{B.22}$$

$$a_t = \rho_A a_{t-1} + s_A \varepsilon_{A,t} \tag{B.23}$$

$$\theta_t = \rho_\theta \theta_{t-1} + s_\theta \varepsilon_{\theta,t} \tag{B.24}$$

$$r_t^{re} = \rho_r r_{t-1}^{re} + (1-\rho_r)\left[\phi_\pi \pi_t + \phi_x x_t\right] + s_r \varepsilon_{r,t} \tag{B.25}$$

$$qe_t = \widehat{re}_t \tag{B.26}$$

$$x_t = y_t - y_t^* \tag{B.27}$$

This is twenty-seven equations in $\{l_t, c_t, \widehat{w}_t, \lambda_{t-1,t}, r_t^s, \pi_t, \lambda_{b,t-1,t}, r_t^b, q_t, \widehat{b}_t^{FI}, \theta_t, \widehat{re}_t, \widehat{s}_t, \omega_t, r_t^{re},$

$\widehat{p}_{*,t}, \widehat{x}_{1,t}, \widehat{x}_{2,t}, \widehat{p}_{m,t}, y_t, a_t, c_{b,t}, \widehat{v}_t^p, \widehat{b}_t^{cb}, \widehat{b}_t, qe_t, x_t\big\}$ in twenty-seven variables.

The model can be reduced to the equations presented in Subsection 2.1 as follows. First, (B.10) can be used to eliminate $r_t^{re}$, so that the policy rule may be written as (2.34) in terms of $r_t^s$. Second, (B.11)-(B.13) can be combined with (B.18), which yields the textbook New Keynesian Phillips curve expressed as a function of marginal cost, where $\gamma = \frac{(1-\phi)(1-\phi\beta)}{\phi}$:

$$\pi_t = \gamma \widehat{p}_{m,t} + \beta\, \mathbb{E}_t\, \pi_{t+1} \tag{B.28}$$

Combining (B.1) with (B.14) and (B.16), making note of the fact that $\widehat{v}_t^p = 0$ around a zero inflation steady state, yields:

$$\widehat{p}_{m,t} = \chi y_t - (1+\chi) a_t + \sigma c_t \tag{B.29}$$

Making use of (B.15) allows us to write this as:

$$\widehat{p}_{m,t} = \frac{\chi(1-z)+\sigma}{1-z} y_t - (1+\chi)a_t - \frac{\sigma z}{1-z} c_{b,t} \tag{B.30}$$

Combining (B.19)-(B.21) with (B.26) allows us to write:

$$c_{b,t} = \frac{b^{FI}}{b}\left(q_t + \widehat{b}_t^{FI}\right) + \frac{b^{cb}}{b} qe_t \tag{B.31}$$

Defining $\bar{b}^{FI} = b^{FI}/b$ and $\bar{b}^{cb} = b^{cb}/b$ (i.e. the fraction of total bonds held by financial intermediaries and the central bank, respectively, in steady state), and making use of the binding leverage constraint, (B.7), leaves:

$$c_{b,t} = \bar{b}^{FI}\theta_t + \bar{b}^{FI} qe_t \tag{B.32}$$

Plugging (B.32) into (B.30) then gives:

$$\widehat{p}_{m,t} = \frac{\chi(1-z)+\sigma}{1-z} y_t - (1+\chi)a_t - \frac{\sigma z}{1-z}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] \tag{B.33}$$

Define the hypothetical natural rate of output, $y_t^*$, as the level of output consistent with flexible prices and no credit market shocks. That is, $y_t^*$ is the level of output consistent with $\widehat{p}_{m,t} = \theta_t = qe_t = 0$, or:

$$y_t^* = \frac{(1+\chi)(1-z)}{\chi(1-z)+\sigma} a_t \tag{B.34}$$

But then, using (B.27), we can write marginal cost as:

$$\widehat{p}_{m,t} = \frac{\chi(1-z)+\sigma}{1-z} x_t + \frac{\sigma z}{1-z}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] \tag{B.35}$$

Plugging (B.34) into (B.28), defining $\zeta = \frac{\chi(1-z)+\sigma}{1-z}$, yields (2.2).

To derive the IS equation, combine (B.2)-(B.4) with (B.6) and (B.16). Doing so yields:

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1}\right) - \frac{z}{\sigma}\left(\mathbb{E}_t\, r_{t+1}^b - \mathbb{E}_t\, \pi_{t+1}\right) \tag{B.36}$$

But from the Euler equation for the impatient household, along with the "full bailout" assumption embodied in (B.21), we can write:

$$\mathbb{E}_t\, r^b_{t+1} - \mathbb{E}_t\, \pi_{t+1} = \sigma\left[\mathbb{E}_t\, c_{b,t+1} - c_{b,t}\right] = \sigma\left[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\right] \tag{B.37}$$

Combining (B.37) with (B.36) yields:

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1-z}{\sigma}\left(r^s_t - \mathbb{E}_t\, \pi_{t+1}\right) - z\left[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\right] \tag{B.38}$$

Note that an alternative, and arguably more intuitive, way to write the IS expression is based on a simple algebraic manipulation of (B.36):

$$y_t = \mathbb{E}_t\, y_{t+1} - \frac{1}{\sigma}\left(r^s_t - \mathbb{E}_t\, \pi_{t+1}\right) - \frac{z}{\sigma}\left(\mathbb{E}_t\, r^b_{t+1} - r^s_t\right) \tag{B.39}$$

(B.39) is the familiar IS/Euler equation, written in terms of output rather than the output gap, appended with a term equal to the long-short interest rate spread, i.e. $\mathbb{E}_t\, r^b_{t+1} - r^s_t$.

The natural rate of interest, $r^*_t$, is defined as the real rate consistent with the IS equation holding at the natural rate of output absent credit shocks. This implies that:

$$r^*_t = \frac{\sigma}{1-z}\left(\mathbb{E}_t\, y^*_{t+1} - y^*_t\right) \tag{B.40}$$

Adding and subtracting $y^*_t$ and $\mathbb{E}_t\, y^*_{t+1}$ from both sides of (B.38) and re-arranging yields (2.1). Making use of (B.23), allows one to write an AR(1) process for $r^*_t$ as in (2.36), where $\rho_f = \rho_A$ and $s_f = \frac{\sigma(\rho_A-1)(1+\chi)}{\chi(1-z)+\sigma}$.

$$r^*_t = \frac{\sigma(\rho_A - 1)}{1-z} y^*_t \tag{B.41}$$

Computing the dynamics of $x_t$, $\pi_t$, and $r^s_t$ does not require keeping track of $y_t$, $y^*_t$,$q_t$, $r^b_t$, $\omega_t$, $\widehat{s}_t$, $\widehat{b}^{FI}_t$, $\widehat{b}_t$, $\widehat{b}_{cb,t}$ or $c_{b,t}$. Given the solution for $x_t$, $\pi_t$, and $r^s_t$, the dynamics of these variables can be computed using the full system, (B.1)-(B.27).

# C Determinacy of Interest Rate Rules

The linearized four equation model as laid out in Section 2 – as captured by the IS equation, (2.1), the Phillips curve, (2.2), and rules for the policy rate, (2.34), and the central bank's long bond portfolio, (2.35) – has exactly the same requirements for equilibrium determinacy as the textbook three equation model when the central bank's long bond portfolio is exogenous.

For the purposes of examining equilibrium determinacy, one can treat purely exogenous variable as constant – e.g. set $r^*_t = \theta_t = 0$, and set $qe_t = 0$ since we are thinking of it as exogenous for now. Define a vector of variables, $\mathbf{z}_t = \left[\pi_t \;\, x_t \;\, r^s_{t-1}\right]'$. This vector of variables evolves according to:

$$\mathbb{E}_t\, \mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t, \tag{C.1}$$

where:

$$\mathbf{A} = \begin{pmatrix} 1 + \frac{1-z}{\sigma}\left(\phi_x(1-\rho_r) + \frac{\gamma\zeta}{\beta}\right) & \frac{1-z}{\sigma}\left(\phi_\pi(1-\rho_r) - \beta^{-1}\right) & \frac{(1-z)\rho_r}{\sigma} \\ -\frac{\gamma\zeta}{\beta} & \beta^{-1} & 0 \\ \phi_x(1-\rho_r) & \phi_\pi(1-\rho_r) & \rho_r \end{pmatrix}. \tag{C.2}$$

The proof of determinacy follows Woodford (2003), Appendix C. The vector $\mathbf{z}_t$ includes two forward-looking variables, $\pi_t$ and $x_t$, and one state variable, $r^s_{t-1}$. For a unique rational expectations equilibrium, there must be exactly two unstable eigenvalues of $\mathbf{A}$. The characteristic roots of $\mathbf{A}$ satisfy:

$$\mathcal{P}(\mu) = \mu^3 + A_2\mu^2 + A_1\mu + A_0 = 0, \tag{C.3}$$

where:

$$A_0 = -\det(\mathbf{A}), \tag{C.4}$$

$$A_1 = \frac{1}{2}\left(\mathrm{tr}(\mathbf{A}^2) - \mathrm{tr}(\mathbf{A})^2\right), \tag{C.5}$$

$$A_2 = -\mathrm{tr}(\mathbf{A}). \tag{C.6}$$

For exactly two unstable eigenvalues, the following conditions must be satisfied:

$$1 + A_2 + A_1 + A_0 > 0, \tag{C.7}$$

$$-1 + A_2 - A_1 + A_0 < 0, \tag{C.8}$$

$$A_0^2 - A_0A_2 + A_1 - 1 > 0. \tag{C.9}$$

(C.8) is automatically satisfied given assumptions on signs of the parameters. A necessary and sufficient condition for (C.7) and (C.9) being satisfied is that:

$$\phi_\pi(1-\rho_r) + \frac{1-\beta}{\gamma\zeta}\phi_x(1-\rho_r) > 1 - \rho_r, \tag{C.10}$$

or, more compactly:

$$\phi_\pi + \frac{1-\beta}{\gamma\zeta}\phi_x > 1. \tag{C.11}$$

This is the same condition required for equilibrium determinacy in the three equation New Keynesian model, subject to the caveat that the slope coefficient in the Phillips curve, $\gamma\zeta$, is slightly different in the four equation model. (C.11) requires that the central bank react sufficiently aggressively to endogenous variables inflation and the output gap. Intuitively, so long as the central bank's long bond portfolio is exogenous, in terms of endogenous variables the four equation model takes exactly the same form as the three equation model, only with slightly different coefficients on the output gap in the Phillips curve and on the real interest rate in the IS equation. Hence, it should not be surprising that the condition on the parameters of the interest rate rule for determinacy is the same as in the three equation model.

When we consider an endogenous QE rule, as in Section 4, the system of variables expands to four: $\mathbf{z}_t = \begin{bmatrix}\pi_t & x_t & r^s_{t-1} & qe_{t-1}\end{bmatrix}'$. The system of endogenous variables may be written:

$$\mathbf{B}_1 \, \mathbb{E}_t \, \mathbf{z}_{t+1} = \mathbf{B}_2 \mathbf{z}_t, \tag{C.12}$$

where:

$$\mathbf{B}_2 = \begin{pmatrix} \beta & 0 & 0 & -\frac{z\gamma\sigma}{1-z}\bar{b}^{cb} \\ \frac{1-z}{\sigma} - z\bar{b}^{cb}(1-\rho_q)\lambda_\pi & 1 - z\bar{b}^{cb}(1-\rho_q)\lambda_x & -\frac{1-z}{\sigma} & z\bar{b}^{cb}(1-\rho_q) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{C.13}$$

$$\mathbf{B}_1 = \begin{pmatrix} 1 & -\gamma\zeta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (1-\rho_r)\phi_\pi & (1-\rho_r)\phi_x & \rho_r & 0 \\ (1-\rho_q)\lambda_\pi & (1-\rho_q)\lambda_x & 0 & \rho_q \end{pmatrix}. \tag{C.14}$$

We then have:

$$\mathbb{E}_t \, \mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t, \tag{C.15}$$

where:

$$\mathbf{A} = \mathbf{B}_1^{-1}\mathbf{B}_2. \tag{C.16}$$

As discussed in Section 4, we numerically characterize restrictions on policy rule reaction coefficients for equilibrium determinacy (i.e. for there to be exactly two unstable eigenvalues in $\mathbf{A}$).

# D Potential, Flexible Price, and Efficient Output

As noted above, we define potential output, $y_t^*$, as the level of output consistent with flexible prices and no credit market shocks. It is given by (B.34). One could alternatively define flexible price output, $y_t^f$, as the level of output consistent with flexible prices only. This would satisfy:

$$y_t^f = \frac{(1-z)(1+\chi)}{\chi(1-z)+\sigma} a_t + \frac{\sigma z}{\chi(1-z)+\sigma}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] \tag{D.1}$$

The difference between (D.1) and (B.34) is that credit market disturbances – both $\theta_t$ as well as $qe_t$ – impact flexible price output but not potential output. If we were to instead define the output gap as $x_t^f = y_t - y_t^f$, the Phillips curve representation would be:

$$\pi_t = \gamma\zeta x_t^f + \beta \, \mathbb{E}_t \, \pi_{t+1} \tag{D.2}$$

In other words, written in terms of flexible price output, the Phillips curve, (D.2), features no endogenous cost-push wedge related to credit market disturbances. The cost-push wedge comes from defining potential output as not featuring credit market disturbances in addition to price flexibility. Credit market shocks drive a wedge between $y_t^*$ and $y_t^f$, in a similar way to how markup shocks are often used to justify a wedge between flexible price and efficient output – and a corresponding cost-push wedge in the Phillips curve – in the standard three equation model.

To be clear, (D.2) and (2.2) are equivalent ways to describe the supply-side of the model. If one prefers to work with (D.2), one simply needs to write the IS equation and exogenous processes in terms of $x_t^f$, $y_t^f$ and $r_t^f$ instead of $x_t$, $y_t^*$, and $r_t^*$. We prefer working with the specification in the text, using our definition of potential output, because it aligns closer to the notion of efficient output, which is ultimately what is relevant for welfare. Our model features two time-varying welfare-relevant distortions, not one as in the standard New Keynesian model.

We can motivate our notion of potential output, $y_t^*$, as coinciding with efficient output under a particular welfare criterion. Suppose that the planner's objective is to maximize the utility of the parent, (2.3), subject to perfectly smoothing the child's consumption at some constant level, $\bar{C}_b$. Absent credit market disturbances, even with price stickiness the full bailout assumption would result in perfectly smooth consumption for the child. Formally, the planner's problem would be:

$$\max_{C_t, L_t} \ \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left[ \frac{C_{t+j}^{1-\sigma} - 1}{1-\sigma} - \psi \frac{L_{t+j}^{1+\chi}}{1+\chi} \right] \tag{D.3}$$

s.t.

$$C_t \leq A_t L_t - \bar{C}_b \tag{D.4}$$

With no endogenous states, the planner's problem is effectively static. The first order condition is:

$$\psi L_t^{\chi} = C_t^{-\sigma} A_t \tag{D.5}$$

Defining $Y_t^e = C_t + \bar{C}_b = A_t L_t$, and linearizing about the steady state in which $\bar{C}_b / Y^e = z$, one obtains:

$$y_t^e = \frac{(1+\chi)(1-z)}{\chi(1-z)+\sigma} a_t \tag{D.6}$$

The linearized expression for $y_t^e$ in (D.6) is identical to that for $y_t^*$, (B.34). In the non-linearized model, $Y_t^*$ would differ from $Y_t^e$ by a constant owing to the markup of price over marginal cost, though this constant drops out in the linearization. This steady state distortion could be eliminated by appealing to a constant Pigouvian subsidy to labor, as is commonplace in the New Keynesian literature on optimal monetary policy.

# E Model Calibration

The parameters of the model are calibrated as follows. The unit of time is a quarter. We assume a zero trend inflation rate, so $\Pi = 1$. This implies that steady state price dispersion is $v^p = 1$ and the steady state relative reset price is $p_* = 1$. We set $\epsilon = 11$, which implies a steady state price markup of ten percent. The discount factor of the parent is set to $\beta = 0.995$, which together with $\Pi = 1$ implies a steady state short term rate of 200 basis points at an annualized frequency (i.e. $R^s = 1.005$). We then target a steady state spread of the return on the long bond over the short term bond of 200 basis points at an annualized frequency, which implies $\beta_b = 0.99$ and $R^b = 1.01$. We set $\kappa = 1 - 40^{-1}$, implying a ten year duration of the long bond. Together with $R^b$, this implies a steady state value of $Q$.

The coefficient of relative risk aversion, $\sigma$, and the inverse Frisch elasticity, $\chi$, are both set to 1. We target a steady state share of child consumption, $z = C_b/Y$, of one-third. We then pick $\psi$ to normalize steady state labor input to unity. Together, these parameters imply a value of the steady state transfer from parent to child, $X^b$. We assume that the Calvo parameter is $\phi = 0.75$, implying a mean duration between price changes of one year. We assume that the size of the central bank's balance sheet is 10 percent of steady state output, i.e. $QE = 0.1 \times Y$. We pick a steady state target of the risk-weighted leverage ratio of $\Theta = 5$. This then implies a value of the steady state equity transfer from the parent to the FI, $X^{FI}$.

For the exercises in Subsection 2.3, we assume that the Taylor rule parameters are $\rho_r = 0.8$, $\phi_\pi = 1.5$, and $\phi_x = 0$. The autoregressive parameter of the QE process, $\rho_q$, is also set to 0.8. The autoregressive parameters for productivity and the credit shock are also both set to 0.8. This implies, as shown below in Appendix B, that the AR parameter in the natural rate process is also 0.8.

Table E.1: Parameter Values of Full Model

| Parameter | Value | Description (Target) |
|---|---|---|
| $\beta$ | 0.995 | Discount factor, parent |
| $\sigma$ | 1 | Inverse elasticity of substitution |
| $\chi$ | 1 | Inverse Frisch elasticity |
| $\psi$ | 1.36 | Labor disutility scaling parameter (target $L = 1$) |
| $\beta_b$ | 0.99 | Discount factor, child (target spread of 200 b.p. annualized) |
| $\Pi$ | 1 | Steady state trend inflation |
| $\epsilon$ | 11 | Elasticity of substitution (target markup ten percent) |
| $\kappa$ | $1 - 40^{-1}$ | Coupon decay (target duration ten years) |
| $\phi$ | 0.75 | Calvo price |
| $\Theta$ | 5 | Steady state risk-weighted leverage |
| $QE$ | 0.1 | Steady state central bank bond portfolio |
| $z$ | 0.33 | Steady state child share of consumption |
| $X^b$ | 0.33 | Steady state parent-child transfer |
| $X^{FI}$ | 0.046 | Steady state parent-FI equity transfer |
| $\rho_r$ | 0.8 | Taylor rule smoothing |
| $\phi_\pi$ | 1.5 | Taylor rule inflation |
| $\phi_x$ | 0 | Taylor rule gap |
| $\rho_A$ | 0.8 | AR productivity |
| $\rho_\theta$ | 0.8 | AR credit |
| $\rho_q$ | 0.8 | AR QE |

Note: this table lists the values of calibrated parameters for the exercises in Subsection 2.3.

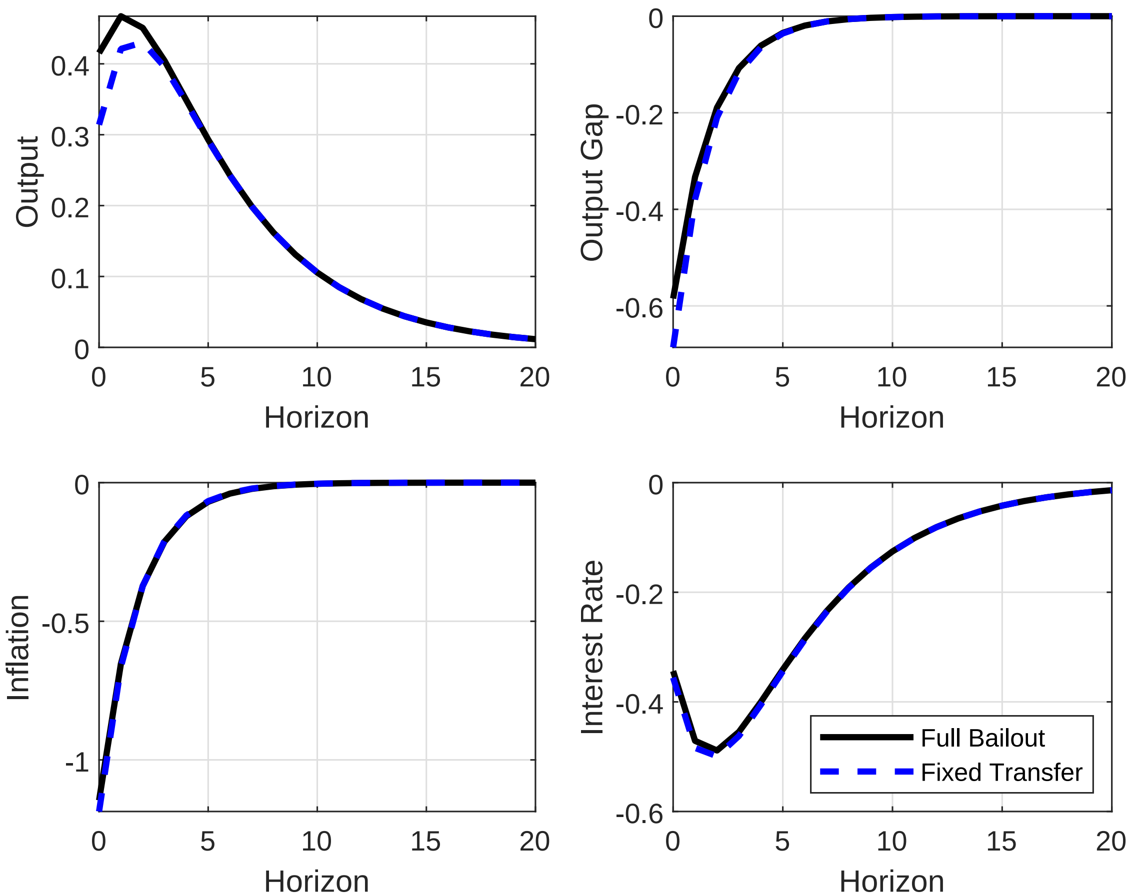
# F Robustness

## F.1 Full Bailout

In the baseline four equation model, we make a "full bailout" assumption that there is a complete payoff from the parent household, each period, of the outstanding debt of children (see (2.32)). This significantly simplifies the model in that it makes the child's consumption equal to the value of bonds, (2.33).

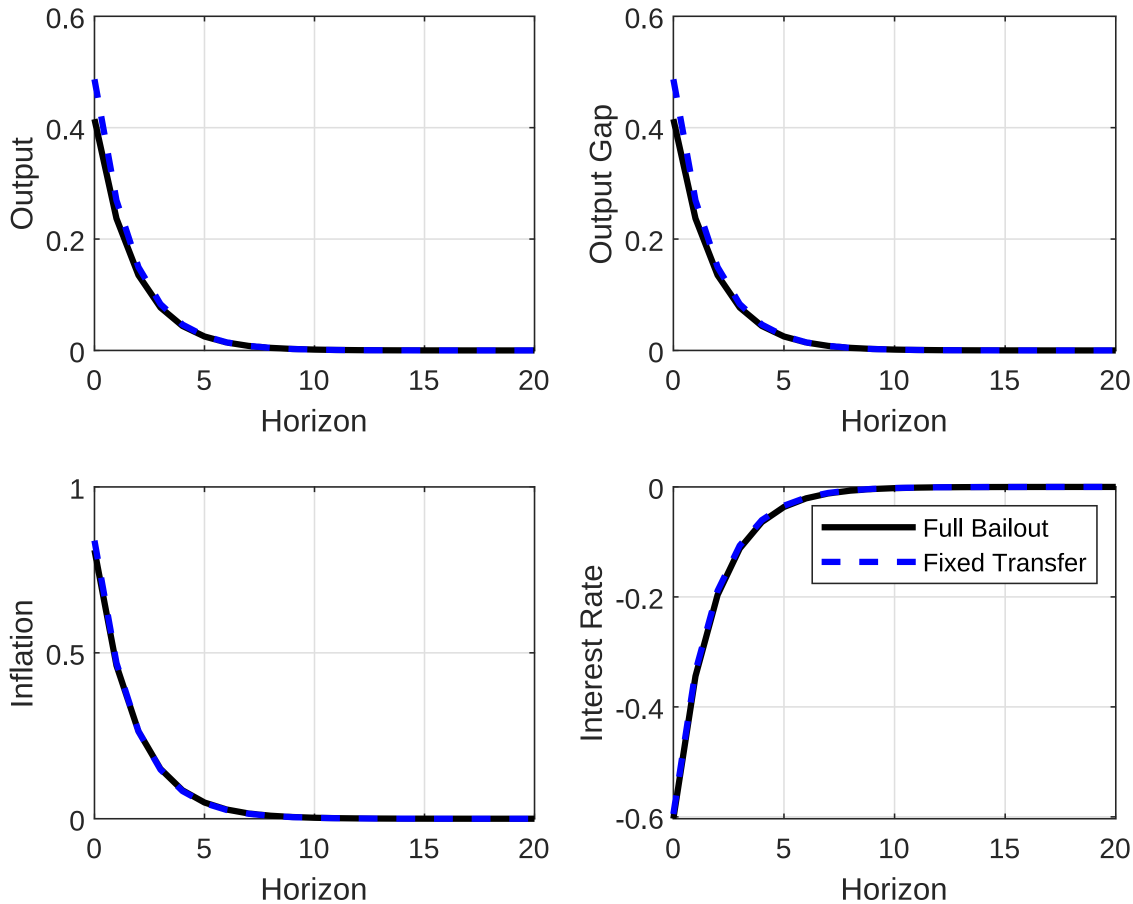
As noted in the text, this assumption on the transfer is not critical for the quantitative characteristics of the model. To see this more cleanly, we assume that the transfer from parent to child is fixed each period, equal to the steady state version of (2.32). This leaves the steady state of the modified model identical to our baseline model, but does affect some dynamics. Figure F.1 shows impulse responses to a natural rate shock, Figure F.2 to a conventional monetary policy shock, and Figure F.3 to a credit/QE shock with and without the bull bailout assumption. The solid lines are responses with the full bailout and are identical to the responses in the baseline model shown in the text. Dashed blue lines are responses in the four equation model with a fixed bailout.

Figure F.1: **IRFs to Shock to Potential Output, Fixed vs. Full Bailout**




*Notes:* Black solid lines: IRFs to a one percentage point shock to potential output in the baseline four equation model. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the four equation model where there is no full bailout from parent to child.

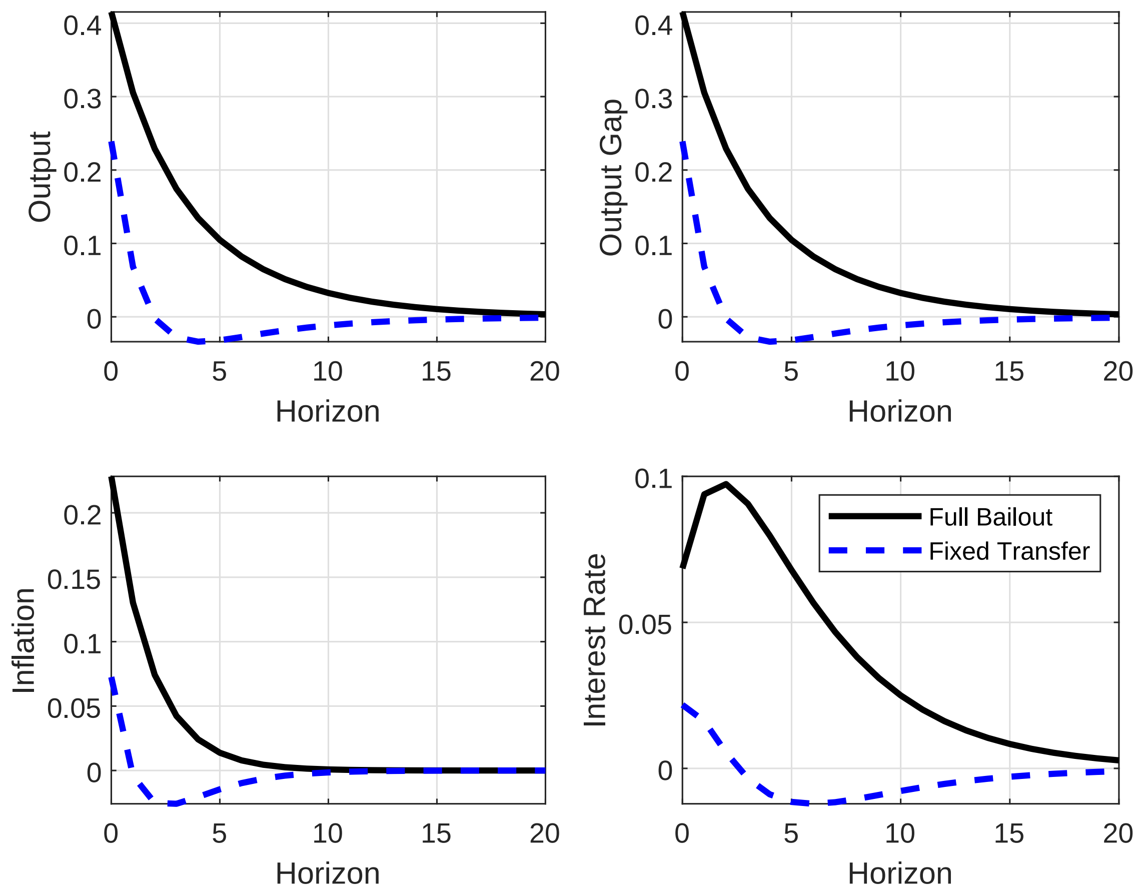
Figure F.2: **IRFs to Policy Shock, Fixed vs. Full Bailout**




*Notes:* Black solid lines: IRFs to a conventional monetary policy shock in the baseline four equation model. The size and sign of the shock are chosen to generate the same impact response of output as in Figure F.1. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized policy shock in the four equation model where there is no full bailout from parent to child.

In the case of natural rate and monetary policy shocks, the differences for the responses of aggregate output, inflation, and the interest rate with or without the full bailout assumption are inconsequential. In comparison to our baseline model, output reacts slightly less to a natural rate shock on impact without the full bailout assumption; it reacts slightly more to a monetary policy shock without the full bailout assumption.

Figure F.3: **IRFs to Credit/QE Shock, Fixed vs. Full Bailout**




*Notes:* Black solid lines: IRFs to a credit ($\theta_t$) or QE ($qe_t$) shock in the baseline four equation model. The size and sign of the shocks are chosen to generate the same impact response of output as in Figure F.1. Because the QE and credit shock only differ according to scale in the linearized model (i.e. $\bar{b}^{FI} \neq \bar{b}^{cb}$) and the AR parameters are the same, the normalized impulse responses are identical. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized credit/QE shock in the four equation model where there is no full bailout from parent to child.

Quantitatively, differences are somewhat more noticeable when it comes to a credit or QE shock. Compared to the baseline model with the full bailout assumption, output and inflation react less on impact to a credit/QE shock when there is a fixed transfer. The responses are also less persistent. But qualitatively, they are in-line with our baseline model.

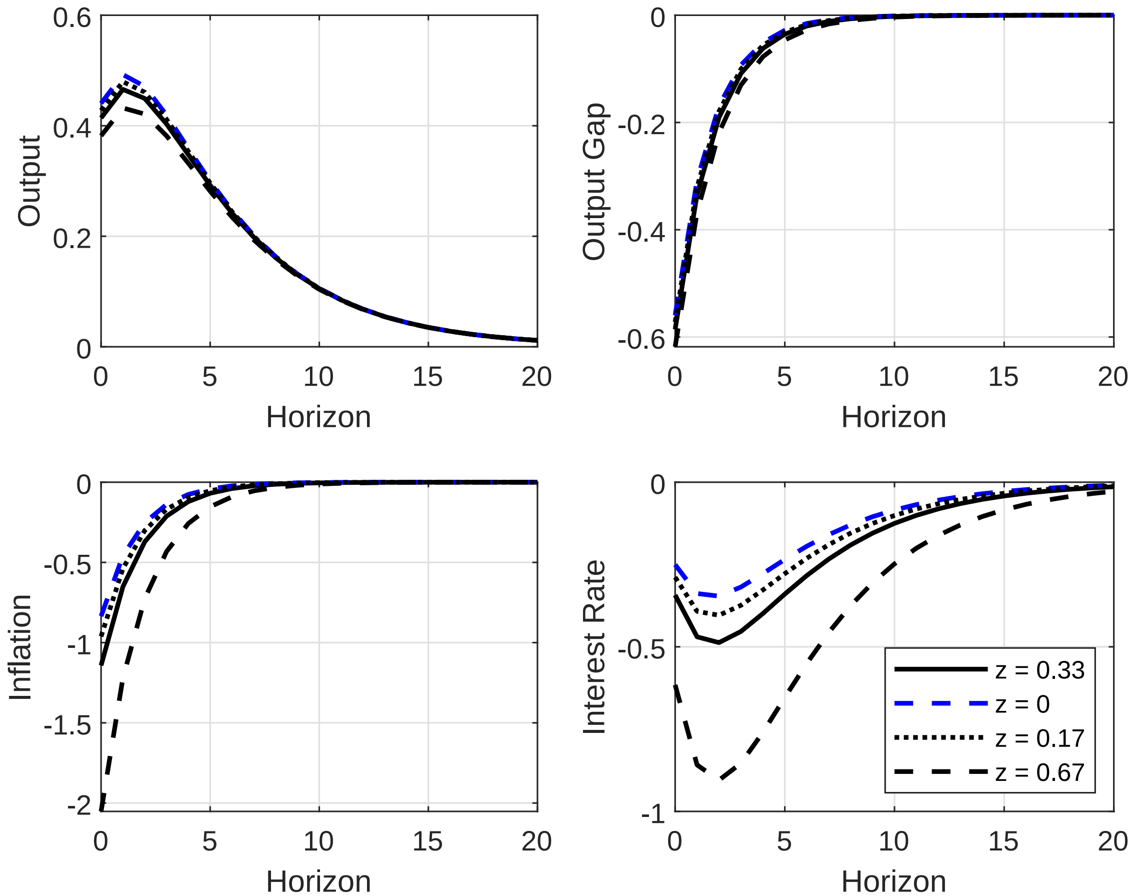
## F.2 Parameterization of $z$

Our linearized four equation model differs from the standard NK model via the parameter $z$; when $z = 0$, the model is equivalent to the standard three equation model. In this subsection, we show impulse responses to natural rate, monetary policy, and credit market shocks for different values of $z$. Responses with our baseline value of $z = 0.33$ are depicted via solid black lines; we also show responses when $z = 0.167$ (dotted black lines) and when $z = 0.67$

(dashed black lines). For point of comparison, the blue dashed lines shows responses in the three equation model ($z = 0$).

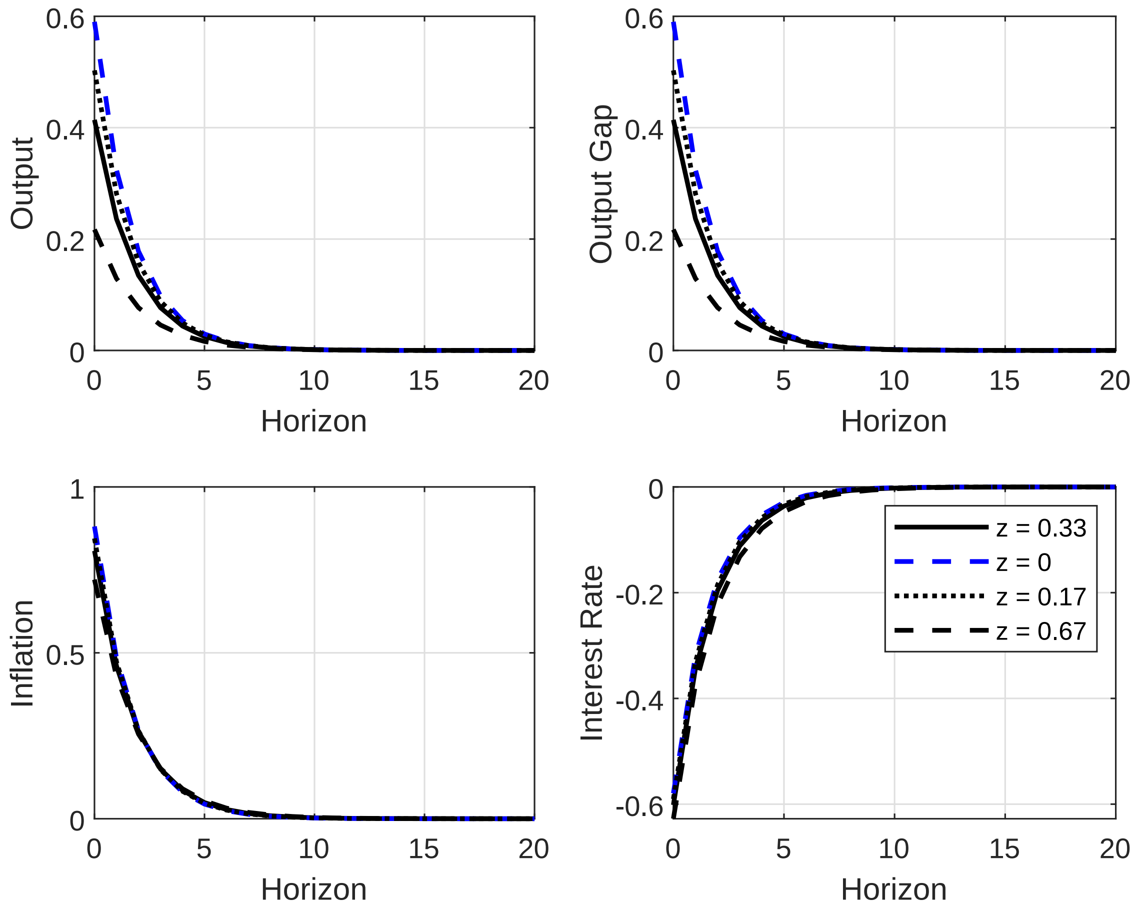
In response to all three shocks, the bigger $z$ is, the more the responses differ from the conventional three equation model. Relative to our baseline case ($z = 0.33$), the impulse responses with $z = 0.167$ or $z = 0.667$ are not very wildly different.

Figure F.4: **IRFs to Shock to Potential Output, Different Values of** $z$



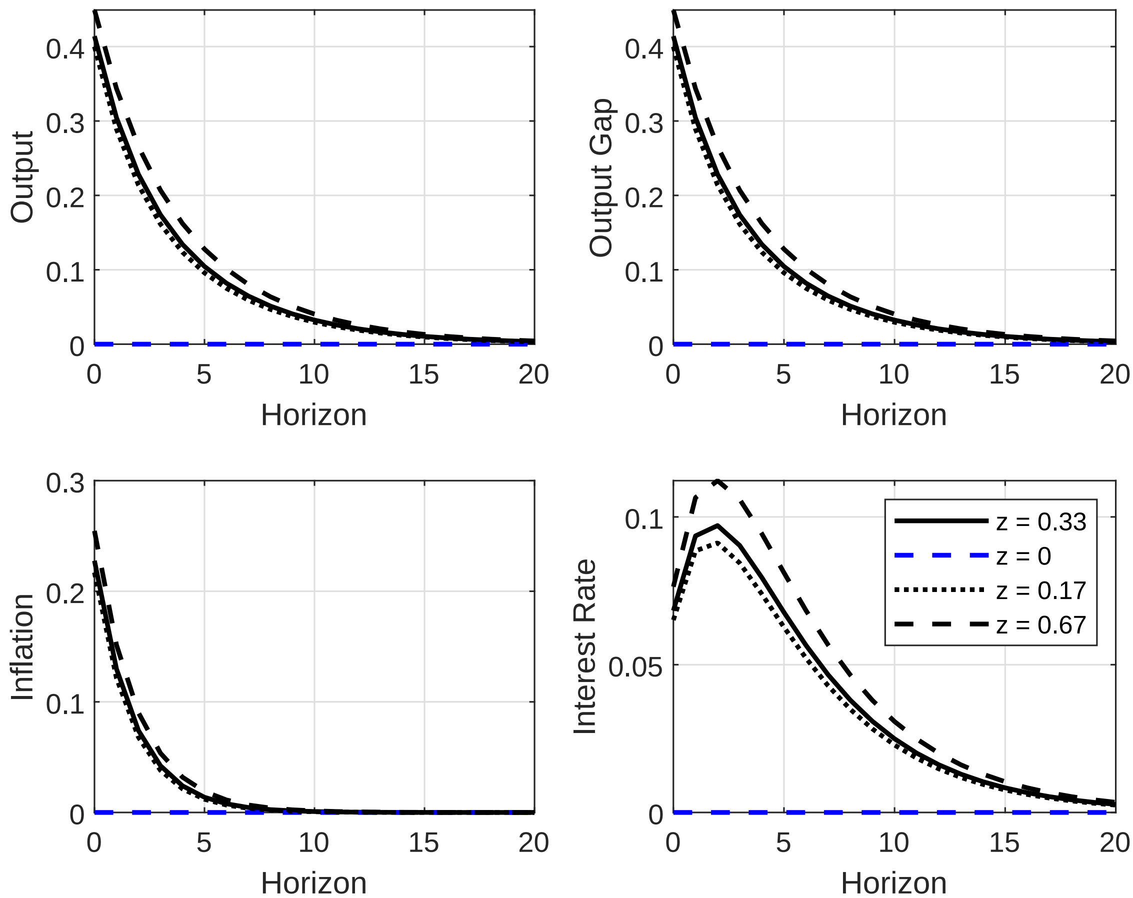

*Notes:* Black solid lines: IRFs to a one percentage point shock to potential output in the baseline four equation model with $z = 0.33$. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the three equation model (equivalent to the four equation model with $z = 0$). Black dotted lines: responses in the four equation model with $z = 0.167$. Black dashed lines: responses in the four equation model with $z = 0.667$.

Figure F.5: **IRFs to Shock to Policy Shock, Different Values of $z$**




*Notes:* Black solid lines: IRFs to a conventional monetary policy shock in the baseline four equation model with $z = 0.33$. The size and sign of the shocks are chosen to generate the same impact response of output as in Figure F.4 when $z = 0.33$. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the three equation model (equivalent to the four equation model with $z = 0$). Black dotted lines: responses in the four equation model with $z = 0.167$. Black dashed lines: responses in the four equation model with $z = 0.667$.

Figure F.6: **IRFs to Shock to Credit/QE Shock, Different Values of $z$**




*Notes:* Black solid lines: IRFs to a credit ($\theta_t$) or QE ($qe_t$) shock in the baseline four equation model with $z = 0.33$. The size and sign of the shocks are chosen to generate the same impact response of output as in Figure F.4 when $z = 0.33$. Because the QE and credit shock only differ according to scale in the linearized model (i.e. $\bar{b}^{FI} \neq \bar{b}^{cb}$) and the AR parameters are the same, the normalized impulse responses are identical. Output and the output gap are expressed in percentage points, while the responses of inflation and the short term interest rate are expressed in annualized percentage points. Blue dashed lines: IRFs to the same-sized natural rate shock in the four equation model where there is no full bailout from parent to child.

# G Optimal Monetary Policy

We consider optimal monetary policy under discretion. The central bank's objective function is to pick instruments, $r_t^s$ and $qe_t$, and endogenous variables, $\pi_t$ and $x_t$, to minimize (3.1) subject to the IS equation, (2.1), the Phillips Curve, (2.2), a ZLB constraint on the policy rate, and a potential constraint on the use of balance sheet policies. Formally:

$$\min_{r_t^s, qe_t, \pi_t, x_t} \quad \mu x_t^2 + \pi_t^2$$

s.t.

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1} - r_t^*\right) - z\Big[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\Big], \quad \text{(G.1)}$$

$$\pi_t = \gamma\zeta x_t - \frac{z\gamma\sigma}{1-z}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] + \beta\, \mathbb{E}_t\, \pi_{t+1}, \tag{G.2}$$

$$r_t^s = 0, \tag{G.3}$$

$$qe_t = 0. \tag{G.4}$$

(G.3) only holds when the policy rate is constrained by the ZLB (i.e. the policy rate is fixed), as in Subsection 3.2, while (G.4) only binds when QE is unavailable, as in Subsection 3.3. In solving the problem under discretion, the policymaker treats expectations of all future variables as given.

A Lagrangian is:

$$\mathbb{L} = \mu x_t^2 + \pi_t^2 + \lambda_{1,t}\left(x_t - \mathbb{E}_t\, x_{t+1} + \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1} - r_t^*\right) + z\left[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\right]\right)$$
$$+ \lambda_{2,t}\left(\pi_t - \gamma\zeta x_t + \frac{z\gamma\sigma}{1-z}\left[\bar{b}^{FI}\theta_t + \bar{b}^{cb} qe_t\right] - \beta\, \mathbb{E}_t\, \pi_{t+1}\right) - \lambda_{3,t} r_t^s - \lambda_{4,t} qe_t.$$

The first order conditions are:

$$\frac{\partial \mathbb{L}}{\partial r_t^s} = \lambda_{1,t}\frac{1-z}{\sigma} - \lambda_{3,t} = 0, \tag{G.5}$$

$$\frac{\partial \mathbb{L}}{\partial qe_t} = -\lambda_{1,t} z \bar{b}^{cb} + \lambda_{2,t}\frac{z\gamma\sigma\bar{b}^{cb}}{1-z} - \lambda_{4,t} = 0, \tag{G.6}$$

$$\frac{\partial \mathbb{L}}{\partial \pi_t} = 2\pi_t + \lambda_{2,t} = 0, \tag{G.7}$$

$$\frac{\partial \mathbb{L}}{\partial x_t} = 2\mu x_t + \lambda_{1,t} - \gamma\zeta\lambda_{2,t} = 0. \tag{G.8}$$

## G.1 Both Instruments Available

When both instruments are available, we have $\lambda_{3,t} = \lambda_{4,t} = 0$. From (G.5), this means that $\lambda_{1,t} = 0$. But then from (G.6), $\lambda_{1,t} = \lambda_{4,t} = 0$ implies that $\lambda_{2,t} = 0$. But then (G.7) then requires that $\pi_t = 0$. Then, from (G.8), we see that $x_t = 0$.

### G.1.1 Proof of Theorem 1

The theorem can be proved by contradiction. Suppose first we can achieve $x_t = \pi_t = 0$ with $qe_t = 0$ for all $t$. Then the Phillips curve may then be written as:

$$0 = -\frac{z\gamma\sigma}{1-z}\bar{b}^{FI}\theta_t, \tag{G.9}$$

which does not hold unless $\theta_t = 0$, which contradicts the assumption. Hence, there is a contradiction.

Second, suppose we can achieve $x_t = \pi_t = \mathbb{E}_t\, x_{t+1} = \mathbb{E}_t\, \pi_{t+1} = 0$ with $r_t^s = 0$ for all $t$.

Then the Phillips curve becomes

$$0 = \frac{z\gamma\sigma}{1-z}\Big[\bar{b}^{FI}\theta_t + \bar{b}^{cb}qe_t\Big]. \tag{G.10}$$

This requires

$$qe_t = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t. \tag{G.11}$$

Note that (G.11) is identical to the equilibrium path for $qe_t$ given in Proposition 1. With the policy rate fixed and the equilibrium path of QE given in this way, the IS curve becomes

$$0 = \frac{1-z}{\sigma}r_t^* - z\Big[\bar{b}^{FI}\left(\mathbb{E}_t\,\theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\,qe_{t+1} - qe_t\right)\Big]. \tag{G.12}$$

Further apply (G.11) on both $qe_t$ and $\mathbb{E}_t\,qe_{t+1}$. This implies

$$0 = \frac{1-z}{\sigma}r_t^*, \tag{G.13}$$

which does not hold unless $z = 1$ (which we have ruled out) or $r_t^* = 0$ (which contradicts the assumption). Hence, we have another contradiction.■

## G.2 The ZLB

### G.2.1 First Order Condition

Next, suppose that the ZLB binds, so $\lambda_{3,t} \geq 0$, but QE is available, so $\lambda_{4,t} = 0$. We will have $\lambda_{1,t} > 0$. We can solve for $\lambda_{1,t}$ in terms of $\lambda_{2,t}$ from (G.6):

$$\lambda_{1,t} = \frac{\gamma\sigma}{1-z}\lambda_{2,t}. \tag{G.14}$$

Then, from (G.8), we have:

$$\frac{2\mu(1-z)}{\gamma\zeta(1-z) - \gamma\sigma}x_t = \lambda_{2,t}. \tag{G.15}$$

Plug (G.15) into (G.7):

$$2\pi_t + \frac{2\mu(1-z)}{\gamma\zeta(1-z) - \gamma\sigma}x_t = 0.$$

Simplifying yields:

$$\pi_t = -\frac{\mu(1-z)}{\gamma\zeta(1-z) - \gamma\sigma}x_t, \tag{G.16}$$

as shown in the main text.

### G.2.2 Proof of Proposition 2

When the ZLB is binding, the IS curve becomes

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-z}{\sigma}\left(-\mathbb{E}_t\, \pi_{t+1} - r_t^*\right) - z\Big[\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right) + \bar{b}^{cb}\left(\mathbb{E}_t\, qe_{t+1} - qe_t\right)\Big], \quad \text{(G.17)}$$

and the Phillips curve remains as (2.2). (3.4) holds and describes balance sheet policy, and the policy rate is fixed, i.e. $r_t^s = 0$.

Guess

$$\pi_t = \omega_1 r_t^* + \omega_3 \theta_t,$$

then (3.4) implies

$$x_t = -\frac{\gamma\zeta(1-z) - \gamma\sigma}{\mu(1-z)}(\omega_1 r_t^* + \omega_3\theta_t) \equiv \omega_2(\omega_1 r_t^* + \omega_3\theta_t).$$

If in period $t+1$, the economy exits the ZLB, then both instruments are available, which implies $\pi_{t+1} = x_{t+1} = 0$. If in period $t+1$, the economy is still at the ZLB, $\pi_{t+1} = \omega_1 r_{t+1}^* + \omega_3\theta_{t+1}, x_{t+1} = \omega_2(\omega_1 r_{t+1}^* + \omega_3\theta_{t+1})$. As a result,

$$\begin{aligned}
\mathbb{E}_t\pi_{t+1} &= \alpha(\rho_f\omega_1 r_t^* + \rho_\theta\omega_3\theta_t), \\
\mathbb{E}_t x_{t+1} &= \alpha\omega_2(\rho_f\omega_1 r_t^* + \rho_\theta\omega_3\theta_t).
\end{aligned}$$

We also guess the equilibrium path of QE is:

$$qe_t = \tau r_t^* + \omega_4\theta_t.$$

If in period $t+1$, the economy exits the ZLB, then everything goes back to the unconstrained case: $qe_{t+1} = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_{t+1}$. Otherwise, $qe_{t+1} = \tau r_{t+1}^* + \omega_4\theta_{t+1}$. Then,

$$\mathbb{E}_t qe_{t+1} = \alpha\rho_f\tau r_t^* + \alpha\rho_\theta\omega_4\theta_t - (1-\alpha)\rho_\theta\frac{\bar{b}^{FI}}{\bar{b}^{cb}}\theta_t.$$

Plugging the guesses into (2.2) and (G.17), we have:

$$\omega_1 r_t^* + \omega_3\theta_t = \gamma\zeta\omega_2(\omega_1 r_t^* + \omega_3\theta_t) - \frac{z\gamma\sigma}{1-z}\Big[\left(\bar{b}^{FI} + \bar{b}^{cb}\omega_4\right)\theta_t + \bar{b}^{cb}\tau r_t^*\Big] + \beta\alpha(\rho_f\omega_1 r_t^* + \rho_\theta\omega_3\theta_t), \quad \text{(G.18)}$$

$$\begin{aligned}
\omega_2(\omega_1 r_t^* + \omega_3\theta_t) &= \alpha\omega_2(\rho_f\omega_1 r_t^* + \rho_\theta\omega_3\theta_t) - \frac{1-z}{\sigma}\left[-\alpha(\rho_f\omega_1 r_t^* + \rho_\theta\omega_3\theta_t) - r_t^*\right] \\
&\quad - z\bar{b}^{FI}\left(\rho_\theta - 1\right)\theta_t - z\bar{b}^{cb}\Big[\alpha\rho_\theta\omega_4 - (1-\alpha)\rho_\theta\frac{\bar{b}^{FI}}{\bar{b}^{cb}} - \omega_4\Big]\theta_t \\
&\quad - z\bar{b}^{cb}(\alpha\rho_f - 1)\tau r_t^*. \qquad \text{(G.19)}
\end{aligned}$$

Matching coefficients, we have the following system of equations:

$$\omega_1 = \gamma\zeta\omega_2\omega_1 - \frac{z\gamma\sigma}{1-z}\bar{b}^{cb}\tau + \beta\alpha\rho_f\omega_1 \tag{G.20}$$

$$\omega_2\omega_1 = \alpha\omega_2\rho_f\omega_1 + \frac{1-z}{\sigma}(\alpha\rho_f\omega_1 + 1) - z\bar{b}^{cb}(\alpha\rho_f - 1)\tau \tag{G.21}$$

$$\omega_3 = \gamma\zeta\omega_2\omega_3 - \frac{z\gamma\sigma}{1-z}\left(\bar{b}^{FI} + \bar{b}^{cb}\omega_4\right) + \beta\alpha\rho_\theta\omega_3 \tag{G.22}$$

$$\omega_2\omega_3 = \alpha\omega_2\rho_\theta\omega_3 + \frac{1-z}{\sigma}\alpha\rho_\theta\omega_3 - z\bar{b}^{FI}(\rho_\theta - 1) - z\bar{b}^{cb}\Big[\alpha\rho_\theta\omega_4 - (1-\alpha)\rho_\theta\frac{\bar{b}^{FI}}{\bar{b}^{cb}} - \omega_4\Big] \tag{G.23}$$

Solve $\tau$ as a function of $\omega_1$ in the first two equations

$$\tau = \frac{(1-z)(\gamma\zeta\omega_2\omega_1 + \beta\alpha\rho_f\omega_1 - \omega_1)}{\bar{b}^{cb}z\gamma\sigma}$$

$$\tau = \frac{\alpha\omega_2\rho_f\omega_1 + \frac{1-z}{\sigma}(\alpha\rho_f\omega_1 + 1) - \omega_2\omega_1}{z\bar{b}^{cb}(\alpha\rho_f - 1)}$$

The first equation is the same as (3.10). Then,

$$\frac{\alpha\omega_2\rho_f\omega_1 + \frac{1-z}{\sigma}(\alpha\rho_f\omega_1 + 1) - \omega_2\omega_1}{z\bar{b}^{cb}(\alpha\rho_f - 1)} = \frac{(1-z)(\gamma\zeta\omega_2\omega_1 + \beta\alpha\rho_f\omega_1 - \omega_1)}{\bar{b}^{cb}z\gamma\sigma}$$

Simplify, we get

$$\omega_1 = \frac{\gamma(1-z)}{(\alpha\rho_f - 1)(1-z)(\gamma\zeta\omega_2 + \beta\alpha\rho_f - 1) - \gamma\sigma\alpha\omega_2\rho_f - \gamma(1-z)\alpha\rho_f + \gamma\sigma\omega_2}$$

This is the same as (3.8).

Next, simplify (G.23)

$$\omega_2\omega_3 = \alpha\omega_2\rho_\theta\omega_3 + \frac{1-z}{\sigma}\alpha\rho_\theta\omega_3 + z(1-\alpha\rho_\theta)(\bar{b}^{FI} + \bar{b}^{cb}\omega_4)$$

Therefore,

$$\bar{b}^{FI} + \bar{b}^{cb}\omega_4 = \frac{\omega_2 - \alpha\omega_2\rho_\theta - \frac{1-z}{\sigma}\alpha\rho_\theta}{z(1-\alpha\rho_\theta)}\omega_3 \tag{G.24}$$

Similarly, rearrange (G.22)

$$\bar{b}^{FI} + \bar{b}^{cb}\omega_4 = (\gamma\zeta\omega_2 + \beta\alpha\rho_\theta - 1)\frac{1-z}{z\gamma\sigma}\omega_3 \tag{G.25}$$

Combining (G.24)-(G.25),

$$\left(\frac{\omega_2 - \alpha\omega_2\rho_\theta - \frac{1-z}{\sigma}\alpha\rho_\theta}{z(1-\alpha\rho_\theta)} - (\gamma\zeta\omega_2 + \beta\alpha\rho_\theta - 1)\frac{1-z}{z\gamma\sigma}\right)\omega_3 = 0$$

This has to hold for any parameter specification. Therefore,

$$\omega_3 = 0.$$

Plug back to (G.24), we get

$$\omega_4 = -\frac{\bar{b}^{FI}}{\bar{b}^{cb}}.$$

■

## G.3 No QE Available

### G.3.1 First Order Condition

When QE is unavailable, we have $\lambda_{4,t} > 0$, but $\lambda_{3,t} = 0$. The later implies from (G.5) that $\lambda_{1,t} = 0$. From (G.7), we have:

$$\lambda_{2,t} = -2\pi_t. \tag{G.26}$$

Plugging into (G.8), we have:

$$2\mu x_t + 2\gamma\zeta\pi_t = 0, \tag{G.27}$$

which implies:

$$\pi_t = -\frac{\mu}{\gamma\zeta}x_t, \tag{G.28}$$

as shown in the text.

### G.3.2 Proof of Proposition 3

When the QE policy is not available, (3.11) holds, and $qe_t = 0$. The IS Phillips curves can be written:

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-z}{\sigma}\left(r_t^s - \mathbb{E}_t\, \pi_{t+1} - r_t^*\right) - z\bar{b}^{FI}\left(\mathbb{E}_t\, \theta_{t+1} - \theta_t\right), \tag{G.29}$$

and

$$\pi_t = \gamma\zeta x_t - \frac{z\gamma\sigma}{1-z}\bar{b}^{FI}\theta_t + \beta\,\mathbb{E}_t\, \pi_{t+1}. \tag{G.30}$$

Substitute (3.11) into (G.30)

$$\pi_t = -\frac{\gamma^2\zeta^2}{\mu}\pi_t + \beta\,\mathbb{E}_t\, \pi_{t+1} - \frac{z\gamma\sigma}{1-z}\bar{b}^{FI}\theta_t. \tag{G.31}$$

Solve the above equation forward, we have

$$\pi_t = \varphi\theta_t \tag{G.32}$$

where

$$\varphi = -\frac{\mu}{\gamma^2\zeta^2 + \mu(1-\beta\rho_\theta)} \frac{z\gamma\sigma}{1-z}\bar{b}^{FI}. \tag{G.33}$$

Combining the solution for $\pi_t$ and (3.11), we obtain

$$x_t = -\frac{\gamma\zeta}{\mu}\varphi\theta_t. \tag{G.34}$$

Plug the solutions for $\pi_t$ and $x_t$ into (G.29):

$$r_t^s = r_t^* + \left[\rho_\theta\varphi + \frac{\sigma(1-\rho_\theta)}{1-z}\frac{\gamma\zeta}{\mu}\varphi + \frac{(1-\rho_\theta)\sigma z\bar{b}^{FI}}{1-z}\right]\theta_t. \tag{G.35}$$

As a result,

$$\eta = \rho_\theta\varphi + \frac{\sigma(1-\rho_\theta)}{1-z}\frac{\gamma\zeta}{\mu}\varphi + \frac{(1-\rho_\theta)\sigma z\bar{b}^{FI}}{1-z}. \tag{G.36}$$

■