Graduate Macro Theory II: A Note On Computing Impulse Responses for an AR(p)

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Spring 2012

Suppose we have a generic AR(p) process:

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \cdots + \rho_p x_{t-p} + \varepsilon_t$$

Computing the impulse response to a shock at time t, i.e. $E_t x_{t+j} - E_{t-1} x_{t+j}, \; \forall j \geq 0$, can be something of a hassle if you do this recursively. By recursively I mean calculating the impulse response on impact, and then in the next period given the response on impact, and then at a forecast horizon of two given the previous responses, and so on. As a concrete example, suppose we have an AR(3) with known coefficients:

$$x_t = 0.8 x_{t-1} + 0.6 x_{t-2} - 0.5 x_{t-3} + \varepsilon_t$$

Calculating the impulse response function to a unit shock recursively, I get:

$$\begin{aligned}
\text{IRF}(0) &= 1 \\
\text{IRF}(1) &= 0.8 \times \text{IRF}(0) = 0.8 \\
\text{IRF}(2) &= 0.8 \times \text{IRF}(1) + 0.6 \times \text{IRF}(0) = 1.24 \\
\text{IRF}(3) &= 0.8 \times \text{IRF}(2) + 0.6 \times \text{IRF}(1) - 0.6 \times \text{IRF}(0) = 0.9720 \\
\text{IRF}(4) &= 0.8 \times \text{IRF}(3) + 0.6 \times \text{IRF}(2) - 0.6 \times \text{IRF}(1) = 1.1216 \\
&\vdots \\
\text{IRF}(j) &= 0.8 \times \text{IRF}(j-1) + 0.6 \times \text{IRF}(j-2) - 0.6 \times \text{IRF}(j-3)
\end{aligned}$$

This is straightforward enough to do, but can be a coding hassle. Before you can automate the loop, you have to manually do p periods. For p large, it's kind of a hassle, and it really becomes a hassle if you want to experiment with different lag lengths.

It turns out that there is an easier way to do this, and that's by transforming this scalar AR(p) into a vector AR(1). Computing impulse responses for AR(1) is easy – it's just the coefficient raised to the power of the horizon. For the arbitrary AR(p), define the following vector:

$$z_t \equiv \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p-1} \end{bmatrix}$$

We can write this is a vector AR(1) as follows, which is sometimes called the companion form:

$$\begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p-1} \end{bmatrix} = \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 & \dots & \rho_p \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ \vdots \\ x_{t-p} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

More compactly:

$$z_t = \Lambda z_{t-1} + \varepsilon_t$$

What's nice here is that the impulse response function is quite easy to calculate. For any horizon $j \geq 0$, the IRF to a unit shock at time t is just the (1,1) element Λ^j.

Let's see this concretely for the AR(3) I wrote above. The companion matrix form is:

$$\begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \end{bmatrix} = \begin{pmatrix} 0.8 & 0.6 & -0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ x_{t-3} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \end{bmatrix}$$

We can calculate what the coefficient matrix raised to various powers as follows:

$$\Lambda^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda^1 = \begin{pmatrix} 0.8000 & 0.6000 & -0.5000 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \end{pmatrix}$$

$$\Lambda^2 = \begin{pmatrix} 1.2400 & -0.0200 & -0.4000 \\ 0.8000 & 0.6000 & -0.5000 \\ 1.0000 & 0 & 0 \end{pmatrix}$$

$$\Lambda^3 = \begin{pmatrix} 0.9720 & 0.3440 & -0.6200 \\ 1.2400 & -0.0200 & -0.4000 \\ 0.8000 & 0.6000 & -0.5000 \end{pmatrix}$$

$$\Lambda^4 = \begin{pmatrix} 1.1216 & -0.0368 & -0.4860 \\ 0.9720 & 0.3440 & -0.6200 \\ 1.2400 & -0.0200 & -0.4000 \end{pmatrix}$$

By picking out the (1,1) element of Λ^j, you recover the impulse response at horizon j to a unit shock at time t. This is much easier from a coding perspective than doing it recursively.