Liquidity Transformation Example

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1 Setup

- There are $X$ households with 1 unit of funds to invest teach. Live for periods $T = 0, 1, \text{ and } 2$
- Fraction $p$ will need to consume in $T = 1$ ("impatient/early"), fraction $1 - p$ in $T = 2$ ("patient/late")
- No aggregate uncertainty – exactly $pX$ will need to consume in $T = 1$, but at the individual level there is uncertainty (in $T = 0$) about when one will need to consume
- Household expected utility: $E[U] = p \ln C_1 + (1 - p) \ln C_2$, where $C_1$ and $C_2$ denote equal the (gross) returns on different savings choices
- Investment opportunity offers $r_1$ if liquidated in $T = 1$ and $r_2$ if liquidated in $T = 2$. $r_2 > r_1$ and assume $r_1 < 1$ while $r_2 > 1$
- Storage offers $r_1 = r_2 = 1$ (i.e. no net return)
- Bank deposit offers $r_1^d$ or $r_2^d$
- Questions: what can bank offer for $r_1^d$ and $r_2^d$? Which options (direct investment, storage, or indirect investment through bank) will households choose?

2 Bank Problem

- Suppose that the bank wants to choose $r_1^d$ and $r_2^d$ to maximize household expected utility, subject to bank have available funds to make such payments. Assume all $X$ households deposit with the bank.
- Bank has to choose how much to store, $S$, and how much to invest, $I$. Budget constraint is:
  \[ S + I = X \]
  If $r_1 > 1$, bank wouldn’t store any, it would invest everything. But we are assuming $r_1 < 1$ while $r_2 > 1$
• $pX$ households will need their funds and are promised $r_1^d$, so bank must store enough to meet this:

\[ r_1^d pX = S \]

• $(1 - p)X$ will be promised $r_2^d$, so bank has to invest to meet this:

\[ r_2^d (1 - p)X = r_2 I \]

• Combine these constraints to write $r_1^d$ and $r_2^d$ as a function of $I$:

\[
\begin{align*}
    r_1^d &= \frac{X - I}{pX} \\
    r_2^d &= \frac{r_2 I}{(1 - p)X}
\end{align*}
\]

• Bank problem is then to pick $I$ to maximize expected household utility:

\[
\max_I \quad p \ln \left( \frac{X - I}{pX} \right) + (1 - p) \ln \left( \frac{r_2 I}{(1 - p)X} \right)
\]

• This can be written:

\[
\max_I \quad p \ln (X - I) - p \ln p - \ln X + (1 - p) \ln r_2 + (1 - p) \ln I - (1 - p) \ln (1 - p) - (1 - p) \ln X
\]

• The FOC is:

\[
\frac{-p}{X - I} + \frac{1 - p}{I} = 0
\]

• Setting equal to zero and simplifying somewhat:

\[
pI = (1 - p)(X - I)
\]

Solving for $I$:

\[
I = (1 - p)X
\]

Which implies:

\[
S = pX
\]

But once we know $S$ and $I$, we can determine $r_1^d$ and $r_2^d$. In particular:

\[
\begin{align*}
    r_1^d &= 1 \\
    r_2^d &= r_2
\end{align*}
\]

• Effectively, what the bank is doing is (i) pooling $X$ funds, (ii) saving enough to give “early” types what they would get from storage, and (iii) investing the rest
• In effect, by pooling resources and eliminating idiosyncratic uncertainty, the bank can offer “late” types what they would get from direct investment while insuring “early” types with what they would get from storage.

3 Numbers and the Benefits of Liquidity Transformation

• Suppose $r_1 = 1/2$ and $r_2 = 5/3$. The measure of liquidity of the investment project is $r_1/r_2 = 3/10$

• Let $p = 1/2$ and $X = 1000$

• Household expected utility from storage:

$$E[U^s] = \frac{1}{2} \ln(1) + \frac{1}{2} \ln(1) = 0$$

• Household expected utility from direct investment:

$$E[U^i] = \frac{1}{2} \ln \left(\frac{1}{2}\right) + \frac{1}{2} \ln \left(\frac{5}{3}\right) = -0.0912$$

• Facing an option among storage or direct investment, household will choose storage, even though direct investment offers positive expected return: $\frac{1}{2} + \frac{1}{2} \times \frac{5}{3} = 1.083$

• Now think about the bank. From above, it will choose $r_1^d = 1$ and $r_2^d = r_2 = 5/3$. This generates expected utility for a household of:

$$E[U^d] = \frac{1}{2} \ln(1) + \frac{1}{2} \ln(5/3) = 0.2554$$

• This is better than expected utility from either of the other alternatives.

• Note the bank has in effect created a more liquid asset than what it is investing in – liquidity of deposits is $r_1^d/r_2^d = 3/5$, whereas liquidity of underlying investment is $r_1/r_2 = 3/10$

• By pooling resources and eliminating aggregate uncertainty, bank has insured household on the downside – they get option of storage if they happen to be early types – while giving households all the upside from direct investment.

• This makes the households clearly better off!

4 To Run or Not?

• Suppose you wake up in $T = 1$ and are revealed to be a “patient” type – i.e. you don’t need to withdraw.
• As long as no one else withdraws who doesn’t need to, you are obviously better off by waiting

• But suppose you think \( N \geq 0 \) people who don’t need to withdraw in \( T = 1 \) will withdraw. What makes sense?

• Bank’s promised repayment:
  \[
  r^d_t(pX + N)
  \]

• Bank has stored \( S = r^d_t pX \) anticipating this many withdrawals.

• Now it needs to come up with additional funds by early liquidating of what it has invested. Additional funds it needs to come up with are \( r^d_t N \). To generate this, will need to sell \( Z \) units of what it invested, where \( Z \) satisfies \( r^d_1 Z = r^d_t N \).

• This leaves \( I - \frac{r^d_t}{r_1} N \) left over, which will generate \( r_2 \) of funds

• There will be \((1 - p)X - N\) households left to distribute this to in \( T = 2 \)

• So then the bank will only be able to repay the following amount, \( r^d_{2,t} \), which will not necessarily equal what was promised. \( r^d_{2,t} \) satisfies:
  \[
  r^d_{2,t}((1 - p)X - N) = r_2 \left( I - \frac{r^d_t}{r_1} N \right)
  \]

• Since \( I = (1 - p)X \) from above, this is:
  \[
  r^d_{2,t}((1 - p)X - N) = r_2 \left( (1 - p)X - \frac{r^d_t}{r_1} N \right)
  \]

• So:
  \[
  r^d_{2,t} = r_2 \frac{(1 - p)X - \frac{r^d_t}{r_1} N}{(1 - p)X - N}
  \]

• Note, if \( N = 0 \), this is the same as what we had above: \( r^d_{2,t} = r_2 \)

• Since \( r^d_1 > r_1 \), if \( N > 0 \) then \( r^d_{2,t} < r^d_2 \)
  
  − In other words, if more than the expected number of people withdraw early, the bank can’t repay patient types as much as was promised

• How big does \( N \) have to be for it to make sense to withdraw in \( T = 1 \)? You will withdraw early if \( r^d_{2,t} < r^d_1 = 1 \). So:
  \[
  \frac{(1 - p)X - \frac{r^d_t}{r_1} N}{(1 - p)X - N} < r^d_1
  \]

• Which implies:
  \[
  \left( \frac{r^d_1}{r_2} - \frac{r^d_1}{r_1} \right) N < (1 - p)X \left( \frac{r^d_1}{r_2} - 1 \right)
  \]
Both terms in parentheses are negative. So multiply both sides by $-1$ to make positive, which flips the inequality:

$$\left(\frac{r^d_1}{r_1} - \frac{r^d_1}{r_2}\right)N > (1 - p)X \left(1 - \frac{r^d_1}{r_2}\right)$$

$$N > \frac{(1 - p)X \left(1 - \frac{r^d_1}{r_2}\right)}{-\frac{r^d_1}{r_1} + \frac{r^d_1}{r_2}}$$

- For the numbers I am using, the cutoff value works out to 148.8571.

- So, if you are a late type and think that 149 or more other late types are going to withdraw early (out of 500 total late types in the example we are using), then it is optimal for you to withdraw early.

- But if everyone who is a patient/late type tries to withdraw early, the bank will fail – it can at most come up with $S + r_1I = pX + r_1(1 - p)X$ of funds, which in our example works out to 750. Since it has promised all early depositors $r^d_1 = 1$, it can only pay the first 750 who try to withdraw, leaving the other 250 with nothing.