

Graduate Macro Theory II: Notes on Determinacy with Interest Rate Rules

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1 Introduction

In models with sticky prices, it is popular to characterize monetary policy with simple interest rate rules instead of exogenous money supply rules. Such rules focus in on the instrument central banks seem to care about (e.g. interest rates, not measures of the money supply), seem to fit the data reasonably well, and often have good normative properties.

A complicating factor with interest rate rules is that issues of determinacy arise. In general, interest rate rules must react sufficiently strongly to *endogenous* variables (like inflation and/or a measure of output) in order to guarantee a determinate rational expectations equilibrium. My “determinate” I mean “unique.” If a rule does not respond aggressively enough to endogenous variables then there may be indeterminacy, which can give rise to non-fundamental “sun-spot” equilibria. If there is an indeterminate equilibrium in these models, then that means that there is no unique non-explosive value of current inflation that satisfies the equilibrium conditions of the model given the current state. In a model with no nominal rigidity this just means there is nominal indeterminacy. But if there is nominal rigidity (e.g. price stickiness), then nominal indeterminacy also gives rise to real indeterminacy in the sense that there may be non-fundamental fluctuations in real quantities. From a welfare perspective real indeterminacy is undesirable, so we’d like to understand the restrictions on policy rules giving rise to determinacy.

2 Taylor’s Original Intuition

The “grandfather” of interest rate rules is widely considered to be John Taylor, after whom the “Taylor Rule” is named. His famous paper on the topic was Taylor (1993), “Discretion Versus Policy Rules in Practice,” which appeared in the 1993 *Carnegie-Rochester Conference Series on Public Policy*. In this paper, he documented that a policy rule of the following form (omitting constants, so we can interpret all variables as deviations from trend/steady state):

$$i_t = \phi_\pi \pi_t + \phi_x x_t$$

Here π_t is inflation, x_t is the gap between actual and potential output, and i_t is the interest rate controlled by the central bank (e.g. the Fed Funds rate).¹ Taylor argued that values of $\phi_\pi = 1.5$ and $\phi_x = 0.5$ fit the data well. He argued that the coefficient on inflation needed to be greater than 1. This came to be known as the “Taylor principle.”

Taylor’s logic for this parameter restriction is loosely as follows. Total aggregate demand depends on the real interest rate, and inflation depends on aggregate demand. The real interest rate is approximately $r = i - \pi$. Technically it is expected future inflation, not current inflation, but assume that it is current inflation. Whenever inflation increases, if $\phi_\pi > 1$, the nominal interest rate increases by more. This means real interest rates increase whenever inflation increases. Higher real interest rates depress aggregate demand, which brings inflation down. In contrast, suppose that $\phi_\pi < 1$. This means that whenever inflation increases the real interest rate declines. This decline in the real rate fuels more inflation, and so inflation can “spiral” out of control.

This is *stabilizing* logic. Implicitly, it sounds like you need a sufficient reaction to inflation to generate a stable root to keep the system from exploding. Though a similar restriction obtains in a forward-looking New Keynesian model, such a restriction is not really about “stabilizing” per se. Rather, we need a sufficient response to endogenous variables in a policy rule to impart a sufficient number of *unstable* roots into the system. This makes the policy functions unique. We rule out explosions by assumption.

3 Determinacy in a Model with Flexible Prices

Suppose that we have a very simple model. There is no capital, so all output must be consumed. Prices are flexible, meaning that the classical dichotomy holds and there is no effect of nominal variables on real variables. Money demand is implicitly generated via money in the utility function, additively separable from consumption. The demand side of the economy is summarized by the Euler/IS equation (all variables are taken to be either percentage deviations from steady state or deviations from steady state):

$$y_t = E_t y_{t+1} - (i_t - E_t \pi_{t+1})$$

Suppose the policy rule just reacts to inflation with a random, mean zero shock, u_t :

$$i_t = \phi_\pi \pi_t + u_t$$

Suppose that u_t follows a stationary (e.g. $0 < \rho < 1$ AR(1) process):

$$u_t = \rho u_{t-1} + e_t$$

To make life as simple as possible, suppose that real output is constant. This means that

¹Taylor’s paper is empirical. He measured the “gap” with a measure of output less a statistical trend, which is different than the theoretical concept of the gap.

$y_t = E_t y_{t+1} = 0$. The Euler equation then becomes:

$$i_t = E_t \pi_{t+1}$$

If we combine these expressions, we get:

$$E_t \pi_{t+1} = \phi_\pi \pi_t + u_t$$

This is a forward-looking difference equation for which there exist many different solutions as a general matter. To get a solution we use the equivalent of a transversality, requiring that $\lim_{T \rightarrow \infty} E_t \pi_{t+T} = 0$. For there to be a unique non-explosive solution, you need the difference equation to be explosive. Basically,, this is a system of one forward-looking variable, π_t , and one state variable, u_t . The eigenvalue associated with u_t will be ρ , which is stable. For saddle point stability, we need an unstable eigenvalue with π_t . This eigenvalue is ϕ_π . If $\phi_\pi < 1$, then there is no unique solution – any value of π_t will have expected inflation go to zero in the limit for any u_t . If $\phi_\pi > 1$ there will exist a unique solution, with that solution given by $\pi_t = -\frac{u_t}{\phi_\pi - \rho}$.

4 Determinacy in a Basic New Keynesian Model

Consider a standard New Keynesian model. There is the Euler/IS equation, the Phillips Curve, and an exogenous process for the flexible price level of output (recall that there are multiple ways to write down the equilibrium conditions). Let all variables denote percentage (or actual) deviations from the non-stochastic steady state without writing tilde over them. The equations of the model are:

$$\begin{aligned} \pi_t &= \gamma(y_t - y_t^f) + \beta E_t \pi_{t+1} \\ y_t &= E_t y_{t+1} - i_t + E_t \pi_{t+1} \\ y_t^f &= \rho y_{t-1}^f + \varepsilon_t \end{aligned}$$

The slope coefficient $\gamma = \frac{(1-\theta)(1-\theta\beta)}{\theta}(1+\phi)$, where 1 is the coefficient of relative risk aversion and ϕ is the inverse Frisch labor supply elasticity. Suppose that the nominal interest rate (in deviation form) obeys a simple Taylor rule of the form:

$$i_t = \phi_\pi \pi_t + \phi_x (y_t - y_t^f)$$

We want to know the following: what restrictions on ϕ_π and ϕ_x must be made in order to ensure a determinate rational expectations equilibrium? To see this, eliminate i_t and form a three variable system. The Euler/IS equation becomes:

$$y_t = E_t y_{t+1} - \phi_\pi \pi_t - \phi_x y_t + \phi_x y_t^f + E_t \pi_{t+1}$$

We can form a vector system:

$$E_t \begin{bmatrix} \pi_{t+1} \\ y_{t+1} \\ y_{t+1}^f \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\gamma}{\beta} & \frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} & -\frac{\gamma}{\beta} - \phi_x \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \\ y_t^f \end{bmatrix}$$

We need to find the eigenvalues of this system. One of the eigenvalues is clearly ρ , which is less than 1 and hence stable. To find the other two eigenvalues we just need to find the eigenvalues of the upper 2×2 block of the coefficient matrix. That is, we need to find the λ which makes:

$$\det \begin{bmatrix} \frac{1}{\beta} - \lambda & -\frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} - \lambda \end{bmatrix} = 0$$

The determinant of a 2×2 matrix is just the difference of the product of the diagonals:

$$\left(\frac{1}{\beta} - \lambda\right) \left(1 + \phi_x + \frac{\gamma}{\beta} - \lambda\right) + \frac{\gamma}{\beta} \left(\phi_\pi - \frac{1}{\beta}\right) = 0$$

Now, two useful facts about eigenvalues and determinants. First, the product of the eigenvalues is just equal to the determinant of the matrix. Second, the sum of the eigenvalues is equal to the trace of the matrix. The determinant and trace of the upper 2×2 matrix are:

$$\begin{aligned} \lambda_1 \lambda_2 &= \det \begin{bmatrix} \frac{1}{\beta} & -\frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} \end{bmatrix} = \frac{1}{\beta} + \frac{\phi_x}{\beta} + \frac{\gamma \phi_\pi}{\beta} \\ \lambda_1 + \lambda_2 &= \text{trace} \begin{bmatrix} \frac{1}{\beta} & -\frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} \end{bmatrix} = \frac{1}{\beta} + 1 + \phi_x + \frac{\gamma}{\beta} \end{aligned}$$

Since both the determinant and trace must be positive given standard assumptions on parameter values, we know that both eigenvalues must be positive as well.

For a unique equilibrium, we need both of these eigenvalues to be explosive (we already have one stable root, and we have two jump variables). Since we know from above that both these eigenvalues must be positive, then (ignoring complex roots), the necessary condition for stability is that:

$$(\lambda_1 - 1)(\lambda_2 - 1) > 0$$

Multiply this out:

$$\begin{aligned} \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) + 1 &> 0 \\ \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) &> -1 \end{aligned}$$

Plug in our expressions from above and simplify:

$$\begin{aligned}\frac{1}{\beta} + \frac{\phi_x}{\beta} + \frac{\gamma\phi_\pi}{\beta} - \left(\frac{1}{\beta} + 1 + \phi_x + \frac{\gamma}{\beta}\right) &> -1 \\ \phi_x \left(\frac{1}{\beta} - 1\right) + \frac{\gamma\phi_\pi}{\beta} - \frac{\gamma}{\beta} &> 0 \\ \phi_x(1 - \beta) + \gamma\phi_\pi - \gamma &> 0\end{aligned}$$

The last line follows from multiplying both sides by β . Now divide both sides by γ and simplify:

$$\phi_x \frac{1 - \beta}{\gamma} + \phi_\pi > 1$$

This is the condition that must be satisfied for there to exist a determinate equilibrium. We can see that $\phi_\pi > 1$ is slightly too strong of a restriction – determinacy also depends on the response to the output gap in the policy rule. But if $\beta \approx 1$, then unless γ is very small the determinacy condition is still roughly $\phi_\pi > 1$.

You can trick up the model along a number of dimensions but something like this basic condition usually emerges. Recall that this condition is needed to pin down a unique, non-explosive equilibrium, which is (quite) different than Taylor’s original intuition.