1 Introduction

In the previous section we studied equilibrium in a particularly simple world – an endowment economy. This was useful for getting intuition about what determined the real interest rate, but lacked realism.

In this section we endogenize supply. The concept of equilibrium is identical to before, but the set up is more interesting. Now we have firms that produce output using capital and labor, and households that consume, save, and supply labor. In addition to the goods market that must clear, we have another market – the labor market. We have several different exogenous variables and can examine how changes in these variables affect the endogenous variables of the model.

2 Firm

As in the Solow Model section, we assume that there are many, identical firms. We can therefore normalize the number of firms to be 1, and call this firm the representative firm. The firm will produce output using capital and labor, and an exogenous productivity shifter which will be called \( A \). We abstract from growth, either in population or in the number of effective workers. The firm exists for two periods, \( t \) and \( t + 1 \), and wakes up with an exogenously given level of current capital, \( K_t \). The firm uses capital and labor to make stuff, and returns its profit to households in the form of dividends.

The production technology is:

\[
Y_t = A_t F(K_t, N_t)
\]

The function \( F(\cdot) \) has the usual properties: increasing and concave in both arguments, and constant returns to scale. An example production function satisfying this functional form is Cobb-Douglas:

\[
F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}, \quad 0 < \alpha < 1.
\]

We assume that the firm hires labor on a period-by-period basis from the household at wage rate \( w_t \), which both the firm and household take as given.
We depart slightly from our earlier assumption and assume that the firm, rather than the household, owns the capital stock and makes investment decisions. Since the household owns the firm and there are no other frictions, the ownership structure on the capital stock ends up being irrelevant – we would get exactly the same equilibrium whether the firm makes capital accumulation decisions or whether the household makes those decisions. This set-up is perhaps more realistic, however.

The firm’s profit function is given by:

$$\Pi_t = Y_t - w_t N_t - I_t$$

Here investment in new capital, $I_t$, comes out of profit. Think of it this way. The firm produces some output and then pays labor. It can either return this net revenue to the household in the form of a dividend or it can re-invest that profit in new capital, which will help the firm produce in the future. Re-investing today, $I_t$, reduces current dividends but increases the productive capacity in the future, and hence should increase profit in the future. The capital accumulation equation is standard, where the firm takes its initial level of capital, $K_t$, as given:

$$K_{t+1} = I_t + (1 - \delta)K_t$$

This expression tells us what capital tomorrow will be given current capital and current investment. In principle this accumulation equation can be iterated forward one period:

$$K_{t+2} = I_{t+1} + (1 - \delta)K_{t+1}$$

Recall that the “world ends” after $t + 1$, so the firm will have no interest in leaving any capital over for period $t + 2$. As long as it can do “negative” investment, it will want to set $K_{t+2} = 0$, or $I_{t+1} = -(1 - \delta)K_{t+1}$. Think about negative investment in the second period as the firm liquidating itself – it produces output in $t + 1$ using $K_{t+1}$, and after that production there is $(1 - \delta)K_{t+1}$ of capital left over, which the firm returns to the household in the form of a dividend. This presumes that the household can consume the capital. This may seem a little hokey, but is standard and we wouldn’t have to make such a strict assumption in a multi-period version of the model.

The firm’s objective is to maximize the present value of the dividends that it returns to households.¹ The firm discounts future profit/dividend by $1 + r_t$, which serves as the discount factor for goods. The discounted value of profit/dividends can be interpreted as the value of the firm. The maximization problem is dynamic because investing in new capital today reduces the current dividend but increases future dividends. The firm can choose labor input and investment in new capital in both periods. The firm wants to maximize:

$$1$$

¹I am using the terms “dividend” and “profit” interchangeably. In practice the firm may not return profit each period, but households get compensated indirectly through capital gains (share price appreciation). As long as there are no credit market imperfections the distinction between a dividend received and a capital gain is immaterial.
\[ \max_{N_t, N_{t+1}, I_t, I_{t+1}} V_t = \Pi_t + \frac{1}{1 + r_t} \Pi_{t+1} \]

s.t.

\[ K_{t+1} = I_t + (1 - \delta)K_t \]
\[ K_{t+2} = I_{t+1} + (1 - \delta)K_{t+1} \]

Plugging in the profit function, we get:

\[ \max_{N_t, N_{t+1}, I_t, I_{t+1}} V_t = AF(K_t, N_t) - w_t N_t - I_t + \frac{1}{1 + r_t} (A_{t+1} F(K_{t+1}, N_{t+1}) - w_{t+1} N_{t+1} - I_{t+1}) \]

s.t.

\[ K_{t+1} = I_t + (1 - \delta)K_t \]
\[ K_{t+2} = I_{t+1} + (1 - \delta)K_{t+1} \]

This is a constrained problem. As before, we can take care of the constraints by substituting them into the objective function. In particular, impose the terminal condition discussed above that \( I_{t+1} = -(1 - \delta)K_{t+1} \), and eliminate \( I_t = K_{t+1} - (1 - \delta)K_t \). This allows us to write the problem as one of choosing future consumption instead of current saving:

\[ \max_{N_t, N_{t+1}, K_{t+1}} V_t = AF(K_t, N_t) - w_t N_t - K_{t+1} + (1 - \delta)K_t + \frac{1}{1 + r_t} (A_{t+1} F(K_{t+1}, N_{t+1}) - w_{t+1} N_{t+1} + (1 - \delta)K_{t+1}) \]

We can characterize the optimum by taking the partial derivatives with respect to the choice variables and setting those partials equal to zero:

\[ \frac{\partial V_t}{\partial N_t} = 0 \Leftrightarrow A_t F_N(K_t, N_t) = w_t \]
\[ \frac{\partial V_t}{\partial N_{t+1}} = 0 \Leftrightarrow A_{t+1} F_N(K_{t+1}, N_{t+1}) = w_{t+1} \]
\[ \frac{\partial V_t}{\partial K_{t+1}} = 0 \Leftrightarrow 1 = \frac{1}{1 + r_t} (A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta)) \]

Two of these first order conditions should look familiar; all have the interpretation as marginal benefit = marginal cost conditions. The first condition simply says that the firm should hire labor up until the point at which the marginal product of labor (marginal benefit) equals the marginal cost of an additional unit of labor (the real wage). The second condition says the same thing for
labor in period \( t + 1 \). Even though profits in period \( t + 1 \) are discounted by \( \frac{1}{1 + r_t} \) in the objective function, because both benefit and cost incur in the same period \( t + 1 \), then no discounting shows up in that first order condition.

The marginal product of labor being equal to the real wage condition (which must hold in the same form in both periods, and is hence sometimes called a static condition) implicitly defines a demand curve for labor. When \( w_t \) goes up, the firm will need to increase the marginal product of labor, which, holding everything else fixed, requires reducing \( N_t \) given the assumed concavity of the production function. The labor demand curve would shift right if \( A_t \) were to increase, while it would shift left if current \( K_t \) were to exogenously decline (say, due to a natural disaster). The labor demand curve, \( N_t^d = N(w_t, A_t, K_t) \), look as follows:

The final optimality condition for the choice of future capital stock looks a little different than what we’ve seen before, but has an intuitive interpretation. It says to equate the marginal cost of investment (foregone current dividends/profits of 1) with the marginal benefit of investment, which is the (i) extra output produced in \( t + 1 \) (the marginal product of future capital, \( A_{t+1} F_K(K_{t+1}, N_{t+1}) \)) and (ii) the salvage or liquidation value of the marginal unit of capital, which is \((1 - \delta)\). Since all of the benefit of extra investment is received in the future, but the cost is born in the present, the future benefit gets discounted by \( \frac{1}{1 + r_t} \). This condition can be re-arranged to yield:

\[
r_t + \delta = A_{t+1} F_K(K_{t+1}, N_{t+1})
\]

For given values of \( N_{t+1} \) and \( A_{t+1} \), this condition says that the desired period \( t + 1 \) capital stock, \( K_{t+1} \), is a decreasing function of the real interest rate, \( r_t \). This is because of the assumption of concavity – when \( r_t \) goes up, the right hand side must go up, which requires making \( K_{t+1} \) smaller. Reducing \( K_{t+1} \), given a current value of \( K_t \), means reducing current investment. Put differently, this first order condition implicitly defines an investment demand curve, \( I_t = I(r_t, A_{t+1}, K_t) \). As we just discussed, investment will be decreasing in the real interest rate. It will be increasing in
future $A_{t+1}$. This is an important point – investment is fundamentally forward-looking – investment today increases productive capacity in the future, so the more productive you expect to be in the future, the more investment you’d like to do in the present. The investment demand curve also depends on current capital, which is effectively exogenous. The condition above implicitly defines an optimal choice of $K_{t+1}$ which is independent of current $K_t$. But if current $K_t$ is, say, low, then investment would have to be high to achieve a given target level of $K_{t+1}$. Hence, desired investment is decreasing in $K_t$, holding everything else fixed.\footnote{Technically, the level of $N_{t+1}$ is also something that should show up in the investment demand function – the higher $N_{t+1}$, the higher the marginal product of future capital, and hence the more future capital a firm would want, given an interest rate. But since the firm can choose $N_{t+1}$, I omit it as an argument.}

We can draw an investment demand curve below. It is downward-sloping in $r_t$, and shifts right if $A_{t+1}$ increases or if $K_t$ decreases exogenously (say, due to a hurricane or other natural disaster):

Note that the real interest rate here serves not as an explicit cost of investment, but rather as an implicit cost of investment. As I’ve written the problem, the firm is not borrowing to finance investment. Rather, investment comes out of future dividends, and since the benefit of investment (more profit) is only received in the future, the benefit gets discounted. So $r_t$ has the interpretation here as the opportunity cost of investment – rather than accumulating more capital, the firm could have returned more current profit to households. I could have also written down the model where I force the firm to finance new capital accumulation through borrowing. It turns out to not matter (under assumptions we have made) how the firm finances itself – essentially here equity (reducing current dividend to yield more in the future) or debt.

3 Household

The representative household lives for two periods, consuming and working in both periods. The household problem is similar to what we had before, with the twist that it gets to choose how much
to work.

Let the household’s endowment of time each period be normalized to 1. Let \( N_t \) denote how much labor it supplies. Therefore its leisure is \( 1 - N_t \). We assume that leisure is an argument in the utility function, with more leisure leading to more utility (in other words, households don’t like to work). Let \( v(1 - N_t) \) be a function that maps leisure into utility; we assume that it is increasing, \( v'(\cdot) > 0 \), and concave, \( v''(\cdot) < 0 \). An example function satisfying this property is the natural log: \( v(1 - N_t) = \ln(1 - N_t) \).

We assume that “period utility” is separable in consumption and leisure; this means that utils in period \( t \) are given by \( u(C_t) + v(1 - N_t) \), where \( u(\cdot) \) has the same properties as before. Lifetime utility is just the weighted sum of period utility, with \( \beta \) the weight on future utils:

\[
U = u(C_t) + v(1 - N_t) + \beta(u(C_{t+1}) + v(1 - N_{t+1}))
\]

The household faces a sequence of two within period budget constraints. Conceptually they are the same as we have seen before, but we are no longer in an endowment economy, so income is not exogenous. The household has two sources of income: wage income, \( w_t N_t \), and distributed profits from firms, \( \Pi_t \), which the household takes as given. The household may also have to pay taxes to a government, \( T_t \) and \( T_{t+1} \). The household can use its first period income to consume or to save in bonds, \( S_t \). In the second period it will want to consume all of its income, where its total income comes from labor income, \( w_{t+1} N_{t+1} \), distributed profits from firms, \( \Pi_{t+1} \), and interest on first period savings, \((1 + r_t)S_t \). The two within period constraints are:

\[
C_t + S_t = w_t N_t - T_t + \Pi_t
\]
\[
C_{t+1} = w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} + (1 + r_t)S_t
\]

As before, \( S_t \) shows up in both period constraints, and so we can eliminate it and combine into one intertemporal budget constraint, which has the same flavor as before but has endogenous total income:

\[
C_t + \frac{C_{t+1}}{1 + r_t} = w_t N_t - T_t + \Pi_t + \frac{w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}
\]

We can write the household problem as one of choosing, at time \( t \), a sequence of consumption, \( (C_t, C_{t+1}) \), and labor, \( (N_t, N_{t+1}) \), to maximize its lifetime utility subject to the intertemporal budget constraint. In actuality the household really chooses saving and solves the problem period-by-period, but it works out easier (and identically) to think about them solving the problem this way. As before, the household is a price-taker, and so takes the real interest rate, \( r_t \), and the real wage rate, \( w_t \), as given. It also takes distributed profits as given.

\[
\max_{C_t, C_{t+1}, N_t, N_{t+1}} U = u(C_t) + v(1 - N_t) + \beta(u(C_{t+1}) + v(1 - N_{t+1}))
\]
s.t.

\[ C_t + \frac{C_{t+1}}{1 + r_t} = w_t N_t - T_t + \Pi_t + \frac{w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t} \]

This is a constrained, multivariate optimization problem. To handle it, we will solve for \( C_{t+1} \) from the constraint, plug that in, and thereby transform the problem to an unconstrained problem in only three choice variables. Solve for \( C_{t+1} \), we get:

\[ C_{t+1} = (1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} \]

Plug into the objective function:

\[
\max_{C_t, N_t, N_{t+1}} U = u(C_t) + v(1 - N_t) + \beta (u((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}) + v(1 - N_{t+1}))
\]

Let’s take the derivatives with respect to the choice variables to find the conditions characterizing optimal behavior:

\[
\frac{\partial U}{\partial C_t} = 0 \iff u'(C_t) + \beta u'((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}) (-1 + r_t)) = 0
\]

\[
\frac{\partial U}{\partial N_t} = 0 \iff -v'(1 - N_t) + \beta u'((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}) (1 + r_t)w_t = 0
\]

\[
\frac{\partial U}{\partial N_{t+1}} = 0 \iff \beta u'((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}) w_{t+1} - \beta v'(1 - N_{t+1}) = 0
\]

These conditions can be simplified and made more interpretable by substituting back in for \( C_{t+1} \).

Doing so:

\[
u'(C_t) = \beta (1 + r_t) u'(C_{t+1})
\]

\[
v'(1 - N_t) = w_t \beta (1 + r_t) u'(C_{t+1})
\]

\[
v'(1 - N_{t+1}) = w_{t+1} u'(C_{t+1})
\]

We can actually simplify the second condition by nothing that \( \beta (1 + r_t) u'(C_{t+1}) = u'(C_t) \). Then we can write the first order conditions as:

\[
u'(C_t) = \beta (1 + r_t) u'(C_{t+1})
\]

\[
v'(1 - N_t) = w_t u'(C_t)
\]

\[
v'(1 - N_{t+1}) = w_{t+1} u'(C_{t+1})
\]

You will note that the first condition is just the consumption Euler equation that we’ve seen
before. This says to equate the marginal benefit of consuming a little more today, \( u'(C_t) \), with the marginal cost, which takes the form of foregone future consumption which reduces future utility, \( \beta(1 + r_t)u'(C_{t+1}) \). The second two conditions take identical forms for each period, \( t \) and \( t + 1 \). These also have the interpretation as marginal benefit equals marginal cost conditions. Suppose that the household chooses to consume one additional unit of leisure in period \( t \). The marginal benefit of this change is the marginal utility of leisure, or \( v'(1 - N_t) \). The marginal cost of this is foregone consumption – working less means you have \( w_t \) fewer units of income to consume, which reduces utility by \( w_t u'(C_t) \). Hence, the right hand side is the marginal cost of an additional unit of leisure today.

We can analyze these first order conditions graphically with an indifference curve-budget line diagram. Because income is now endogenous, we have to hold everything else fixed outside of the graph. The first indifference curve-budget line diagram is the familiar one with current consumption, \( C_t \), on the horizontal axis and future consumption, \( C_{t+1} \), on the vertical axis. The budget line can be derived from the intertemporal budget constraint, holding \( N_t \) and \( N_{t+1} \) fixed. The slope is \( -(1 + r_t) \), and the optimality condition is where the indifference curve and the budget line are tangent.

We can also draw an indifference curve-budget line diagram for the choice of leisure/labor and consumption. We will do the picture with \( C_t \) on the vertical axis and \( 1 - N_t \) on the horizontal axis. The plot is complicated by the fact that there is an upper bound on leisure – it cannot be in excess of 1, which occurs when \( N_t = 0 \). This is going to introduce a kink into the budget constraint. Solve for \( C_t \) from the budget constraint in terms of everything else:

\[
C_t = -\frac{C_{t+1}}{1 + r_t} + w_t N_t - T_t + \Pi_t + \frac{w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}.
\]

The derivative of \( C_t \) with respect to \( N_t \) is: \( \frac{\partial C_t}{\partial N_t} = w_t \). Now this is itself complicated by the fact that we don’t have \( N_t \) on the horizontal axis in the budget line diagram we want to draw – we
have $1 - N_t$. This does not complicate things much, since $\frac{\partial C_t}{\partial (1-N_t)} = -\frac{\partial C_t}{\partial N_t}$. Hence, the slope of the budget line is $-w_t$, which is the price of leisure in terms of foregone consumption – if you consume an additional unit of leisure, you earn $w_t$ fewer units, and thus have $w_t$ fewer units available to consume.

The budget line ends up having a kink, at $1 - N_t = 1$, or $N_t = 0$. This means that the household could locate at the kink, in which case the first order condition derived above is no longer necessary. Basically, if the household were sufficiently willing to substitute leisure across time (essentially $v(\cdot)$ close to linear) and there were a big differential in wages across time, the household may choose to work all in one period. We are going to rule that out, and assume that we are always at an interior solution away from the kink – the household always works some. The graphical optimality condition is shown below:

What happens when there is an increase in the real wage, $w_t$? This would lead the budget line to become steeper, but the kink occurs at the same point. Similarly to the effect of an increase in the real interest rate in the current/future consumption diagram, there turn out to be competing income and substitution effects at work. The substitution effect of a higher real wage says to work more (consume less leisure). But the income effect says to work less – a higher real wage makes you feel richer, which makes you want to consume more goods and more leisure (work less). Hence, there are competing income and substitution effects, and the effect of a change in the real wage on labor income is ambiguous. This means that the labor supply curve could be upward or downward sloping.

We are going to circumvent this problem by graphically studying what is called the Frisch labor supply curve. The Frisch labor supply curve isolates the substitution effect only – it tells us how labor hours would change as the wage changes, holding the marginal utility of consumption constant (which is effectively holding income constant). Recall the first order condition from above:

$$v'(1 - N_t) = u'(C_t)w_t$$
The Frisch labor supply curve plots $w_t$ against $N_t$, holding $u'(C_t)$ (and hence $C_t$) fixed. Suppose that $w_t$ goes up, which makes the right hand side bigger. This means that the left hand side must get bigger. Since we are holding $u'(C_t)$ fixed and since $v(\cdot)$ is concave, then $v'(\cdot)$ is decreasing in its argument (which is leisure). This means that, when $w_t$ gets bigger holding $u'(C_t)$ fixed, then $1 - N_t$ must get smaller, which means $N_t$ must get bigger. Put differently, the Frisch labor supply curve must be upward-sloping. Mathematically, denote the labor supply curve by $N^s = N(C_t, w_t)$, where the dependance on $C_t$ makes clear that $C_t$ is held fixed when drawing it:

To re-iterate, when we draw the Frisch labor supply curve, we are holding consumption fixed. In other words, it is a hypothetical construct that tells us how labor supply would change as the wage changes holding consumption fixed. Movements in consumption will result in shifts of the Frisch labor supply curve. For example, suppose that the real interest rate goes up. This would make consumption decline (assuming, as we do, that the substitution effect dominates). Consumption declining would make $u'(C_t)$ go up. For a given real wage, this must mean that $v'(1 - N_t)$ would have to go up, which would necessitate $1 - N_t$ getting smaller, or $N_t$ going up. Put differently, a higher real interest rate would lead to the entire labor supply curve shifting out to the right.

Many other variables will make the labor supply curve shift. Movements in basically anything other than $r_t$ will cause the budget line in the consumption-leisure indifference curve diagram to shift up or down, and hence to lead to movements in $C_t$ and $1 - N_t$ of the same direction. For example, suppose that there is a known increase in future total factor productivity, $A_{t+1}$, which operates like an increase in future income. This makes the household feel richer, which leads it to want to consume more goods and more leisure immediately (equivalently, to work less). This would lead to $C_t$ rising, holding everything else fixed, which would then lead to $u'(C_t)$ smaller, which would shift the labor supply curve in. Increases in either current or future government spending will both shift the labor supply curve to the right – holding other factors fixed, these make the

\footnote{You may (rightly) wonder how $w_t$ could go up with $C_t$ being affected. Implicitly, this thought experiment would have an increase in $w_t$ offset by a reduction in $\Pi_t$ so as to leave the household with the same overall level of income.}
household feel poorer, and hence to desire to work less.

4 The Government

The government problem is simple and is identical to what we’ve already had. The government exogenously picks a sequence of spending, \( G_t \) and \( G_{t+1} \), and taxes must adjust to make the government budget constraint hold.

The government faces two within period constraints:

\[
G_t + S_t^G = T_t
\]
\[
G_{t+1} = T_{t+1} + (1 + r_t)S_t^G
\]

These can be combined into one intertemporal budget constraint:

\[
G_t + \frac{G_{t+1}}{1 + r_t} = T_t + \frac{T_{t+1}}{1 + r_t}
\]

Since the household knows that the government budget constraint must hold, the household intertemporal budget constraint can be written:

\[
C_t + \frac{C_{t+1}}{1 + r_t} = w_t N_t - G_t + \Pi_t + \frac{w_{t+1} N_{t+1} - G_{t+1} + \Pi_{t+1}}{1 + r_t}
\]

In other words, the household will behave as though the government balances its budget period-by-period, just as in the earlier equilibrium notes. That is, Ricardian equivalence continues to hold. This means that changes in taxes that are not matched by changes in the time path of government spending will have no impact on the economy.

5 The Output Supply and Demand Curves and Equilibrium

Now we are going to take the decision problems of the household and firm and combine them to derive aggregate demand and supply curves. The demand side is slightly more complicated than what we previously saw due to the presence of investment, but the derivation is basically the same. An endogenous supply curve is what is new.

The aggregate supply curve, or what we will call the \( Y^s \) curve, is the set of \((r_t, Y_t)\) pairs consistent with household and firm optimization on the production side of the economy and with labor market-clearing. The labor market clearing means that the quantity of labor demanded must equal the quantity of labor supplied, given a real interest rate. We can characterize this graphically by combining the labor demand and supply curves from above:
As noted above, the real interest rate enters into the production side of the economy through the effect on labor supply. An increase in the real interest rate, holding everything else fixed, leads households to want to reduce consumption and increase labor supply. An increase in labor supply leads to more labor input in equilibrium. More labor input leads to more output. Hence, the $Y^s$ curve will be upward sloping.

The figure above shows a graphical derivation of the $Y^s$ curve. This is a four part graph. First, start in the upper right diagram with some real interest rate, $r_t^0$. This gives a value of consumption and hence a position of the labor supply curve, which determines a value of total employment in the upper left diagram, $N_t^0$. This can be read down to a graph of the production function in the lower left diagram, where that graph holds $A_t$ and $K_t$ fixed. This gives a value of $Y_t^0$. The lower
right diagram is just a 45 degree line diagram with \( Y \) on both axes and is used to “reflect up” the value of \( Y \) that one gets from the production function diagram. Hence, \( (r^0_t, Y^0_t) \) is a point on the \( Y^s \) curve.

To derive a new point on the curve, suppose that the real interest rate is higher, say \( r^1_t \). A higher real interest rate will lead the labor supply curve to shift right. This leads to a higher level of employment, \( N^1_t \), and hence a higher level of output, \( Y^1_t \). We could do the same for a lower real interest rate, and we would find a lower level of output. Connecting the points, we get an upward-sloping \( Y^s \) curve.

The derivation of the \( Y^d \) curve is similar to before. It is again complicated by the fact that the total amount of consumption depends on the total amount of income. In particular, the consumption function will again take the form: \( C_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) \), where government spending replaces taxes because of Ricardian equivalence.\(^4\) In this problem income is endogenous, but to maintain consistency with what we’ve done before we write consumption as a function of total income (which ultimately gets paid to households anyway either in the form of wages or distributed profit/dividends). The total demand for goods comes from combing the period budget constraints of the different actors in the economy. These are:

\[
C_t + S_t = w_t N_t - T_t + \Pi_t \\
\Pi_t = Y_t - w_t N_t - I_t \\
G_t + S_t^G = T_t
\]

In any equilibrium it must be that \( S_t^G = -S_t \). Hence, imposing this, along with the definition of profits, we get a standard looking accounting identity:

\[
Y^d_t = C_t + I_t + G_t
\]

Now plug in the optimal decision rules derived above:

\[
Y^d_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) + I(r_t, A_{t+1}, K_t) + G_t
\]

As noted, we again have the thorny issue that \( Y^d_t \) depends on \( Y_t \). In any equilibrium, however, \( Y_t = Y^d_t \). We can graphically plot \( Y^d_t \) as a function of \( Y_t \), relying on the fact that we know the MPC, or the derivative of the consumption function with respect to current income, must be between 0 and 1. This means that there will graphically exist a point where \( Y^d_t = Y_t \). We can derive the \( Y^d \) curve, which is the set of \( (r_t, Y_t) \) points consistent with agents optimizing and \( Y^d_t = Y_t \), but picking out different real interest rates. A higher real interest rate causes the expenditure line to shift down, which lowers the point where \( Y_t = Y^d_t \). The reverse is true for a lower real interest rate.

\(^4\)Note that I also am putting total income, \( Y_t \), as the argument in the consumption function, instead of breaking it into its constituent components. This is to maintain congruity with what we’ve already done, but also the household ultimately receives all income in the economy, since it owns the firm. Household consumption decisions depend on total income, not just labor income.
Connecting the dots, you get a downward-sloping $Y^d$ curve.

The full, general equilibrium of the economy occurs where we are on both the $Y^d$ and $Y^s$ curves. One way to think about it is that each of these curves represents a partial equilibrium in a sub-market. The $Y^s$ curve is based off of the labor market clearing, and traces out different levels of output that obtain for different real interest rates, which works through the labor supply curve. The $Y^d$ curve represents the “goods” market clearing, tracing out different points where income equals expenditure, $Y_t = Y^d_t$. In general equilibrium, both the goods and the labor markets must clear. Put differently, we must be on both the $Y^s$ and $Y^d$ curves. This occurs at the point where they intersect:
The way to think about this figure is as follows. The plot with the $Y^d$ and the $Y^s$ curves is the “main” diagram, with each curve coming from a different sub-market, with the real interest rate being the thing that connects the sub-markets. The $Y^s$ curve shows all $(r_t, y_t)$ pairs where the labor market is in equilibrium, $N^s = N^d$, while the $Y^d$ curve shows all $(r_t, Y_t)$ pairs where the goods market is in equilibrium, $Y_t^d = Y_t$.

6 Responses to Shocks

Now that we have derived a graphical depiction of the competitive equilibrium in the economy, we are in a position to look at how that equilibrium changes when any of the exogenous variables changes. Although one could entertain multiple exogenous variables changing at the same time, we will study these changes in isolation. The four exogenous variables that we’ll look at are current and future total factor productivity, $A_t$ and $A_{t+1}$, and current and future government expenditure, $G_t$ and $G_{t+1}$. We could also look at exogenous reductions in the current capital stock ($K_t$) or changes in uncertainty over the future, but will not do so in the notes.

These exercises can get rather difficult, because there are many curves shifting, and in most cases the exogenous changes do not neatly align solely into only “supply” or “demand.” For this reason I think it is important to slowly walk through these changes in the graphs, and to follow an ordered approach to looking at how the changes work their way through the economy. I always do these in the following steps:
1. Always start in the labor market. Ask yourself whether the labor supply curve, $N^s$, and/or labor demand curve, $N^d$, will shift holding the real interest rate fixed and the total level of current income fixed. This will tell you what the new level of $N_t$ would be for a given real interest rate. Things that will shift the labor supply curve are things which would shift consumption, holding current income and the current real interest rate fixed.

2. Take the new hypothetical level of $N_t$ for a given interest rate, and “bring it down” to the production function graph. This will tell you the level of output that would be produced for a given interest rate.

3. Given the new level of $Y_t$, “take it across” to the 45 degree line and then “reflect it up” to figure out how the $Y^s$ curve shifts. If your work shows that $Y_t$ would be higher for a given interest rate, then the $Y^s$ shifts out.

4. Next, ask yourself how, if at all, the $Y^d$ curve shifts. Remember, $Y^d$ shifts occur whenever the quantity demanded would change for a given interest rate.

5. Combine the shifts in $Y^d$ and $Y^s$ to find the new equilibrium levels of $Y_t$ and $r_t$. At this stage stop and ask yourself what happens to the components of output – consumption and investment. In some cases this is pretty easy, in other it requires some more thought.

6. “Work your way back” down and to the left to get back to the labor market, making sure that all the quantities in the four part picture make sense. In particular, changes consumption (for example, through the real interest rate) will work into labor supply shifts in such a way that all the quantities line up.

7. In some instances there will be some ambiguity about what is happening. Often times you can use math to resolve the ambiguity – in particular, the labor supply first order condition often comes in handy.

6.1 Increase in $A_t$

Let’s first do an exogenous increase in $A_t$, holding everything else fixed. Follow the formula above. We start in the labor market. An increase in $A_t$ will shift the labor demand curve out and to the right – this is because a higher $A_t$ raises the marginal product of labor. There is no direct effect on the labor supply curve, since we are engaging in the hypothetical thought experiment of keeping the real interest rate and the level of income fixed, and hence consumption is fixed. Labor demand shifting out with no change in supply means that the quantity of labor would increase for a given real interest rate.

Next, “bring this down” to the production function graph, also noting that the production function itself will shift up when $A_t$ increases. More employment plus a higher production function means a higher quantity of $Y_t$. “Taking this across” to the 45 degree line and then “reflecting up” we see that the $Y^s$ curve must shift to the right.
There is no shift in the $Y^d$ curve. This may seem non-intuitive, but nothing in the $Y^d$ curve depends directly on $A_t$ (only indirectly through $Y_t$, which would represent a movement along the curve). This means that, in the new equilibrium, $r_t$ will be lower and $Y_t$ will be higher. $r_t$ lower and $Y_t$ higher mean that both consumption and investment will be higher. Now we have to “work our way back” to the labor market graph to make everything consistent. The actual change in output is smaller than the horizontal shift of the $Y^s$ curve. Working back, we see that this must mean that the labor supply curve shifts in relative to its initial position such that that the new equilibrium quantity of output is consistent with the production function. Effectively, the lower real interest rate discourages people from working as much.

The graph above shows how everything works out, albeit the graphs don’t show how we have started in the upper left and worked our way down and around and back. To figure out all the ultimate effects, we can see that output ends up higher and the real interest rate ends up lower. Output/income higher and the real interest rate lower mean that consumption will be higher. A lower real interest rate means that investment is also higher. In the labor market we can see that the real wage is also definitely higher. The equilibrium effect on employment is technically ambiguous – supply shifts in, demand shifts out, leading to an ambiguous change in quantity. In all likelihood total hours worked will rise.\(^5\) We can think about the intuition for this as follows. Since the change

\(^5\)We know that output must rise; this could occur even with a drop in $N_t$ because $A_t$ is higher.
in $A_t$ is only transitory (for one period), the change in income is only transitory (for one period).\(^6\) Hence, the effect on consumption ought to be relatively small, which means that the inward shift in the labor supply curve should be relatively weak. This means it is likely that the shift in labor demand “dominates” the shift in supply such that $N_t$ increases in the new equilibrium.

6.2 Increase in $A_{t+1}$

Next, consider an exogenous but anticipated increase in $A_{t+1}$, which becomes known in period $t$. As before, start in the labor market. Because the marginal product of labor only depends on current $A_t$, an anticipated increase in $A_{t+1}$ does not shift the labor demand curve. It does, however, lead to a shift in labor supply – in particular, labor supply shifts in. This occurs because, holding current income and the real interest rate fixed, households would like to increase current consumption and leisure because their future income will be higher. Hence, labor supply shifts in, which leads to a reduction in the quantity of labor for a given real interest rate.

“Bringing down” the reduction in the quantity of labor to the production function, we see that less labor means less output. “Reflecting this over” to the 45 degree line and then bringing it up, we see that the entire $Y^s$ curve actually shifts inward. In words, this says that, when the future is expected to be better, the economy as a whole would like to produce less today. This may seem a bit counterintuitive, but it’s a direct result of smoothing – if the future is going to be better, people feel richer and would like to work less today, and hence the $Y^s$ curve shifts in.

Now we need to ask ourselves: what happens to the $Y^d$ curve? Here, we see that the $Y^d$ curve must shift right, for two reasons. Higher $A_{t+1}$ means higher $Y_{t+1}$, which means the household would like to consume more for every given level of current income. Higher $A_{t+1}$ also increases firm demand for investment, since the future marginal product of capital will be higher. Both of these things make the $Y^d$ curve shift right.

The demand curve shifting out with the supply curve shifting in means that $r_t$ is definitely higher. This makes sense from our earlier discussion in the endowment economy – the future is plentiful relative to the present, so the real interest rate rises in equilibrium.

\(^6\)Technically this is not quite right. If investment today goes up, then the economy will have a higher capital stock tomorrow, and hence, holding everything else fixed, higher income tomorrow. Capital accumulation is the one way that this model can “propagate” the effects of a transitory shock like this through time. However, in practice, this effect is very small, and hence we ignore it always. The capital stock relative to investment is very large; hence even very large swings in investment have relatively small effects on the capital stock. Hence, we can safely ignore this effect and treat $Y_{t+1}$ as effectively exogenous with respect to current investment.
If we look at the graph, everything but the higher real interest rate appears to be ambiguous. There are competing effects at work: higher future productivity makes household want to consume more and firms want to do more investment; but higher real interest rates have the opposite effect. Higher future productivity makes households want to work less; but higher real interest rates work to reverse that. Hence, everything is naturally ambiguous. I have drawn the picture where the competing effects offset, such that output is unchanged, but that need not be the case in general.

Even though we cannot sign the effects on many of the endogenous variables, the model does allow us to say something about the co-movements. In particular, look at the mathematical condition for labor market equilibrium, which is given by:

$$v'(1 - N_t) = u'(C_t)A_tF_N(K_t, N_t)$$

Here, I have replaced $w_t$ with $A_tF_N(K_t, N_t)$, which comes from the firm optimality condition. Suppose that $N_t$ were to increase following an increase in $A_{t+1}$. This would mean that $1 - N_t$ decreases, meaning that $v'(1 - N_t)$ would go up. An increase in $N_t$ would lower $A_tF_N(K_t, N_t)$. For the optimality condition to hold, the right hand side of the above expression must increase, which means that $u'(C_t)$ would have to go up, which requires $C_t$ to fall. $N_t$ higher would mean $Y_t$ is higher – $K_t$ and $A_t$ aren’t changing, and output is increasing in $N_t$. Higher $Y_t$ along with lower $C_t$ means higher $I_t$ (since government spending is not changing). Hence, we could have consumption falling, with output, hours, and investment rising. The reverse could also be true – consumption could rise,
but then (if you do the similar thought experiments above) output, hours, and investment would have to decline.

This discussion brings up a broader point which is useful to think about in this model. Given our assumptions on utility and the production function, consumption and labor input must move in opposite directions \textit{absent a change in} \( A_t \) \textit{or} \( K_t \). If \( A_t \) and \( K_t \) are not moving, then \( N_t \) and \( Y_t \) must move in the same direction, which then means that consumption must move in the opposite direction of output. This does not hold for a change in \( A_t \) – there consumption and labor input can move in the same direction because labor demand is shifting (the marginal product of labor is higher). This result is basically hard-wired into the model because of our assumptions on preferences, the concavity of the production function, and because of the assumption that the labor market always clears.

\textbf{6.3 Increase in} \( G_t \)

Next consider an exogenous increase in government spending, \( G_t \). We’ve already studied this, but in a context in which supply was fixed. Here, the dynamics are more interesting.

As always, start in the labor market. For a given level of current income and the current real interest rate, an increase in \( G_t \) would cause consumption to fall. This means that \( u'(C_t) \) would rise. \( u'(C_t) \) bigger means that \( u'(1 - N_t) \) must get smaller, so \( N_t \) must rise for a given real wage. This means that the labor supply curve shifts out. Intuitively, what is happening is that the increase in government spending makes the household feel poorer, and it wants to react by decreasing consumption and increasing labor input. There is no effect on labor demand. The rightward shift in the labor supply curve means that, for a given real interest rate, there will be higher employment. This translates into an outward shift in the \( Y^s \) curve.

As in the previous section of the course, there is also an effect on the \( Y^d \) curve. In particular, an increase in \( G_t \) increases the overall demand for goods and services, leading to an outward shift of the \( Y^d \) curve. We actually know something about the magnitude of this shift – in particular, the horizontal shift in the demand curve is one for one, which one can show using the total derivative (this is exactly the same as in the endowment economy notes).

The figure below shows the shifts at work:
Now, if you are paying attention, there also appears to be some ambiguity here. In particular, with the $Y^s$ curve shifting out and the $Y^d$ curve also shifting out, we know that equilibrium output must be higher, but the effect on the real interest rate appears ambiguous. We can actually go beyond the graph to show that the effect is not ambiguous – in particular, the real interest rate must rise, as I have drawn it in the figure above.

To show this, we have to delve into some stuff we know about the equations underlying the graphs. First, since output is higher, it must be the case that employment is higher. After all, the only way to have more output is to have more inputs or higher productivity, and productivity isn’t changing here and the capital stock is fixed. Hence, $N_t$ must be higher. From the arguments above about the labor market equilibrium, since neither $A_t$ nor $K_t$ are changing, it must be the case that $C_t$ is falling then – without a change in $A_t$ or $K_t$, our standard assumptions on preferences require that $C_t$ and $N_t$ move in opposite directions. Finally, look at the consumption function: $C_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)$. Suppose that $r_t$ falls. $r_t$ falling would graphically mean that $Y_t$ has to go up by more than $G_t$. We know this because we know that the horizontal shift in the $Y^d$ curve is equal to the change in government spending – if output were to rise by more than the increase in government spending in the new equilibrium, then we would have to be at a point to the “southeast” on the $Y^d$ curve relative to the point with no change in the real interest rate. Hence, if $r_t$ fell, then $Y_t$ would go up by more than $G_t$. But $Y_t$ going up by more than $G_t$ and $r_t$ falling would mean consumption rises, which we know that it does not. Hence, $r_t$ cannot fall – it must rise, and output must rise by less than the increase in government spending. Put differently,
the equilibrium multiplier is positive but less than 1. Finally, since \( r_t \) rises and \( A_{t+1} \) and \( K_t \) are unaffected, we know that investment must decline.

These results imply that an increase in government expenditure “crowds out” private expenditure – both consumption and investment decline, with the real interest rate rising. The “crowding out” is not complete – total output rises, so the declines in consumption and investment sum up to less than the increase in government spending. Total hours/employment rises. At this point, it is worth saying something about whether or not the increase in government spending is beneficial. Even though it increases output, it turns out (under this simple framework) that households are worse off – consumption is lower, and hours are higher so leisure is lower, so utility is lower. As I noted earlier, this probably overstates the welfare effect – government expenditure is usually on things that make households better off, and indeed we could write down a model in which government spending enters the household utility function. Hence, we can’t really say whether households are better or worse off following the increase in \( G_t \) – the important thing here is that there is a negative wealth effect that works to make households worse off, which must be balanced against any beneficial impact of the spending increase in evaluating the overall welfare impact of government purchases.

### 6.4 Increase in \( G_{t+1} \)

Finally, suppose that we have an increase in \( G_{t+1} \) that is anticipated by agents in the model in period \( t \).

As always, start in the labor market. The increase in \( G_{t+1} \), much like the increase in \( G_t \), makes households feel poorer. For a given level of income and a given real interest rate, they would like to reduce current consumption. This means \( u'(C_t) \) would rise, which would necessitate \( v'(1 - N_t) \) rising for a given \( w_t \). In other words, the labor supply curve must shift out, similarly to how it did for an increase in current \( G_t \). There is no effect on labor demand. Labor supply shifting out with no change in labor demand means a higher quantity of labor for a given real interest rate, which will translate into an outward shift of the \( Y^s \) curve.

Next, let’s think about what happens to the \( Y^d \) curve. From our earlier analysis we know that the \( Y^d \) curve will shift in – households will want to reduce consumption because they feel poorer, which leads to an inward shift in the quantity of goods demanded. An outward shift of the \( Y^s \) curve but an inward shift of \( Y^d \) means that the real interest rate definitely falls, but the quantity of output looks to be ambiguous. The figures are shown below:
We can once again resolve the ambiguity by appealing to a little bit of math. A lower real interest rate means that investment will be higher – since $A_{t+1}$ and $K_t$ are not changing, a lower $r_t$ definitely means that $I_t$ must rise. Since neither $A_t$ nor $K_t$ are changing, from our previous argument we know that $C_t$ and $N_t$ cannot move in the same direction. Suppose that $C_t$ rises – since $I_t$ rises and $G_t$ does not change, this would have to mean that $Y_t$ rises. But the only way for $Y_t$ to rise is for $N_t$ to rise, but if $I_t$ rises, and $A_t$ and $K_t$ do not change, then $N_t$ cannot rise. Hence, this is a contradiction – $C_t$ must fall, and $N_t$ must therefore rise. This also means that output goes up.

6.5 Summarizing Effects

Below is a table that summarizes the qualitative effects of changes in each of these four exogenous variables (where the changes occur in isolation). “+” means that the endogenous variable increases, a “-” means that the endogenous variable decreases, and a “?” means that the effect is ambiguous.
<table>
<thead>
<tr>
<th>Variable</th>
<th>$\uparrow A_t$</th>
<th>$\uparrow A_{t+1}$</th>
<th>$\uparrow G_t$</th>
<th>$\uparrow G_{t+1}$</th>
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