1 Introduction

In our discussion of consumption, we analyzed the problem of an optimizing household that takes the real interest rate as given. In this set of notes we endogenize the real interest rate. In particular, the real interest rate serves as a price that clears markets, and will play a central role in a more sophisticated model of the economy.

We endogenize the real interest rate here in a particularly simple environment in which “supply” (income, or real GDP, $Y$) is exogenously fixed. This is a gross simplification but the analysis turns out to have important insights into a more sophisticated model in which supply is endogenous.

2 The Environment

The economic environment here is identical to the environment we analyzed when looking at the consumption decisions of a particular household. We assume that there are a large number of identical households. The assumption that there are a large number of households is important insofar that there being a “large” number of them ensures that they act as price-takers: when making decisions, households behave as though they have no influence on any of the prices they face. Assuming that they are identical means that we can act as though there is only one household – the “representative” household.

As in the consumption notes, there are two periods: $t$ and $t+1$. Income is known and exogenously given: $Y_t$ and $Y_{t+1}$. This means that “supply” is fixed – the total amount of resources available in the economy is given, and does not respond to changes in how many of those resources the household wants to consume. Sometimes I’ll use the term “endowment” economy to describe this set up. It’s not the way the real world works, but it’s very useful to use this fiction to gain some intuition for more complicated models.

The economic problem of the representative household is:

$$\max_{C_t, C_{t+1}} u(C_t) + \beta u(C_{t+1})$$
\begin{align*}
\text{s.t.} \quad C_t + \frac{C_{t+1}}{1 + r_t} &= Y_t + \frac{Y_{t+1}}{1 + r_t} \\
\text{The first order condition characterizing the optimal allocation is the standard Euler equation:} \quad u'(C_t) &= \beta (1 + r_t) u'(C_{t+1})
\end{align*}

The consumption function is generically given by $C_t = C(Y_t, Y_{t+1}, r_t)$, where we know that consumption is increasing in both current and future income, and we assume that the substitution effect dominates, so that consumption is decreasing in the real interest rate.

There are no other actors in the economy. There are just households, which are all identical. They all have an exogenously given level of income, and endowment of fruit in each period, $Y_t$ and $Y_{t+1}$.

### 3 Equilibrium

In the consumption notes, we analyzed how consumption would react to changes in current and future income, holding the real interest rate fixed. The real interest rate, you will recall, has the interpretation as the “price” of current consumption relative to future consumption. Now we look at how the real interest rate adjusts when current and future income change, both of which trigger changes in desired consumption.

We are now in a situation to define a competitive equilibrium. A competitive equilibrium is a set of prices and allocations such that (i) all agents in a market are behaving optimally, taking all prices as given; and (ii) all markets clear. The adjective “competitive” means that agents take prices as given. The “equilibrium” component refers to agents (i) behaving optimally and (ii) markets clearing. If agents weren’t behaving optimally or if markets didn’t clear (loosely, that demand not equal supply), then there would be an incentive for the actors in the economy to alter their behavior along some dimension – in other words, it wouldn’t be an equilibrium.

The above is a general definition of a competitive equilibrium that we will use throughout the course. Let’s be clear about some terminology relevant to the specific example economy under study here. “Allocations” refer to how much first and second period consumption the household undertakes. The “price” in this economy is the real interest rate – since it is the return on saving / cost on borrowing, it represents the price of current consumption relative to future consumption. The market here is the market for consumption. Market-clearing means that total consumption must equal total income. Resources available for consumption today are given by $Y_t$. There is no way to transfer these resources across time through investment, since we have assumed that the economy is endowed with a fixed amount of fruit each period. This means that, even though households have access to a savings vehicle, markets clearing mean that households cannot save or borrow in equilibrium. In other words, market-clearing requires that all income each period be consumed. You can intuit why this is necessary in an equilibrium – if there is no way to transfer
resources across time, leaving some resources unconsumed would be leaving utility on the table, while consuming more resources than are physically available is infeasible.

In the context of this simple model of household behavior, the consumption function is given by \( C_t = C(Y_t, Y_{t+1}, r_t) \). This consumption function reflects optimal behavior on the part of the household. Let \( Y_t^s \) denote the aggregate supply of resources in the economy – as noted above, in this endowment economy this is just a fixed exogenous amount. Let \( Y_t^d \) denote aggregate demand for resources in the economy. The demand for resources comes from just consumption (in a more complicated model, investment, government spending, and potentially net exports would also be there): \( Y_t^d = C(Y_t, Y_{t+1}, r_t) \). In equilibrium, market-clearing will require that \( Y_t = Y_t^d = Y_t^s \).

Characterizing the demand side of the economy becomes complicated because desired spending, \( C \), depends on total income, \( Y \), and total income must equal total spending, \( Y = C \). Put differently, the aggregate demand side of the economy is given by: \( Y_t^d = C(Y_t, Y_{t+1}, r_t) \). This becomes complicated because \( Y_t^d \) shows up on both the left and the right hand side. What we want is to express \( Y_t^d \) as a function of \( r_t \) and \( Y_{t+1} \). The expression \( Y_t^d = C(Y_t, Y_{t+1}, r_t) \) implicitly defines \( Y_t^d \) as a function only of future income and the real interest rate, \( Y_t^d = Y^d(Y_{t+1}, r_t) \). Without making a functional form assumption on the consumption function, we can only qualitatively characterize this demand curve. A nice and convenient way to do so uses a graph that you may have seen from principles of macro. The graph plots \( Y_t^d \) as a function of \( Y_t \), not necessarily imposing that they be equal: \( Y_t^d = C(Y_t, Y_{t+1}, r_t) \). Holding \( r_t \) and \( Y_{t+1} \) fixed, this graph is upward sloping, with slope less than 1: \( \frac{\partial Y_t^d}{\partial Y_t} = \frac{\partial C_t}{\partial Y_t} \). As we argued in the last set of notes, because households have a desire to smooth consumption, the partial of consumption with respect to current income, or what we will heretofore call the MPC, is between 0 and 1. \( Y_t^d \) will also in general be greater than 0 even if current income is 0; as long as households have some future income coming their way, they will want to do positive consumption. In words then, a plot of \( Y_t^d \) against \( Y_t \) will have a positive vertical axis intercept and positive slope less than one. This means that the plot of \( Y_t^d \) must cross a 45 degree line, which is a line starting at the origin with slope 1 which shows all points where \( Y_t^d = Y_t \).

We can graph this as follows:
As noted in the previous paragraph, in any equilibrium the quantity of goods demand has to equal total income. Hence, we must have $Y_t^d = Y_t$. Graphically, this occurs at exactly one point, where the $Y_t^d$ graph crosses the 45 degree line. This point of intersection shows the quantity of resources demanded, given future resources, $Y_{t+1}$, and the real interest rate, $r_t$. The aggregate demand curve, or what we will also sometimes simply call the “$Y^d$ curve,” plots out how the quantity of goods demanded in aggregate varies with the real interest rate. We can graphically derive this by varying the real interest rate. Suppose that the real interest rate is initially $r_t^0$ and the level of output where $Y_t^d = Y_t^d$ at that interest rate is $Y_t^0$ (this holds $Y_{t+1}$ fixed). Because we assume that consumption is decreasing in the real interest rate, an increase in the real interest, say from $r_t^0$ to $r_t^1$, will shift the $Y_t^d$ line down in $(Y_t^d, Y_t)$ space, which means that the quantity demanded falls when $r_t$ increases, say to $Y_t^1$. Equivalently, a decrease in $r_t$, say from $r_t^0$ to $r_t^2$, shifts the $Y_t^d$ line up in $(Y_t^d, Y_t)$ space. This means that that quantity of $Y_t^d$ increases when $r_t$ decreases, say to $Y_t^2$. If we connect the dots to a figure below, one with $Y_t$ on the horizontal axis and $r_t$ on the vertical axis, we get a downward-sloping demand curve:
The demand curve, or $Y^d$ curve, shows how the quantity of goods demanded varies with the real interest rate. This demand curve holds fixed future income (or expectations of future income). When future income changes (or is expected to change), this will shift the demand curve – households will demand more current goods at every given interest rate. We can derive this shift using the same graphical device we used to derive the downward-sloping $Y^d$ curve in the first place. An increase in $Y_{t+1}$ will increase the amount of consumption people want to do for a given level of current income, $Y_t$, and the interest rate. This has the effect of shifting up the $Y^d$ curve in $(Y^d_t, Y_t)$ space, holding the interest rate fixed. Put differently, at any given interest rate, the amount of resources demanded increases, meaning that the entire $Y^d$ curve in $(Y_t, r_t)$ space shifts out to the right.
The supply side of the economy is very simple. Income is exogenously given – this is an endowment economy. Hence the supply curve is just \( Y_s = Y_t \), where \( Y_t \) is given. Equilibrium occurs where \( Y_s = Y^d \), or graphically where the curves intersect:

The real interest rate plays the role of price here. The real rate adjusts so as to bring quantity demanded equal to quantity supplied. Once we know the real interest rate, we can plug it into the consumption function to back out the quantity of consumption. For the endowment economy that ends up being a trivial exercise, because we know in equilibrium that \( C_t = Y_t \).

Now that we graphically have this set up, we can look at how the equilibrium changes when
something changes. Here we have one source of a shift in the supply curve – an exogenous change in $Y_t$. We also have one source of a shift in the demand curve – an exogenous change in $Y_{t+1}$. A known (or expected) change in $Y_{t+1}$ is a future supply shock, but because of the forward-looking nature of consumption, it is a source of change in current demand.

Let’s first consider what happens when current supply increases; i.e. $Y_t$ increases exogenously and therefore the $Y^s$ curve shifts out. There is no shift in demand. We can see that the equilibrium real interest rate must fall. We can think about the intuition for this effect as follows. When current income increases, households would like to increase their consumption, but by less than the change in current income. Put differently, they would like to save some of the change in current income. In equilibrium, market-clearing dictates that consumption rise one-for-one with income, however. In order to make households willing to increase their consumption one-for-one with the change in income, the real interest rate has to fall. Think about it the following way: the household would like to increase its saving, but the real interest rate must fall to dissuade them from increasing their saving, since in equilibrium all output must be consumed.

Next, consider what happens when there is a change in future income. Note that this future supply change works as a current demand shifter. Households would like to increase their consumption in response to the change in future income by borrowing. In equilibrium this is not possible, so the real interest rate has to rise to dissuade them from borrowing.
The important point in this analysis is that the real interest rate adjusts to equate current demand for goods with the current supply of goods. Because consumption is forward-looking, in equilibrium the real interest rate ends up being a measure of how plentiful the future is relative to the present. If current income is high relative to the future, the household would like to save to smooth out its income; this is not possible, so the interest rate must fall to dissuade the household from doing this saving.

3.1 Algebraic Example

Let’s use a specific functional form for consumption to solve for the competitive equilibrium. In particular, suppose that the within period utility function is the natural log, \( u(C_t) = \ln C_t \). We derived the consumption function for this utility function in the last set of notes:

\[
C_t = \frac{1}{1 + \beta} Y_t + \frac{1}{1 + \beta} \frac{Y_{t+1}}{1 + \beta} \frac{1}{1 + r_t}
\]

Given this consumption function, the demand side of the economy is characterized by:

\[
Y_t^d = C_t = \frac{1}{1 + \beta} Y_t + \frac{1}{1 + \beta} \frac{Y_{t+1}}{1 + r_t}
\]

Set \( Y_t^d = Y_t \) and solve:
The last line gives an algebraic expression for the demand curve. We can see that it is decreasing in $r_t$ and increasing in $Y_{t+1}$, just as we argued graphically above. Now, to solve for the final equilibrium, we bring in the supply side. $Y_t^s = Y_t$. Set $Y_t^s = Y_t^d = Y_t$ and solve for $r_t$:

$$1 + r_t = \frac{1}{\beta} \frac{Y_{t+1}}{Y_t}$$

This says that the real interest rate that equates demand with exogenously given supply is equal to $\frac{1}{\beta}$ times the ratio of future income to current income. We can plug this market-clearing price back into the consumption function to solve for the market-clearing allocations:

$$C_t = Y_t$$
$$C_{t+1} = Y_{t+1}$$

In equilibrium, all output must be consumed. This is a direct consequence of the assumption that we have an endowment economy with exogenous supply. As argued qualitatively above, the real interest rate has to move around to make people willing to just consume their endowment each period. An increase in current income, $Y_t$, would make people want to increase their saving. This is not possible in equilibrium, so the real interest rate has to fall to dissuade people from saving. In contrast, if future income goes up, households would want to increase current consumption by borrowing. Again, this is not possible in equilibrium, so the real interest rate must rise to offset the desired increase in borrowing such that households are willing to simply consume their endowment.

### 4 Adding a Government

Next we’re going to augment the basic model to include a government. The household side of the economy is identical, subject to the caveat that they have to pay taxes to the government. We still assume that total income is exogenous. The only difference is to include another actor into the economy.

The government chooses to consume resources in each period, which we denote by $G_t$ and $G_{t+1}$. 

\[ Y_t^d = \frac{1}{1+\beta} Y_t^d + \frac{1}{1+\beta} Y_{t+1} + \frac{1}{1+\beta} Y_t^d (1 - \frac{1}{1+\beta}) = \frac{1}{1+\beta} Y_{t+1} \]

\[ Y_t^d (\frac{\beta}{1+\beta}) = \frac{1}{1+\beta} Y_{t+1} \]

\[ Y_t^d = \frac{Y_{t+1}}{\beta(1+r_t)} \]
We do not model the government’s problem – it simply chooses its spending exogenously. We also assume that the household gets no benefit from government spending. In that sense, government expenditure is wasteful. In reality this is too strong of an assumption – most government spending – on public goods like roads, sewage, defense, etc. – is valuable to households. At the expense of complicating the model we could modify household preferences to value government spending, but (at least under some reasonable specifications) it would not fundamentally alter any of our conclusions, so we abstract from this here.

The government must pay for its spending via taxes, which are instituted lump sum. By “lump sum” I mean that the taxes just take a fixed amount of household income – the amount of taxes does not vary with the level of income, as would be the case in a proportional tax system. Just like the household faces a budget constraint, so too does the government. It takes the same basic form, with $S^G$ denoting government saving/borrowing (saving has $S^G > 0$, borrowing has $S^G < 0$). The two within period constraints are:

\[
G_t + S^G = T_t \\
G_{t+1} = T_{t+1} + (1 + r_t)S^G_t
\]

As with the household, the interest rate on government saving/borrowing is $r_t$, which the government takes as given. We make the following assumptions on the environment. The government chooses, at time $t$, a path of its spending, $G_t$ and $G_{t+1}$. Then it has to come up with a sequence of taxes, $T_t$ and $T_{t+1}$, to make both of these constraints hold. There is more than one sequence of taxes that makes both constraints hold: you could run deficits in the first period, $S_t < 0$, and have high taxes in the second period, or a surplus in the first period, $S_t > 0$, and low taxes in the second. An important result that we are about to show is that the sequence of taxes does not matter in equilibrium, a result which is sometimes called Ricardian Equivalence.

Just as we did for the household, we can combine these two constraints into one, what we will call the government’s intertemporal budget constraint:

\[
G_t + \frac{G_{t+1}}{1+r_t} = T_t + \frac{T_{t+1}}{1+r_t}
\]

This has the same interpretation as the household’s intertemporal budget constraint: the present value of the stream of government spending must equal the present value of the stream of government revenue.

The household’s objective is the same – it wants to maximize lifetime utility, $U = u(C_t) + \beta u(C_{t+1})$. Its two within-period budget constraints have to be re-cast slightly to account for taxes. Each period the household has exogenously given income, $Y_t$, and must pay $T_t$ to the government. The within-period constraints are:
\[ C_t + S_t = Y_t - T_t \]
\[ C_{t+1} = Y_{t+1} - T_{t+1} + (1 + r_t)S_t \]

These can be combined to yield the household’s intertemporal budget constraint:

\[ C_t + \frac{C_{t+1}}{1 + r_t} = Y_t - T_t + \frac{Y_{t+1} - T_{t+1}}{1 + r_t} \]

In words this says that the present value of the stream of consumption must equal the present value of the stream of net income, or income minus taxes.

The generic definition of the competitive equilibrium is the same: it is a set of prices and allocations where all agents are optimizing and all markets clear. Here the allocations are a little more interesting, because total income can be split between household consumption and government spending. Markets clearing means that both the household and the government budget constraints must hold. If we combine the two intertemporal budget constraints into one, something pretty cool happens:

\[ C_t + \frac{C_{t+1}}{1 + r_t} = Y_t - T_t + \frac{Y_{t+1} - T_{t+1}}{1 + r_t} \]

The last line reveals a critical result: the household behaves as though the government balances its budget every period, whether the government actually does that or not. Another way to put this is as follows: given a time path of spending, the timing of taxes is irrelevant to the household. Think about this as follows. If the government runs a deficit today, it must pay for the deficit with higher taxes in the future. These higher taxes must cover the first period deficit plus interest plus next period’s spending. Put differently, \( T_{t+1} = G_{t+1} + (1 + r_t)(G_t - T_t) \). In other words, a deficit in the first period \( (G_t - T_t > 0) \), just increases taxes in the future by the amount of the deficit accounting for interest. The household’s period \( t + 1 \) net income is reduced by the amount of the deficit plus interest, which discounted back to the present is just the amount of the deficit. In other words, because the household is forward-looking, the household is indifferent between the government having low taxes in the present and running a deficit versus having high taxes in the present with no deficit. All that matters for the household is the time path of government expenditure.

We call this result **Ricardian equivalence**: the household only cares about the time path of government spending; conditional on that time path, the timing of taxes is irrelevant. This has some potentially important policy implications. Among other things, it tells us that a tax cut, if it
is not matched with some kind of change in spending, will have no effect on household consumption decisions. If the government cuts current taxes, runs a deficit, and makes no promise to change future spending, then households will anticipate higher future taxes. They will react by simply saving the tax cut today to pay for the future tax cut, and so consumption will not be stimulated. At a basic level, a tax cut that is not matched by a spending change requires higher future taxes, and hence does not change the present value of the household’s income, which is ultimately what consumption depends upon.

The household’s Euler equation looks the same as in the model without a government: \( u'(C_t) = \beta(1 + r_t)u'(C_{t+1}) \). To get its consumption function, we would need a functional form assumption. Then we could plug the Euler equation into the intertemporal budget constraint and isolate \( C_t \) as a function of things the household takes as given. Doing this will yield the following consumption function:

\[
C_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)
\]

Here I have imposed the Ricardian Equivalence result: both household and government budget constraints holding means that the household behaves as though the budget is balanced period-by-period, with \( T_t = G_t \). This is what allows me to replace \( T_t \) with \( G_t \) in the consumption function. Consumption will be increasing its first two arguments and decreasing in the real interest rate (the latter by assumption).

Market-clearing requires that household saving be equal to government borrowing: \( S_t = -S^{G}_t \). Plug this into the household’s first period constraint:

\[
C_t - S^{G}_t = Y_t - T_t
\]

Now plug in the definition of government saving/borrowing and simplify:

\[
C_t - (T_T - G_t) = Y_T - T_t
\]

\[
C_t = Y_t - G_t
\]

\[
Y_t = C_t + G_t
\]

This is just the standard accounting identity. The aggregate demand relationship is:

\[
Y^d_t = C(Y^d_t - G_t, Y_{t+1} - G_{t+1}, r_t) + G_t
\]

Here again we see Ricardian Equivalence in action: taxes, \( T_t \) and \( T_{t+1} \), do not show up in the demand relationship.

The derivation of the of the aggregate demand, or \( Y^d \) curve, is the same as above. We could graphically plot \( Y^d \) against \( Y \), and finding possible equilibrium points at the intersection of the 45 degree line at which \( Y^d = Y \), which must be true in any equilibrium. Tracing out levels of \( Y \)
consistent with different interest rates would yield a downward-sloping \( Y^d \) curve, just as we had before.

The addition of government spending – both present and future – brings in some additional sources of demand shifts. First, suppose that there is an increase in \( G_t \). Graphically, in \( (Y^d, Y) \) space we know that this will shift the total demand curve upwards. Even though an increase in \( G_t \) reduces desired \( C_t \) for a given \( Y_t \), the drop in desired \( C_t \) is less than one since the marginal propensity to consume, MPC (the first derivative of the consumption function) is less than one. This means that, on net, the expenditure line shifts up, which works to shift the \( Y^d \) curve out to the right. We can see this graphically below:

Mathematically we can actually figure out by how much the \( Y^d \) curve will shift using the total derivative. Totally differentiating the demand relationship:

\[
dY^d_t = \frac{\partial C_t}{\partial Y_t}(dY_t - dG_t) + \frac{\partial C_t}{\partial Y_{t+1}}(dY_{t+1} - dG_{t+1}) + \frac{\partial C_t}{\partial r_t}dr_t + dG_t
\]

Here \( dY^d_t = Y^d_t - Y^d_0 \), \( dG_t = G_t - G_0 \), etc. – i.e. deviations from an initial point, which I have labeled 0. The partial derivatives, e.g. \( \frac{\partial C_t}{\partial Y_t} \), are the partials evaluated at that point 0. For the though experiment of how much the \( Y^d \) curve shifts to the right, we hold future income, future government spending, and the real interest rate fixed – i.e. \( dr_t = dG_{t+1} = dY_{t+1} = 0 \). Simplifying, and letting \( \frac{\partial C_t}{\partial Y_t} = MPC \), we get:
\[ dY^d_t = MPC(dY^d_t - dG_t) + dG_t \]
\[ (1 - MPC)dY^d_t = (1 - MPC)dG_t \]
\[ dY^d_t = dG_t \]

In other words, this tells us that the horizontal shift in the \( Y^d \) curve is one – i.e., total desired spending increases one-for-one with an increase in government spending, holding the real interest rate constant. There is no “multiplier” effect; for more on this, see the subsection below.

An anticipated increase in future government expenditure will also have an effect on current aggregate demand through its influence on current desired consumption. An increase in future government spending, \( G_{t+1} \), acts like a reduction in future net income. Given the forward-looking nature of households, this leads them to want to reduce their current consumption. In \((Y^d, Y)\) space this shifts the desired spending line down, which in turns shifts the \( Y^d \) curve in and to the right.

Finally, note that changes in taxes, \( T_t \) and \( T_{t+1} \), do not show up anywhere in the demand relationship. This is a result of Ricardian Equivalence, which is an equilibrium phenomenon. When we talked about permanent versus transitory tax cuts in the consumption notes, it was in a non-equilibrium setting. Here, because the government’s budget constraint must hold, forward-looking households behave as though the budget is balanced each period, with \( G_t = T_t \), so that taxes disappear from the consumption function.
Now that we know government spending changes shift the $Y^d$ curve, we are now in a position to examine the equilibrium consequences of changes in current and future government spending. We continue to assume that supply is fixed. First, consider an increase in $G_t$. As noted above, this shifts the $Y^d$ curve out to the right horizontally. In equilibrium, the real interest rate must rise. Since equilibrium output/income is unchanged, consumption falls one-for-one with the increase in government spending. In other words, the increase in government spending simply “crowds out” private spending. The reason the real interest rate rises is that households do not want to cut their consumption back one-for-one with the increase in spending: they would like to reduce their consumption, but only by a fraction of the increase in spending. The real interest rate has to rise to deter them from doing that, so that in equilibrium consumption falls by the full amount of the increase in government spending.

Next, consider an increase in future government spending, $G_{t+1}$. As shown above, this leads households to try to reduce their current consumption, which works to shift the $Y^d$ curve in. The household wants to reduce consumption because they feel poorer – higher government spending means higher taxes, either in the present or in the future. Since current government spending does not change, there is no countervailing force that would shift the $Y^d$ curve out. Hence, the $Y^d$ curve shifts in. Since the $Y^s$ curve is vertical, the real interest rate must decline and the equilibrium level of $Y$ must remain unchanged. Intuitively, what is going on is that households would like to reduce their consumption/increase their saving when they anticipate higher future government spending. In equilibrium this cannot happen, so the real interest rate must decline to discourage the household from engaging in this additional saving. Since current government spending and current output do not change, current consumption cannot change either. The only effect of an increase in $G_{t+1}$ is a lower real interest rate – there is no change in the equilibrium allocations.
4.1 A Note on the Government Spending “Multiplier”

You may recall from principles of macro the notion of the multiplier. The multiplier concept can be applied to any change in desired spending, but is most often associated with government spending. In many principles textbooks you will see a statement to the effect of “An increase in government expenditure will lead to an increase in GDP of a multiple of the increase in government expenditure equal to \( \frac{1}{1 - MPC} \), or one over one minus the marginal propensity to consume.” Since the the MPC is less than 1, the multiplier must be greater than one.

It turns out that there is something similar going on in the model we have laid out, but there are two things that are different. First, forward-looking optimal behavior by households means that there is a direct negative effect of government spending on consumption; this works to lower the actual multiplier. Second, supply is inelastic here, so the real interest rate has to react to the change in government spending so as to leave total output/income unchanged. The derivation of a multiplier of \( \frac{1}{1 - MPC} \) relies on the following two assumptions: (i) there is no direct response of consumption to government spending, and (ii) the real interest rate is fixed.

To see this, let’s start with the demand side of the economy. Mathematically, we have:

\[
Y^d_t = C(Y^d_t - G_t, Y^d_{t+1} - G_{t+1}, r_t) + G_t
\]

We want to mathematically determine how much desired spending would change following an increase in government spending. Take the total derivative of the demand relationship:

\[
dY^d_t = \frac{\partial C_t}{Y^d_t} \left( dY^d_t - dG_t \right) + \frac{\partial C_t}{Y^d_{t+1}} (dY^d_{t+1} - dG_{t+1}) + \frac{\partial C_t}{\partial r_t} dr_t + dG_t
\]

\[dY^d_t = Y^d_t - Y^d_0, \quad dr_t = r_t - r_0, \quad dG_t = G_t - G_0, \text{ etc.} \quad \text{i.e. the changes relative to a particular point, which I’m labeling 0. The partial derivatives - e.g. } \frac{\partial C_t}{\partial Y^d_t} \text{ - are evaluated at that point. Partial derivatives evaluated at a point are just numbers: call } MPC \text{ the partial of consumption with}
\]
respect to current income, the marginal propensity to consume; and let $MPC_{t+1}$ be the martial with respect to future income. Simplifying:

$$dY^d_t = MPC \left( dY^d_t - dG_t \right) + MPC_{t+1} (dY_{t+1} - dG_{t+1}) + \frac{\partial C_t}{\partial r_t} dr_t + dG_t$$

Now, suppose that future income, future government spending, and the real interest rate are held fixed: this means that $dY_{t+1} = dG_{t+1} = dr_t$. Simplify a little further:

$$dY^d_t = MPC \left( dY^d_t - dG_t \right) + dG_t$$

Now solve for $dY^d_t$:

$$(1 - MPC)dY^d_t = (1 - MPC)dG_t$$

$$dY^d_t = dG_t$$

$$\frac{dY^d_t}{dG_t} = 1$$

In words, this says that, when government spending increases by one unit, the total demand for goods also increases by one unit. Put differently, the $Y^d$ curve shifts horizontally one-for-one with the increase in $G_t$. If the real interest rate were held fixed (so that the $Y^s$ curve were perfectly elastic as opposed to perfectly inelastic as here), then output would increase one-for-one with government spending – i.e., the multiplier would be one. Of course, as we saw above, the actual multiplier turns out to be zero in this specification of the model – an increase in government spending does not stimulate output, it just reduces consumption one-for-one. That is an artifact of the assumption that we have an endowment economy in which supply is fixed. If the $Y^s$ curve were upward-sloping (and did not shift in response to an increase in $G$), then output would react to a change in spending, but the multiplier would be bound between 0 and 1 (0 occurring with a vertical $Y^s$ curve, 1 with a horizontal $Y^s$ curve).

How does one arrive at a multiplier greater than one? To get this, assume that the household does not respond directly to government spending:

$$Y^d_t = C(Y^d_t, Y_{t+1}, r_t) + G_t$$

Totally differentiate as above:

$$dY^d_t = MPCdY^d_t + MPC_{t+1} dY_{t+1} + \frac{\partial C_t}{\partial r_t} dr_t + dG_t$$

As above, assume that future income and the real interest rate are held fixed, and then simplify:
\[ dY_t^d = MPCdY_t^d + dG_t \]
\[ (1 - MPC)dY_t^d = dG_t \]
\[ dY_t^d = \frac{1}{1 - MPC}dG_t \]
\[ \frac{dY_t^d}{dG_t} = \frac{1}{1 - MPC} \]

This says that the total demand for goods in the economy would increase by \( \frac{1}{1 - MPC} \) if government spending were to increase by one unit. If the MPC were, say, one-half, the multiplier would be 2, for example. This statement still relies on the real interest rate being fixed. If supply is inelastic as it is in the endowment economy, the multiplier in equilibrium would still be 0 here. To get the government spending multiplier in excess of one, you need (i) the real interest rate to not move and (ii) people to not be forward-looking to the point where they react directly to changes in government spending. In the real world, people are probably not so forward-looking that they react fully and directly to changes in government spending, and supply is completely inelastic. Hence, our finding of a multiplier of 0 in equilibrium is probably too low. But a multiplier of \( \frac{1}{1 - MPC} \) is too high as well.