1 Introduction

In the previous section we studied equilibrium in a particularly simple world – an endowment economy. This was useful for getting intuition about what determined the real interest rate, but lacked realism.

In this section we endogenize supply. The concept of equilibrium is identical to before, but the set up is more interesting. Now we have firms that produce output using capital and labor, and households that consume, save, and supply labor. In addition to the goods market that must clear, we have another market – the labor market. We have several different exogenous variables and can examine how changes in these variables affect the endogenous variables of the model.

2 Firm

As in the Solow Model section, we assume that there are many, identical firms. We can therefore normalize the number of firms to be 1, and call this firm the representative firm. The firm will produce output using capital and labor, and an exogenous productivity shifter which will be called $A$. We abstract from growth, either in population or in the number of effective workers. The firm exists for two periods, $t$ and $t+1$, and wakes up with an exogenously given level of current capital, $K_t$. The firm uses capital and labor to make stuff, and returns its profit to households in the form of dividends.

The production technology is:

$$Y_t = A_t F(K_t, N_t)$$

The function $F(\cdot)$ has the usual properties: increasing and concave in both arguments, and constant returns to scale. An example production function satisfying this functional form is Cobb-Douglas: $F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, $0 < \alpha < 1$. We assume that the firm hires labor on a period-by-period basis from the household at wage rate $w_t$, which both the firm and household take as given.
We depart slightly from our earlier assumption and assume that the firm, rather than the household, owns the capital stock and makes investment decisions. Since the household owns the firm and there are no other frictions, the ownership structure on the capital stock ends up being irrelevant – we would get exactly the same equilibrium whether the firm makes capital accumulation decisions or whether the household makes those decisions. This set-up is perhaps more realistic, however.

The firm’s profit function is given by:

$$\Pi_t = Y_t - w_t N_t - I_t$$

Here investment in new capital, $I_t$, comes out of profit. Think of it this way. The firm produces some output and then pays labor. It can either return this net revenue to the household in the form of a dividend or it can re-invest that profit in new capital, which will help the firm produce in the future. Re-investing today, $I_t$, reduces current dividends but increases the productive capacity in the future, and hence should increase profit in the future. The capital accumulation equation is standard, subject to one twist to be discussed below, where the firm takes its initial level of capital, $K_t$, as given:

$$K_{t+1} = qI_t + (1 - \delta)K_t$$

This expression tells us what capital tomorrow will be given current capital and current investment. The only difference relative to what we saw in the Solow model is the presence of the exogenous variable $q$, which was implicitly normalized to 1 before. $q$ is what we will call an “investment-specific productivity shock.” An increase in $q$ makes the economy more efficient at transforming investment into new capital goods (likewise a decrease in $q$ means the economy is less good at transforming investment into capital). One thing that $q$ could represent is the health of the financial system. The financial system exists to transform investment into capital. When the system isn’t functioning well, $q$ will be low. By not putting a time subscript on it, I am implicitly assuming that $q$ takes on the same value in periods $t$ and $t + 1$. In principle this accumulation equation can be iterated forward one period:

$$K_{t+2} = qI_{t+1} + (1 - \delta)K_{t+1}$$

Recall that the “world ends” after $t + 1$, so the firm will have no interest in leaving any capital over for period $t + 2$. As long as it can do “negative” investment, it will want to set $K_{t+2} = 0$, or $I_{t+1} = -\frac{(1 - \delta)K_{t+1}}{q}$. Think about negative investment in the second period as the firm liquidating itself – it produces output in $t + 1$ using $K_{t+1}$, and after that production there is $(1 - \delta)K_{t+1}$ of capital left over, which the firm returns to the household in the form of a dividend. This presumes that the household can consume the capital. This may seem a little hokey, but is standard and we wouldn’t have to make such a strict assumption in a multi-period version of the model.

The firm’s objective is to maximize the the present value of the dividends that it returns to
The firm discounts future profit/dividend by $1 + r_t$, which serves as the discount factor for goods. The discounted value of profit/dividends can be interpreted as the value of the firm. The maximization problem is dynamic because investing in new capital today reduces the current dividend but increases future dividends. The firm can choose labor input and investment in new capital in both periods. The firm wants to maximize:

$$\max_{N_t, N_{t+1}, I_t, I_{t+1}} V_t = \Pi_t + \frac{1}{1 + r_t} \Pi_{t+1}$$

s.t.

$$K_{t+1} = qI_t + (1 - \delta)K_t$$
$$K_{t+2} = qI_{t+1} + (1 - \delta)K_{t+1}$$

Plugging in the profit function, we get:

$$\max_{N_t, N_{t+1}, I_t, I_{t+1}} V_t = A_t F(K_t, N_t) - w_t N_t - I_t + \frac{1}{1 + r_t} (A_{t+1} F(K_{t+1}, N_{t+1}) - w_{t+1} N_{t+1} - I_{t+1})$$

s.t.

$$K_{t+1} = qI_t + (1 - \delta)K_t$$
$$K_{t+2} = qI_{t+1} + (1 - \delta)K_{t+1}$$

This is a constrained problem. As before, we can take care of the constraints by substituting them into the objective function. In particular, impose the terminal condition discussed above that $I_{t+1} = -\frac{1}{q}(1 - \delta)K_{t+1}$, and eliminate $I_t = \frac{1}{q}(K_{t+1} - (1 - \delta)K_t)$. This allows us to write the problem as one of one of choosing $K_{t+1}$ instead of investment (similarly to how we wrote the household problem as one of choosing future consumption instead of current saving):

$$\max_{N_t, N_{t+1}, K_{t+1}} V_t = A_t F(K_t, N_t) - w_t N_t - \frac{1}{q} (K_{t+1} - (1 - \delta)K_t) + \frac{1}{1 + r_t} \left( A_{t+1} F(K_{t+1}, N_{t+1}) - w_{t+1} N_{t+1} + \frac{(1 - \delta)K_{t+1}}{q} \right)$$

We can characterize the optimum by taking the partial derivatives with respect to the choice variables and setting those partials equal to zero:

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I am using the terms “dividend” and “profit” interchangeably. In practice the firm may not return profit each period, but households get compensated indirectly through capital gains (share price appreciation). As long as there are no credit market imperfections the distinction between a dividend received and a capital gain is immaterial.
\[
\frac{\partial V_t}{\partial N_t} = 0 \iff A_t F_N(K_t, N_t) = w_t \\
\frac{\partial V_t}{\partial N_{t+1}} = 0 \iff A_{t+1} F_N(K_{t+1}, N_{t+1}) = w_{t+1} \\
\frac{\partial V_t}{\partial K_{t+1}} = 0 \iff 1 = \frac{q}{1 + r_t} A_{t+1} F_K(K_{t+1}, N_{t+1}) + \frac{1 - \delta}{1 + r_t}
\]

Two of these first order conditions should look familiar; all have the interpretation as marginal benefit = marginal cost conditions. The first condition simply says that the firm should hire labor up until the point at which the marginal product of labor (marginal benefit) equals the marginal cost of an additional unit of labor (the real wage). The second condition says the same thing for labor in period \(t + 1\). Even though profits in period \(t + 1\) are discounted by \(\frac{1}{1 + r_t}\) in the objective function, because both benefit and cost incur in the same period \((t + 1)\), then no discounting shows up in that first order condition.

The marginal product of labor being equal to the real wage condition (which must hold in the same form in both periods, and is hence sometimes called a static condition) implicitly defines a demand curve for labor. When \(w_t\) goes up, the firm will need to increase the marginal product of labor, which, holding everything else fixed, requires reducing \(N_t\) given the assumed concavity of the production function. The labor demand curve would shift right if \(A_t\) were to increase, while it would shift left if current \(K_t\) were to exogenously decline (say, due to a natural disaster). The labor demand curve, \(N_t^d = N(w_t, A_t, K_t)\), look as follows:

The final optimality condition for the choice of future capital stock looks a little different than what we’ve seen before, but has an intuitive interpretation. It says to equate the marginal cost of investment (foregone current dividends/profits of 1) with the marginal benefit of an extra unit of investment, which is the (i) extra output produced in \(t + 1\) (the marginal product of future capital,
$A_{t+1}F_K(K_{t+1}, N_{t+1})$ and (ii) the salvage or liquidation value of the marginal unit of capital, which is $\frac{1}{q}(1 - \delta)$. Because 1 unit of investment generates $q$ units of future capital, the right hand side is multiplied by $q$. Since all of the benefit of extra investment is received in the future, but the cost is born in the present, the future benefit gets discounted by $\frac{1}{1+r}$. This condition can be re-arranged to yield:

$$r_t + \delta = qA_{t+1}F_K(K_{t+1}, N_{t+1})$$

For given values of $N_{t+1}$ and $A_{t+1}$, this condition says that the desired period $t+1$ capital stock, $K_{t+1}$, is a decreasing function of the real interest rate, $r_t$. This is because of the assumption of concavity – when $r_t$ goes up, the right hand side must go up, which requires making $K_{t+1}$ smaller. Reducing $K_{t+1}$, given a current value of $K_t$, means reducing current investment. Put differently, this first order condition implicitly defines an investment demand curve, $I_t = I(r_t, A_{t+1}, q, K_t)$. As we just discussed, investment will be decreasing in the real interest rate. It will be increasing in future $A_{t+1}$. This is an important point – investment is fundamentally forward-looking – investment today increases productive capacity in the future, so the more productive you expect to be in the future, the more investment you’d like to do in the present. Investment demand will also be increasing in $q$: the bigger is $q$, the more efficient you are at transforming investment into productivity capital.

The investment demand curve also depends on current capital, which is effectively exogenous. The condition above implicitly defines an optimal choice of $K_{t+1}$ which is independent of current $K_t$. But if current $K_t$ is, say, low, then investment would have to be high to achieve a given target level of $K_{t+1}$. Hence, desired investment is decreasing in $K_t$, holding everything else fixed.\(^2\)

We can draw an investment demand curve below. It is downward-sloping in $r_t$, and shifts right if $A_{t+1}$ or $q$ increase or if $K_t$ decreases exogenously (say, due to a hurricane or other natural disaster):

\(^2\)Technically, the level of $N_{t+1}$ is also something that should show up in the investment demand function – the higher $N_{t+1}$, the higher the marginal product of future capital, and hence the more future capital a firm would want, given an interest rate. But since the firm can choose $N_{t+1}$, I omit it as an argument.
Note that the real interest rate here serves not as an *explicit* cost of investment, but rather as an *implicit* cost of investment. As I’ve written the problem, the firm is not borrowing to finance investment. Rather, investment comes out of future dividends, and since the benefit of investment (more profit) is only received in the future, the benefit gets discounted. So \( r_t \) has the interpretation here as the opportunity cost of investment – rather than accumulating more capital, the firm could have returned more current profit to households. I could have also written down the model where I force the firm to finance new capital accumulation through borrowing. It turns out to not matter (under assumptions we have made) how the firm finances itself – essentially here equity (reducing current dividend to yield more in the future) or debt.

### 3 Household

The representative household lives for two periods, consuming and working in both periods. The household problem is similar to what we had before, with the twist that it gets to choose how much to work.

Let the household’s endowment of time each period be normalized to 1. Let \( N_t \) denote how much labor it supplies. Therefore its leisure is \( 1 - N_t \). We assume that leisure is an argument in the utility function, with more leisure leading to more utility (in other words, households don’t like to work). Let \( v(1 - N_t) \) be a function that maps leisure into utility; we assume that it is increasing, \( v'(-) > 0 \), and concave, \( v''(-) < 0 \). An example function satisfying this property is the natural log: \( v(1 - N_t) = \ln(1 - N_t) \). Quick note to avoid confusion: if utility is increasing and concave in leisure, then it is decreasing and convex in labor input (first derivative with respect to \( N_t \) is \( < 0 \), second derivative with respect to \( N_t \) is \( > 0 \)).

We assume that “period utility” is separable in consumption and leisure; this means that utils in period \( t \) are given by \( u(C_t) + v(1 - N_t) \), where \( u(·) \) has the same properties as before. Lifetime utility is just the weighted sum of period utility, with \( \beta \) the weight on future utils:

\[
U = u(C_t) + v(1 - N_t) + \beta(u(C_{t+1}) + v(1 - N_{t+1}))
\]

The household faces a sequence of two within period budget constraints. Conceptually they are the same as we have seen before, but we are no longer in an endowment economy, so income is not exogenous. The household has two sources of income: wage income, \( w_tN_t \), and distributed profits from firms, \( \Pi_t \), which the household takes as given. The household may also have to pay taxes to a government, \( T_t \) and \( T_{t+1} \). The household can use its first period income to consume or to save in bonds, \( S_t \). In the second period it will want to consume all of its income, where its total income comes from labor income, \( w_{t+1}N_{t+1} \), distributed profits from firms, \( \Pi_{t+1} \), and interest on first period savings, \( (1 + r_t)S_t \). The two within period constraints are:
As before, $S_t$ shows up in both period constraints, and so we can eliminate it and combine into one intertemporal budget constraint, which has the same flavor as before but has endogenous total income:

$$C_t + S_t = w_t N_t - T_t + \Pi_t$$

$$C_{t+1} = w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} + (1 + r_t) S_t$$

We can write the household problem as one of choosing, at time $t$, a sequence of consumption, $(C_t, C_{t+1})$, and labor, $(N_t, N_{t+1})$, to maximize its lifetime utility subject to the intertemporal budget constraint. In actuality the household really chooses saving and solves the problem period-by-period, but it works out easier (and identically) to think about them solving the problem this way. As before, the household is a price-taker, and so takes the real interest rate, $r_t$, and the real wage rate, $w_t$, as given. It also takes distributed profits as given.

$$\max_{C_t, C_{t+1}, N_t, N_{t+1}} U = u(C_t) + v(1 - N_t) + \beta(u(C_{t+1}) + v(1 - N_{t+1}))$$

s.t.

$$C_t + \frac{C_{t+1}}{1 + r_t} = w_t N_t - T_t + \Pi_t + \frac{w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}$$

This is a constrained, multivariate optimization problem. To handle it, we will solve for $C_{t+1}$ from the constraint, plug that in, and thereby transform the problem to an unconstrained problem in only three choice variables. Solve for $C_{t+1}$, we get:

$$C_{t+1} = (1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}$$

Plug into the objective function:

$$\max_{C_t, N_t, N_{t+1}} U = u(C_t) + v(1 - N_t) + \beta(u((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}) + v(1 - N_{t+1}))$$

Let’s take the derivatives with respect to the choice variables to find the conditions characterizing optimal behavior:
\[ \frac{\partial U}{\partial C_t} = 0 \iff u'(C_t) + \beta u'((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1})(-1 + r_t)) = 0 \]

\[ \frac{\partial U}{\partial N_t} = 0 \iff -v'(1 - N_t) + \beta u'((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1})(1 + r_t) w_t = 0 \]

\[ \frac{\partial U}{\partial N_{t+1}} = 0 \iff \beta u'((1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}) w_{t+1} - \beta v'(1 - N_{t+1}) = 0 \]

These conditions can be simplified and made more interpretable by substituting back in for \( C_{t+1} \).

Doing so:

\[ u'(C_t) = \beta (1 + r_t) u'(C_{t+1}) \]

\[ v'(1 - N_t) = w_t \beta (1 + r_t) u'(C_{t+1}) \]

\[ v'(1 - N_{t+1}) = w_{t+1} u'(C_{t+1}) \]

We can actually simplify the second condition by nothing that \( \beta (1 + r_t) u'(C_{t+1}) = u'(C_t) \). Then we can write the first order conditions as:

\[ u'(C_t) = \beta (1 + r_t) u'(C_{t+1}) \]

\[ v'(1 - N_t) = w_t u'(C_t) \]

\[ v'(1 - N_{t+1}) = w_{t+1} u'(C_{t+1}) \]

You will note that the first condition is just the consumption Euler equation that we’ve seen before. This says to equate the marginal benefit of consuming a little more today, \( u'(C_t) \), with the marginal cost, which takes the form of foregone future consumption which reduces future utility, \( \beta (1 + r_t) u'(C_{t+1}) \). The second two conditions take identical forms for each period, \( t \) and \( t + 1 \). These also have the interpretation as marginal benefit equals marginal cost conditions. Suppose that the household chooses to consume one additional unit of leisure in period \( t \). The marginal benefit of this change is the marginal utility of leisure, or \( v'(1 - N_t) \). The marginal cost of this is foregone consumption – working less means you have \( w_t \) fewer units of income to consume, which reduces utility by \( w_t u'(C_t) \). Hence, the right hand side is the marginal cost of an additional unit of leisure today.

We can analyze these first order conditions graphically with an indifference curve-budget line diagram. Because income is now endogenous, we have to hold everything else fixed outside of the graph. The first indifference curve-budget line diagram is the familiar one with current consumption, \( C_t \), on the horizontal axis and future consumption, \( C_{t+1} \), on the vertical axis. The budget line can be derived from the intertemporal budget constraint, holding \( N_t \) and \( N_{t+1} \) fixed. The slope is \(- (1 + r_t)\), and the optimality condition is where the indifference curve and the budget line are tangent.
We can also draw an indifference curve-budget line diagram for the choice of leisure/labor and consumption. We will do the picture with $C_t$ on the vertical axis and $1 - N_t$ on the horizontal axis. The plot is complicated by the fact that there is an upper bound on leisure – it cannot be in excess of 1, which occurs when $N_t = 0$. This is going to introduce a kink into the budget constraint. Solve for $C_t$ from the budget constraint in terms of everything else:

$$C_t = \frac{C_{t+1}}{1 + r_t} + w_t N_t - T_t + \Pi_t + \frac{w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}$$

The derivative of $C_t$ with respect to $N_t$ is: $\frac{\partial C_t}{\partial N_t} = w_t$. Now this is itself complicated by the fact that we don’t have $N_t$ on the horizontal axis in the budget line diagram we want to draw – we have $1 - N_t$. This does not complicate things much, since $\frac{\partial C_t}{\partial (1 - N_t)} = -\frac{\partial C_t}{\partial N_t}$. Hence, the slope of the budget line is $-w_t$, which is the price of leisure in terms of foregone consumption – if you consume an additional unit of leisure, you earn $w_t$ fewer units, and thus have $w_t$ fewer units available to consume.

The budget line ends up having a kink, at $1 - N_t = 1$, or $N_t = 0$. This means that the household could locate at the kink, in which case the first order condition derived above is no longer necessary. Basically, if the household were sufficiently willing to substitute leisure across time (essentially $v(\cdot)$ close to linear) and there were a big differential in wages across time, the household may choose to work all in one period. We are going to rule that out, and assume that we are always at an interior solution away from the kink – the household always works some. The graphical optimality condition is shown below:

What happens when there is an increase in the real wage, $w_t$? This would lead the budget line to become steeper, but the kink occurs at the same point. Similarly to the effect of an increase in the real interest rate in the current/future consumption diagram, there turn out to be competing income and substitution effects at work. The substitution effect of a higher real wage says to work more (consume less leisure). But the income effect says to work less – a higher real wage $w_t$ makes
you feel richer, which makes you want to consume more goods and more leisure (work less). Hence, there are competing income and substitution effects, and the effect of a change in the real wage on labor income is ambiguous. This means that the labor supply curve could be upward or downward sloping.

We are going to assume that the substitution effect dominates, so that labor supply is increasing in the real wage. We can see this for a particular utility function, logarithmic over both consumption and leisure. The optimality condition would be:

\[ \frac{1}{1 - N_t} = \frac{1}{C_t} w_t \]

Now, suppose that there is an increase in the real wage, \( w_t \). For a given level of labor input, this would represent an increase in income of \( w_t \) units. We know from early that the marginal propensity to consume out of current income is less than one. This means that, if \( w_t \) increases, \( C_t \) would increase for a given real interest rate, but less than one for one. Put differently, when \( w_t \) goes up, the right hand side gets bigger since \( C_t \) goes up by less \( w_t \). If the right hand side goes up, the left hand side must go up for the optimality condition to hold. To get the left hand side to go up, we’d need \( 1 - N_t \) to get smaller, which requires \( N_t \) getting bigger. Hence, \( N_t \) would rise when \( w_t \) goes up, holding everything else fixed. For the remainder of the course, we assume that the utility function is such that this substitution effect dominates so that \( N_t \) is increasing in \( w_t \).

We plot the labor supply curve as upward-sloping in \( W_t \), with \( N_t \) on the horizontal axis. In principle, the curve will shift with anything other than \( w_t \) which affects \( C_t \). The thing on which we are going to focus that shifts the curve is the real interest rate, \( r_t \). An increase in \( r_t \) would make \( C_t \) fall, other things being equal. This would make \( u'(C_t) \) bigger, other things being equal. For a given \( w_t \), this would necessitate \( u'(1 - N_t) \) getting bigger, so \( 1 - N_t \) needs to get smaller, or \( N_t \) needs to get bigger. In other words, holding \( w_t \) fixed, an increase in \( r_t \) leads to an increase in the quantity of labor supplied – the labor supply curve shifts to the right. you can think about the intuition for this as follows. When \( r_t \) goes up, the return on savings goes up. This makes you want to increase your saving. In the endowment economy model, the only way to do this was to reduce consumption. Now that income is endogenous, in addition to cutting consumption, you can also increase income by working more. Hence, you respond to an increase in the real interest rate by wanting to reduce consumption and increase labor supply.

To be fully correct, anything else which would affect \( C_t \) holding \( w_t \) fixed would also shift the curve. This would include things like future income, government spending, etc – anything which affects non-wage income. We are going to assume these issues away. This is not quite correct, but it makes our lives much easier in that the labor supply curve won’t shift unless productivity changes, and it does not change any of our primary conclusions. Note: this is different than I have done this in the past, where I assumed that we went with the “full bore” labor supply curve. What we are doing now is simpler – I’m trying to make your lives easier.

Graphically, the labor supply curve is given by \( N^*(w_t, r_t) \) – it is upward-sloping in the wage (because we assume that the substitution effect dominates) and it shifts right when \( r_t \) increases:
4 The Government

The government problem is simple and is identical to what we’ve already had. The government exogenously picks a sequence of spending, $G_t$ and $G_{t+1}$, and taxes must adjust to make the government budget constraint hold.

The government faces two within period constraints:

\[
G_t + S_t^G = T_t \\
G_{t+1} = T_{t+1} + (1 + r_t)S_t^G
\]

These can be combined into one intertemporal budget constraint:

\[
G_t + \frac{G_{t+1}}{1 + r_t} = T_t + \frac{T_{t+1}}{1 + r_t}
\]

Since the household knows that the government budget constraint must hold, the household intertemporal budget constraint can be written:

\[
C_t + \frac{C_{t+1}}{1 + r_t} = w_tN_t - G_t + \Pi_t + \frac{w_{t+1}N_{t+1} - G_{t+1} + \Pi_{t+1}}{1 + r_t}
\]

In other words, the household will behave as though the government balances its budget period-by-period, just as in the earlier equilibrium notes. That is, Ricardian equivalence continues to hold. This means that changes in taxes that are not matched by changes in the time path of government spending will have no impact on the economy.

As noted earlier, we are defining household wealth, $B_t$, as anything which affects household income other than current wage income. From above, this is given by:
\[ B_t = -G_t + \Pi_t + \frac{w_{t+1}N_{t+1} - G_{t+1} + \Pi_{t+1}}{1 + r_t} \]

In other words, household wealth depends negatively on current or future government spending (because this implies an increase in taxes), positively on expected future labor income, and positively on current and expected dividends.

5 The Output Supply and Demand Curves and Equilibrium

Now we are going to take the decision problems of the household and firm and combine them to derive aggregate demand and supply curves. The demand side is slightly more complicated than what we previously saw due to the presence of investment, but the derivation is basically the same. An endogenous supply curve is what is new.

The aggregate supply curve, or what we will call the \( Y^s \) curve, is the set of \( (r_t, Y_t) \) pairs consistent with household and firm optimization on the production side of the economy and with labor market-clearing. The labor market clearing means that the quantity of labor demanded must equal the quantity of labor supplied, given a real interest rate. We can characterize this graphically by combining the labor demand and supply curves from above:

As noted above, the real interest rate enters into the production side of the economy through the effect on labor supply. An increase in the real interest rate, holding everything else fixed, would lead households to want to reduce consumption and increase labor supply. An increase in labor supply leads to more labor input in equilibrium. More labor input leads to more output. Hence, the \( Y^s \) curve will be upward sloping.
The figure above shows a graphical derivation of the $Y^s$ curve. This is a four part graph. First, start in the upper right diagram with some real interest rate, $r^0_t$. This gives a position of the labor supply curve, which determines a value of total employment in the upper left diagram, $N^0_t$, at the intersection of labor demand and supply. To save on notation, in labeling the curves I suppress explicit dependence on other variables. The quantity of employment at which the labor market is in equilibrium can be read down to a graph of the production function in the lower left diagram, where that graph holds $A_t$ and $K_t$ fixed. This gives a value of $Y^0_t$. The lower right diagram is just a 45 degree line diagram with $Y_t$ on both axes and is used to “reflect up” the value of $Y_t$ that one gets from the production function diagram. Hence, $(r^0_t, Y^0_t)$ is a point on the $Y^s$ curve.

To derive a new point on the curve, suppose that the real interest rate is higher, say $r^1_t$. A higher real interest rate will lead the labor supply curve to shift right. This leads to a higher level of employment consistent with labor market clearing, $N^1_t$, and hence a higher level of output, $Y^1_t$, from the production function. We could do the same for a lower real interest rate, and we would find a lower level of output. Connecting the points, we get an upward-sloping $Y^s$ curve. I draw it as a (more or less) straight line, but it need not be. Qualitatively, we just know that it is upward-sloping.

The derivation of the $Y^d$ curve is similar to before. It is again complicated by the fact that the total amount of consumption depends on the total amount of income. In particular, the consumption function will again take the form: $C_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)$, where government
spending replaces taxes because of Ricardian equivalence. In this problem income is endogenous, but to maintain consistency with what we’ve done before we write consumption as a function of total income (which ultimately gets paid to households anyway either in the form of wages or distributed profit/dividends). The total demand for goods comes from combing the period budget constraints of the different actors in the economy. These are:

\[ C_t + S_t = w_t N_t - T_t + \Pi_t \]
\[ \Pi_t = Y_t - w_t N_t - I_t \]
\[ G_t + S^G_t = T_t \]

In any equilibrium it must be that \( S^G_t = -S_t \). Hence, imposing this, along with the definition of profits, we get a standard looking accounting identity:

\[ Y^d_t = C_t + I_t + G_t \]

Now plug in the optimal decision rules derived above:

\[ Y^d_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) + I(r_t, A_{t+1}, q, K_t) + G_t \]

As noted, we again have the thorny issue that \( Y^d_t \) depends on \( Y_t \). In any equilibrium, however, \( Y_t = Y^d_t \). We can graphically plot \( Y^d_t \) as a function of \( Y_t \), relying on the fact that we know the MPC, or the derivative of the consumption function with respect to current income, must be between 0 and 1. This means that there will graphically exist a point where \( Y^d_t = Y_t \). We can derive the \( Y^d \) curve, which is the set of \( (r_t, Y_t) \) points consistent with agents optimizing and \( Y^d_t = Y_t \), but picking out different real interest rates. A higher real interest rate causes the expenditure line to shift down, because higher \( r_t \) lowers demand both for consumption and investment for any level of current income. This shifts the expenditure line down, which means the point at which \( Y_t = Y^d_t \) is smaller. The reverse is true for a lower real interest rate. Connecting the dots, you get a downward-sloping \( Y^d \) curve.

Note that I also am putting total income, \( Y_t \), as the argument in the consumption function, instead of breaking it into its constituent components. This is to maintain congruity with what we’ve already done, but also the household ultimately receives all income in the economy, since it owns the firm. Household consumption decisions depend on total income, not just labor income.
The full, general equilibrium of the economy occurs where we are on both the $Y^d$ and $Y^s$ curves. One way to think about it is that each of these curves represents a partial equilibrium in a sub-market. The $Y^s$ curve is based off of the labor market clearing, and traces out different levels of output that obtain for different real interest rates, which works through the labor supply curve. The $Y^d$ curve represents the “goods” market clearing, tracing out different points where income equals expenditure, $Y_t = Y^d_t$. In general equilibrium, both the goods and the labor markets must clear. Put differently, we must be on both the $Y^s$ and $Y^d$ curves. This occurs at the point where they intersect:
The way to think about this figure is as follows. The plot with the $Y^d$ and the $Y^s$ curves is the “main” diagram, with each curve coming from a different sub-market, with the real interest rate being the thing that connects the sub-markets. The $Y^s$ curve shows all $(r_t, Y_t)$ pairs where the labor market is in equilibrium, $N^s = N^d$, while the $Y^d$ curve shows all $(r_t, Y_t)$ pairs where the goods market is in equilibrium, $Y^d_t = Y_t$.

### 6 Responses to Shocks

Now that we have derived a graphical depiction of the competitive equilibrium in the economy, we are in a position to look at how that equilibrium changes when any of the exogenous variables changes. Although one could entertain multiple exogenous variables changing at the same time, we
will study these changes in isolation. The five exogenous variables that we’ll look at are current
and future total factor productivity, \( A_t \) and \( A_{t+1} \); current and future government expenditure, \( G_t \) and \( G_{t+1} \); and investment-specific productivity, \( q \). We could also look at exogenous reductions in
the current capital stock (\( K_t \)) or changes in uncertainty over the future, but will not do so in the
notes.

These exercises can get rather difficult, because there are many curves shifting, and in most
cases the exogenous changes do not neatly align solely into only “supply” or “demand.” For this
reason I think it is important to slowly walk through these changes in the graphs, and to follow an
ordered approach to looking at how the changes work their way through the economy. I always do
these in the following steps:

1. Always start in the labor market and hold the interest rate fixed. Ask yourself whether the
labor demand curve shifts. This occurs if current total factor productivity, \( A_t \), or the current
capital stock, \( K_t \) were to change. If the labor demand curve shifts, figure out the new level
of \( N_t \) that would clear the labor market (demand=supply) for a given interest rate.

2. Take the new hypothetical level of \( N_t \) for a given interest rate, and “bring it down” to the
production function graph. The production function itself will shift up or down if \( A_t \) or \( K_t \)
change. This will tell you the level of output that would be produced for a given interest rate.

3. Given the new level of \( Y_t \), “take it across” to the 45 degree line and then “reflect it up” to
figure out how the \( Y^s \) curve shifts (if at all). If your work shows that \( Y_t \) would be higher for
a given interest rate, then the \( Y^s \) shifts out.

4. Next, ask yourself how, if at all, the \( Y^d \) curve shifts. Remember, \( Y^d \) shifts occur whenever
the quantity demanded would change for a given interest rate. This will happen if there is a
change in current or future government spending, a change in future total factor productivity,
or a change in investment-specific productivity.

5. Combine the shifts in \( Y^d \) and \( Y^s \) to find the new equilibrium levels of \( Y_t \) and \( r_t \) at the new
point of intersection. At this stage stop and ask yourself what happens to the components of
output – consumption and investment.

6. “Work your way back” down and to the left to get back to the labor market, making sure
that all the quantities in the four part picture make sense. In particular, the change in the
equilibrium real interest rate in the will induce a shift in labor supply. This shift in labor
supply makes the quantity of \( N_t \) coming from the labor market consistent with the equilibrium
quantity of output.

7. “Work your way up” up to the \( Y^d - Y \) graph. The change in the interest rate will shift the
expenditure line in such a way that the intersection of the expenditure line with with \( Y^d = Y \)
line will coincide with the equilibrium level of \( Y_t \) found at the intersection of the \( Y^d \) and \( Y^s \)
curves.
8. In some instances there will be some ambiguity about what is happening. Often times you can use math to resolve the ambiguity.

9. Even though I’ve described all this as if it happens sequentially, which is good for getting intuition, in reality this all happens (more or less) simultaneously.

In doing all this I’m going to be shifting a bunch of curves simultaneously. To make things easier to read, I’m going to employ the following color coding scheme. Black lines denote the “pre-shock” state of the system. Blue lines represent curve shifts due to the immediate impact of the exogenous change (steps (1)-(4)). Red lines represent curve shifts in labor supply or the expenditure line that result because of a change in the interest rate (steps (6)-(7)). Original quantities and prices are represented by superscript 0. New equilibrium quantity and supply are represented by superscript 1.

One final thing that we should note when thinking about this. Any shock which affects \( I_t \) in equilibrium will lead to more or less \( K_{t+1} \), which affects \( Y_{t+1} \). In principle, this would have an effect on \( C_t \) in the present, which would then filter into an effect on the \( Y^d \) curve. We assume this away, appealing to the notion that any change in investment today will have a sufficiently small effect on the capital stock tomorrow so as to be ignored. In other words, think about \( Y_{t+1} \) as effectively being exogenous.

### 6.1 Increase in \( A_t \)

Let’s first do an exogenous increase in \( A_t \), holding everything else fixed. Follow the formula above. We start in the labor market. An increase in \( A_t \) will shift the labor demand curve out and to the right – this is because a higher \( A_t \) raises the marginal product of labor. There is no direct effect on the labor supply curve. Labor demand shifting out with no change in supply means that the quantity of labor would increase for a given real interest rate.

Next, “bring this down” to the production function graph, also noting that the production function itself will shift up when \( A_t \) increases. More employment plus a higher production function means a higher quantity of \( Y_t \). “Taking this across” to the 45 degree line and then “reflecting up” we see that the \( Y^s \) curve must shift to the right.

There is no shift in the \( Y^d \) curve. This may seem non-intuitive, but nothing in the \( Y^d \) curve depends directly on \( A_t \) (only indirectly through \( Y_t \), which would represent a movement along the curve, not a shift of it). This means that, in the new equilibrium, \( r_t \) will be lower and \( Y_t \) will be higher. \( r_t \) lower and \( Y_t \) higher mean that both consumption and investment will be higher; this means that the expenditure line will shift up, intersecting the \( Y^d = Y \) further to the right at the new equilibrium level of \( Y_t \). Now we have to “work our way back” to the labor market graph to make everything consistent. The actual change in output is smaller than the horizontal shift of the \( Y^s \) curve, due to the fact that the \( Y^d \) curve is not perfectly horizontal. Working back, we see that this must mean that the labor supply curve shifts in relative to its initial position such that that the new equilibrium quantity of output is consistent with the production function. Effectively, the lower real interest rate discourages people from working as much.
The graph above shows how everything works out. Remember: black lines are original positions, blue lines are the “intermediate” step of showing how the increase in $A_t$ directly affects the curves, and red lines show the final positions of the curves after the real interest rate is determined. We can see that output ends up higher and the real interest rate ends up lower. Output/income higher and the real interest rate lower mean that consumption will be higher. A lower real interest rate means that investment is also higher. In the labor market we can see that the real wage is also definitely higher. The equilibrium effect on employment is ambiguous – supply shifts in, demand shifts out, leading to an ambiguous change in quantity. In the picture I actually show $N_t$ not changing it all, though it could rise or fall.\footnote{We know that output must rise; this could occur even with a drop in $N_t$ because $A_t$ is higher.}
To summarize the analysis:

1. We start in the labor market. Higher $A_t$ leads to more demand for labor. Holding the real interest rate fixed, there is no direct effect on labor supply. This means more $N_t$ for a given $r_t$.

2. The production function itself shifts up when $A_t$ increases. More $A_t$ plus more $N_t$ means higher $Y_t$ for a given $r_t$.

3. We “bring this over” to the 45 degree line and “reflect it up” and see that the $Y^s$ curve must shift right.

4. There is no direct effect of $A_t$ on $Y^d$. The demand for goods depends on current income, but that’s picked up in the way we derived the curve. We assume that the effects of higher $A_t$ today on future income are sufficiently small to ignore them, and so $Y^d$ does not shift.

5. $Y^s$ shifting out, along with no shift in $Y^d$, means that $r_t$ falls and $Y_t$ goes up.

6. Lower $r_t$ causes the labor supply curve to shift inward. The real wage definitely rises with an ambiguous effect on employment.

7. The lower $r_t$ causes the expenditure line to rise, so that it intersects the $Y^d – Y$ line at the correct equilibrium level of $Y_t$.

### 6.2 Increase in $A_{t+1}$

Next, consider an exogenous but anticipated increase in $A_{t+1}$, which becomes known in period $t$. As before, start in the labor market. Because the marginal product of labor only depends on current $A_t$, an anticipated increase in $A_{t+1}$ does not shift the labor demand curve. This means there is no effect on the labor market. Since the production function does not depend on $A_{t+1}$ either, then the $Y^s$ curve is not affected.

Now we need to ask ourselves: what happens to the $Y^d$ curve? Here, we see that the $Y^d$ curve must shift right. Higher $A_{t+1}$ increases firm demand for investment, since the future marginal product of capital will be higher. This makes the $Y^d$ curve shift right.\(^5\)

The demand curve shifting out with the supply curve not shifting means that $r_t$ is higher. This makes sense from our earlier discussion in the endowment economy – the future is plentiful relative to the present, both the real interest rate and output rise in the new equilibrium. In order to produce more output, there must be more labor input (since $A_t$ and $K_t$ are not changing). The higher interest rate induces an outward shift in labor supply. So we end up with higher $N_t$ and a lower $w_t$.

The picture below shows the effects at work:

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\(^5\)Higher $A_{t+1}$ would also translate into higher $Y_{t+1}$, which would stimulate current consumption demand.
We know from the picture that output is higher, the real interest rate is higher, the wage is lower, and labor input is higher. What about consumption and investment? We can’t necessarily sign those effects – we know that at least one of them must be higher (since $G_t$ is not changed and $Y_t$ is higher), but it’s possible that one or the other could fall – higher $A_{t+1}$ makes $I_t$ higher, other things being equal, and higher $Y_t$ (and expected higher $Y_{t+1}$) would make $C_t$ bigger. but the higher real interest rate works in the opposite direction for both. Hence, the effects on these variables are ambiguous.

To summarize:

1. Start in the labor market. There is no direct effect of a change in $A_{t+1}$.

2. There is also no change in the production function. Hence, there is no effect on the $Y^*$ curve.
3. $Y^d$ shifts out. Firms would like to do more investment (for a given interest rate) because $A_{t+1}$ is higher.

4. $Y^d$ shifting out with no change in $Y^s$ means that $Y_t$ and $r_t$ both rise.

5. $r_t$ rising shifts labor supply out, so that we end up with $N_t$ higher and $w_t$ lower.

6. It is ambiguous what happens to $C_t$ and $I_t$, though we know that $C_t + I_t$ must rise, since $Y_t$ rises with no change in $G_t$.

6.3 Increase in $q$

Now, let’s consider an increase in $q$, the level of investment-specific productivity. This has no direct effect on the labor market – it doesn’t affect labor demand, and it doesn’t affect labor supply. There is also no effect on the production function. Hence, there is no effect on the $Y^s$ curve. There is an effect on the $Y^d$ curve. Higher $q$ stimulates demand for investment. This shifts up the desired expenditure line, which leads to an outward shift in $Y^d$. Combining the outward shift in $Y^d$ with no change in $Y^s$, we observe that the equilibrium level of output must rise along with the real interest rate. The higher real interest rate stimulates labor supply, so that the new quantity of labor is higher (which must be the case to produce more output). This means that the real wage is lower. The higher real interest rate partially offsets the upward shift in the expenditure line, so that it shifts slightly down so that the quantities of $Y$ line up. So we see an increase in output, a higher real interest rate, a lower wage, and more labor. The picture below shows things going on:
Now, with a higher real interest rate but also higher $Y_t$, the effect on $C_t$ appears ambiguous. Likewise, higher $r_t$ will work to offset the effect of $q$ on $I_t$, so that appears ambiguous. We actually know that $I_t$ has to increase, however. We can make this argument graphically. Suppose that the $Y^s$ curve were perfectly vertical – then we would know that $Y_t$ wouldn’t be affected. Higher $r_t$ with no change in $Y_t$ would mean $C_t$ would be lower. But $C_t$ lower with $Y_t$ unchanged would mean $I_t$ is higher. So even in the extreme case of a perfectly vertical $Y^s$ curve, $I_t$ would rise. This means that we can confirm that $I_t$ must rise as long as the $Y^s$ curve is not downward-sloping (which we rule out by assumption). We cannot say with certainty what happens to $C_t$.

Summing up:

1. There is no direct effect of $q$ in the labor market. Hence there is no shift in $Y^s$. 

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2. Higher $q$ stimulates demand for investment, which shifts the $Y^d$ curve right.

3. This leads to higher output and a higher real interest rate in the new equilibrium.

4. A higher real interest rate leads to an outward shift of labor supply. Hence, we end up with more $N_t$ and a lower $w_t$.

5. The higher $r_t$ partially undoes the upward shift in the expenditure line; it shifts down slightly so that the quantities of $Y$ line up.

6. The effects on $C_t$ is ambiguous, but we can reason that $I_t$ must rise.

6.4 Increase in $G_t$

Next consider an exogenous increase in government spending, $G_t$. We’ve already studied this, but in a context in which supply was fixed. Here, the dynamics are more interesting.

As always, start in the labor market. There is no effect on labor demand, and no effect on the production function. Hence, there is no effect on the $Y^s$ curve.

We know that there is an effect on the $Y^d$ curve. In particular, an increase in $G_t$ increases the overall demand for goods and services, leading to an outward shift of the $Y^d$ curve. We actually know something about the magnitude of this shift – in particular, the horizontal shift in the demand curve is one for one, which one can show using the total derivative (this is exactly the same as in the endowment economy notes). Hence, the $Y^d$ curve shifts out. This means that $Y_t$ must increase and $r_t$ must increase as well. Higher $r_t$ stimulates labor supply, which is necessary to produce more output. Hence, we end up with $Y_t$ higher, $r_t$ higher, $N_t$ higher, and $w_t$ lower.

The picture below shows all this working out:
Now, what happens to the components of output? Since \( r_t \) rises and \( A_{t+1} \) and \( q \) don’t change, we know that \( I_t \) must fall. What about \( C_t \)? Here, we need to make use of a result we had from earlier. In particular, mathematically, we know that there is a one-for-one outward shift in the \( Y^d \) curve when \( G_t \) goes up. This means that, if \( r_t \) did not change (e.g. the \( Y^s \) curve were perfectly flat), then \( Y_t \) would go up one-for-one with \( G_t \). Since we observe that \( r_t \) does go up, we know that \( Y_t \) must go up by less than the increase in \( G_t \). In other words, the equilibrium “multiplier” must be less than one – \( Y_t \) goes up by less than the increase in \( G_t \). This is a direct result of Ricardian Equivalence in which the household behaves as though the government balances its budget each period. This means that the first argument in the consumption function, \( Y_t - G_t \), goes down when \( G_t \) goes up. Combining this with \( r_t \) going up, we know then know that \( C_t \) must fall. In other words, the increase in government spending must “crowd out” private expenditure, with both \( C_t \)
and \( I_t \) declining.

To sum up:

1. Start in the labor market. The increase in \( G_t \) has no direct effect.

2. Hence, there is no effect on the \( Y^s \) curve.

3. We know from earlier that the \( Y^d \) curve also must shift right. In fact, mathematically we can show that it shifts right one-for-one with the increase in \( G_t \).

4. This means that \( r_t \) and \( Y_t \) must be higher.

5. Higher \( r_t \) shifts the labor supply curve right, so we end up with higher \( N_t \) and lower \( w_t \).

6. Higher \( r_t \) means that \( I_t \) must decline. Since \( Y_t \) goes up by less than \( G_t \) and \( r_t \) increases, we also know that \( C_t \) must decline.

### 6.5 Increase in \( G_{t+1} \)

Finally, suppose that we have an increase in \( G_{t+1} \) that is anticipated by agents in the model in period \( t \).

As always, start in the labor market. There is no direct effect on either labor demand or supply. There is also no effect on the production function. Hence, there is no shift in the \( Y^s \) curve.

Next, let’s think about what happens to the \( Y^d \) curve. From our earlier analysis we know that the \( Y^d \) curve will shift in – households will want to reduce consumption because they feel poorer, which leads to an inward shift in the quantity of goods demanded. Since \( Y^d \) shifts in, we know that \( r_t \) and \( Y_t \) must both fall. Lower \( Y_t \) means lower \( N_t \), which occurs because the lower \( r_t \) discourages labor supply.

We can see these effects below:
Since \( r_t \) is lower but \( A_{t+1} \) and \( q \) are unchanged, it must be the case that \( I_t \) is higher. But since \( G_t \) is unaffected and \( Y_t \) is lower, it has to be the case that \( C_t \) is lower – even though \( r_t \) is lower, the combination of lower \( Y_t \) and higher future \( G_{t+1} \) work to reduce consumption on net.

In summary:

1. The increase in future government spending has no direct effect on the labor market, and hence no effect on \( Y^s \)

2. The \( Y^d \) curve shifts in – households feel poorer, and would therefore like to lower \( C_t \) for a given \( r_t \).

3. This means that \( r_t \) and \( Y_t \) both fall
4. Lower $r_t$ means that $I_t$ is higher. But since $Y_t = C_t + I_t + G_t$, this means that $C_t$ must fall.

6.6 Summarizing Effects

Below is a table that summarizes the qualitative effects of changes in each of these four exogenous variables (where the changes occur in isolation). “+” means that the endogenous variable increases, a “−” means that the endogenous variable decreases, and a “?” means that the effect is ambiguous.

<table>
<thead>
<tr>
<th>Variable:</th>
<th>↑ $A_t$</th>
<th>↑ $A_{t+1}$</th>
<th>↑ $q$</th>
<th>↑ $G_t$</th>
<th>↑ $G_{t+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>Hours</td>
<td>?</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>Consumption</td>
<td>+</td>
<td>?</td>
<td>?</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>Investment</td>
<td>+</td>
<td>?</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>Real interest rate</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>Real wage</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

For negative changes in the exogenous variables (declines), you just switch the sign in the table.