Extra Questions: Growth

Intermediate Macroeconomics, Fall 2015
The University of Notre Dame
Professor Sims

Instructions: The following are extra questions for practice and review. The topics covered are growth and the Solow Model.

1. If a series, \( X_t \), grows at a constant rate \( g \), then in \( h \) periods its value will be:

\[
X_{t+h} = (1 + g)^h X_t
\]

(a) Suppose that \( g = 0.02 \) and \( h = 4 \). If \( X_t = 1 \), what will \( X_{t+4} \) be equal to?

(b) What is the cumulative growth rate between \( t \) and \( t+h \) in this example (i.e. \( \frac{X_{t+h}-X_t}{X_t} \))? 

(c) Suppose that you took logs and used the approximation that \( \ln(X_{t+h}) \approx \ln(X_t) + g \). If \( g = 0.02 \) and \( h = 4 \), what would this approximate formula predict the growth rate to be? Is it close to what you get in (b)? What accounts for any discrepancy?

(d) Suppose that \( g = 0.02 \) but \( h = 40 \). If \( X_t = 1 \), what will \( X_{t+40} \) be equal to? What is the cumulate growth between \( t \) and \( t+40 \)?

(e) Use the log approximation to calculate the approximate percentage growth between \( t \) and \( t+40 \). Is the discrepancy in the approximation here bigger or smaller than what you found in (c)? Why?

2. Write down the five stylized time series growth facts.

3. Write down the three stylized cross-sectional growth facts.

4. Assume that the production function is Cobb-Douglas over capital and labor. Assume that labor supply is constant at 1 and that there is no exogenous productivity growth. The central equation of the Solow model is:

\[
K_{t+1} = sAK_t^\alpha + (1 - \delta)K_t, \quad 0 < \alpha < 1, \quad 0 < s < 1
\]

Consumption and output are given by:

\[
C_t = sY_t
\]

\[
Y_t = AK_t^\alpha
\]

The wage and rental rate are given by:

\[
w_t = (1 - \alpha)AK_t^\alpha
\]

\[
R_t = \alpha AK_t^{\alpha-1}
\]

(a) Define the steady state and solve for the steady state values of \( K^*, C^*, Y^*, w^* \), and \( R^* \).

(b) What is the value of \( s \) that maximizes \( Y^* \)?
(c) Derive an analytic expression for the value of \( s \) that maximizes \( C^* \).

(d) Graphically analyze the Solow model’s central equation. Plot \( K_{t+1} \) against \( K_t \). Include a 45 degree line showing all points where \( K_{t+1} = K_t \). Argue that there exists a steady state in which \( K_{t+1} = K_t \). Also argue that this point is stable in the sense that starting from any initial value of \( K_t \), you will eventually converge to \( K^* \).

(e) Take logs of the production function (i.e. the expression for \( Y \) as a function of \( K \)). Take first differences and derive an approximate expression for the growth rate of \( Y \) as a function of the growth rate of \( K \).

(f) Suppose that \( K_t < K^* \). From your graph, what will be true about the growth rate of \( K \) for a while? What does this imply about the growth rate of \( Y \)? Do you expect output to grow faster or slower than normal if \( K_t < K^* \)?

(g) Create an Excel spreadsheet. Suppose that \( s = 0.2 \), \( \alpha = 0.33 \), \( \delta = 0.1 \), and \( A = 1 \). What is the steady state value of \( K \)? Suppose that initial \( K_t = \frac{1}{2}K^* \). Use Excel to figure out how \( K_t, Y_t, w_t, C_t, \) and \( R_t \) should evolve from this starting position.

(h) Suppose that the economy initially sits in steady state. There is an unexpected and permanent increase in \( s \). Qualitatively show how this will affect steady state \( K \) in your graphical diagram. Show qualitatively impulse response functions of \( Y, K, C, w, \) and \( R \) to this change in \( s \).

(i) Do this exercise quantitatively in Excel. Use your values from part (g). Suppose that you initially sit in a steady state associated with \( s = 0.2 \). Then \( s \) increases to \( s = 0.3 \). Plot out how \( Y, K, C, w, \) and \( R \) react to this change in the saving rate.

(j) Now suppose that you initially sit in a steady state, but \( \delta \) unexpectedly and permanent increases. Qualitatively show how this will impact steady state \( K \). Show qualitative impulse response functions of capital and the other variables to the increase in the depreciation rate.

(k) Do this exercise quantitatively in Excel. Use the parameter values listed in part (g). Suppose that the depreciation rate increases from 0.1 to 0.15.

5. Suppose that you have two different countries, \( i = 1 \) or \( i = 2 \), which are characterized by the Solow model. These countries are identical except for their saving rates. The central equation of the Solow model is:

\[
K_{i,t+1} = s_i AK_{i,t}^\alpha + (1 - \delta)K_{i,t}, \quad i = 1, 2
\]

\( K_{i,t} \) is capital in country \( i \). The countries have the same \( A, \alpha, \) and \( \delta \). They only differ in their saving rates. Suppose that \( A = 1, \alpha = 0.33, \) and \( \delta = 0.1 \). Suppose that the saving rate in country 1 is \( s_1 = 0.2 \). Suppose that both economies have converged to their steady states. Suppose that steady state output in country 1 is twice as big as in country 2, i.e. \( Y_1^* / Y_2^* = 2 \).

(a) What would the saving rate in country 2 have to be for this to be true?

(b) Calculate the steady state values of \( R \) in each economy with these saving rates (the saving rate given to you for country 1, and the saving rate for country 2 you solved for in (a)).

(c) If capital were mobile (i.e. people in country 1 could send their capital to country 2, and vice-versa), given your answer on (b), what do you think would happen? What capital flow from country 1 to 2, from 2 to 1, or not at all? Why?
(d) Given that capital is fairly mobile in the 21st century, what would you conclude about the plausibility that different saving rates account for different standards of living across the globe?

(e) If it’s not differences in $s$ that account for different standards of living, what is the most plausible candidate?

6. What are some real-world factors that would manifest themselves in the exogenous variable $A$ in the Solow model?

7. Suppose that we have the standard Solow model, but augment it to include population and productivity growth. The production function is:

$$Y_t = AK_t^\alpha (Z_tN_t)^{1-\alpha}$$

$Z_t$ and $N_t$ grow at constant rates over time, given by:

$$\frac{Z_{t+1}}{Z_t} = 1 + g_z$$

$$\frac{N_{t+1}}{N_t} = 1 + g_n$$

The central equation of the Solow model is:

$$K_{t+1} = sAK_t^\alpha (Z_tN_t)^{1-\alpha} + (1 - \delta)K_t$$

(a) Derive expressions for $w_t$ and $R_t$ with this production function.

(b) Define $\hat{k}_t = \frac{K_t}{Z_tN_t}$. Re-write the central equation in terms of this variable.

(c) Derive an expression for the steady state value of $\hat{k}^*$.

(d) Once $\hat{k}_t \to \hat{k}^*$, what will be true about the growth rates of $K_t$ and $k_t = \frac{K_t}{N_t}$?

(e) Derive an expression for the gross growth rate of output, $\frac{Y_{t+1}}{Y_t}$. Re-write this in terms of $\hat{k}_t$ and $\hat{k}_{t+1}$. Once $\hat{k}_t \to \hat{k}^*$, what will be true about the growth rate of output?

(f) Re-write the expressions for $w_t$ and $R_t$ that you derived above in terms of $\hat{k}_t$. Derive expressions for the gross growth rates of these variables, i.e. $\frac{w_{t+1}}{w_t}$ and $\frac{R_{t+1}}{R_t}$. What will these growth rates equal once $\hat{k}_t \to \hat{k}^*$?

(g) Are your answers on this problem when $\hat{k}_t \to \hat{k}^*$ consistent with the stylized growth facts?

8. Suppose that you have a Solow model with government spending. The government consumed $G_t$ of output each period, and pays for this with taxes on households of $T_t = G_t$ (so the government balances its budget each period). The household consumes a constant fraction of its net income:

$$C_t = (1 - s)(Y_t - T_t)$$

Capital accumulates according to the standard law of motion:

$$K_{t+1} = I_t + (1 - \delta)K_t$$

Investment equals output less consumption less government spending:

$$I_t = Y_t - C_t - G_t$$
Output is produced according to:

\[ Y_t = AK_t^\alpha \]

There is no productivity or population growth.

(a) Combine these equations into one central equation describing the dynamics of \( K_t \).
(b) Assume that \( G_t \) is fixed at \( G_0 \). Derive an expression for \( K^* \). It will depend on \( G_0 \).
(c) How will a permanent increase in government spending, say from \( G_0 \) to \( G_1 \), affect \( K^* \)?