Graduate Macro Theory II: Notes on Investment

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Spring 2011

1 Introduction

These notes introduce and discuss modern theories of firm investment. While much of this is done as a decision rule problem of the firm, it is easily incorporated into a general equilibrium structure.

2 Tobin's Q

Jim Tobin (1969) developed an intuitive and celebrated theory of investment. He reasoned that if the market value of physical capital of a firm exceeded its replacement cost, then capital has more value "in the firm" (the numerator) than outside the firm (the denominator). Formally, Tobin's Q is defined as:

$$Q = \frac{\text{Market Value of Firm Capital}}{\text{Replacement Cost of Capital}} \tag{1}$$

Tobin reasoned that firms should accumulate more capital when $Q > 1$ and should draw down their capital stock when $Q < 1$. That is, net investment in physical capital should depend on where Q is in relation to one.

How would one measure Q in the data? Typically this is done by using the value of the stock market in the numerator and data on corporate net worth in the denominator – here net worth is defined as the replacement cost of capital goods. For example, suppose that a company is in the business of bulldozing. It owns a bunch of bulldozers. Suppose that it owns 100 bulldozers, which cost 100 each to replace. This means its replacement cost is $10,000$. Suppose it has 1000 outstanding shares of stock, valued at 15 per share. The market value of the firm is thus $15,000$ and the replacement cost is $10,000$, so Tobin's Q is 1.5. Since bulldozers are worth more in the firm than outside the firm (the market evidently values each unit of capital inside the firm at 1.5 times its market value), the firm should evidently buy more bulldozers.

One might be able to immediately point out an issue with this logic. Economic decisions should be made at the margin – the firm should buy another bulldozer if the marginal value of the bulldozer in the firm exceeds the amount it costs to buy it. Tobin's Q is an average value – it is the average value of capital inside the firm relative to capital on the outside – e.g. normalizing the price of the capital good to be one, it is $\frac{V}{k}$, where V is total firm value. What really ought to matter for investing decisions is $\frac{\partial V}{\partial k}$ – i.e. *marginal* Q, not average Q.

3 Convex Adjustment Costs: Partial Equilibrium

Consider the problem of a firm that owns its capital stock. The real interest rate is fixed at r, and firms discount future profits by $\frac{1}{1+r}$. The firm produces output using capital and labor according to a constant returns to scale Cobb-Douglas production function. It takes the real wage as given. The firm owns its capital stock. Aside from the foregone profits of 1 (the relative price of investment is 1), there is potentially an additional cost of (net) investment, given by the following function:

$$\Phi\left(\frac{I_t}{k_t}\right)k_t$$

We assume that $\Phi(\cdot)$ has the following properties: $\Phi'(\cdot) > 0$, $\Phi''(\cdot) \geq 0$, $\Phi(\delta) = 0$, and $\Phi'(\delta) = 0$. In words, the function is increasing, concave, is equal to 0 at δ, and has first derivative equal to 0 at δ. In words, the firm must pay an increasing and convex cost of *net* investment. Net investment is given by $I_t - \delta k_t$ – i.e. investment over and above what is necessary to replace depreciated capital. This cost is measured in terms of units of current capital. If the firm has a lot of capital, then the cost of net investment will be high relative to if the firm does not have much capital. The idea here is that is the firm has to not run some of its machines while it installs new machines (i.e. it can't produce as much). The more machines its has, the more costly this is.

The firm's problem can be written:

$$\max_{I_t,k_{t+1},n_t} \quad V_0 = E_0 \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)\left(a_t f(k_t,n_t) - w_t n_t - I_t - \Phi\left(\frac{I_t}{k_t}\right)k_t\right)$$

s.t.

$$k_{t+1} = I_t + (1-\delta)k_t$$

Set up a Lagrangian. Denote the multiplier on the constraint by q_t.

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)\left(a_t f(k_t,n_t) - w_t n_t - I_t - \Phi\left(\frac{I_t}{k_t}\right)k_t + q_t(I_t + (1-\delta)k_t - k_{t+1})\right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Leftrightarrow a_t f_n(k_t, n_t) = w_t \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Leftrightarrow q_t = 1 + \Phi'\left(\frac{I_t}{k_t}\right) \quad (3)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow q_t = \frac{1}{1+r} E_t \left(a_{t+1} f_k(k_{t+1}, n_{t+1}) - \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}} + (1-\delta) q_{t+1} \right) \quad (4)$$

$$TV: \quad \lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T E_t q_{t+T} k_{t+T+1} = 0 \quad (5)$$

What exactly is q_t and what does it measure? It is the "shadow value" of the constraint, it is how much the firm would value having the constraint relaxed. Suppose that the firm has optimally chosen k_{t+1} given the state variables. Having the constraint relaxed means having one more unit of installed capital, or one more unit of k_{t+1}. Since this doesn't affect the optimal choice of k_{t+1} since the firm has already optimized, this means the firm will simply reduce current investment by 1 (this is an envelope type argument). This is valuable to the firm for two reasons. First, it means the firm has to do one fewer unit of investment today, which directly increases current profits by 1. Secondly, doing one fewer unit of investment reduces costs associated with adjustment by $\Phi'(\cdot)$, which also raises current profits. The total value in terms of current profits of having an extra unit of k_{t+1} is thus given by (2). In other words, $q_t = \frac{\partial V_0}{\partial k_{t+1}}$ along an optimal path – i.e. the marginal value of an additional unit of capital. The transversality condition says that the present value of any capital "left over at the end of time" is equal to zero.

(3) gives another expression for q which is very intuitive. This can be solved forward using the transversality condition to get:

$$q_t = E_t \frac{1}{1-\delta} \sum_{j=1}^{\infty} \left(\frac{1-\delta}{1+r}\right)^j \left(a_{t+j} f_k(k_{t+j}, n_{t+j}) - \Phi\left(\frac{I_{t+j}}{k_{t+j}}\right) + \Phi'\left(\frac{I_{t+j}}{k_{t+j}}\right)\frac{I_{t+j}}{k_{t+j}} \right)$$

This says that current q must also equal the present discounted value of the marginal profits of having an extra unit of k_{t+1}. The effective discount rate is by both the depreciation rate and r because the extra unit of k_{t+1} depreciates through time. The direct benefit of an extra unit of k_{t+1} is to have $(1-\delta)^{j-1} a_{t+j} f_k(k_{t+j}, n_{t+j})$ extra units of revenue at time $t+j$ – you have $(1-\delta)^{j-1}$ extra units of capital at time $t+j$, which yields marginal product $a_{t+j} f_k(k_{t+j}, n_{t+j})$. There is also an indirect effect - having more k_{t+1} reduces the installation costs. Having $(1-\delta)^{j-1}$ extra units of capital at time $t+j$ reduces installation costs by $\frac{\partial \Phi}{\partial k_{t+j}} = \Phi'\left(\frac{I_{t+j}}{k_{t+j}}\right)\frac{I_{t+j}}{k_{t+j}} - \Phi\left(\frac{I_{t+j}}{k_{t+j}}\right)$. Hence, q_t is also the present discounted value of having an extra unit of capital, measured in terms of future marginal products and reductions in installation costs.

3.1 q and Investment

The first order condition (2) above implicitly defines an investment demand function relating net investment to q_t. If $q_t > 1$, then $\Phi'(\cdot) > 0$, which requires $I_t > \delta k_t$. Suppose that we have the following functional form of the adjustment cost function:

$$\Phi(\cdot) = \frac{\phi}{2}\left(\frac{I_t}{k_t} - \delta\right)^2$$

It is easy to verify that this satisfies the properties laid out above. Plug this into (2):

$$q_t = 1 + \phi\left(\frac{I_t}{k_t} - \delta\right)$$

Now solve for $\frac{I_t}{k_t}$:

$$\frac{I_t}{k_t} = \frac{1}{\phi}(q_t - 1) + \delta \tag{6}$$

This says that net investment will be positive if and only if $q_t \geq 1$. ϕ governs how sensitive investment is to q_t. Put differently, a firm's investment as a fraction of its size (where size is measured in terms of capital stock) should only be a function of q_t and parameters. In other words, q_t should be a sufficient statistic for investment.

3.2 Tobin's Q, q, and the Hayashi Theorem

Recall from above that Tobin's Q is the ratio of the market value of a firm divided by the replacement cost of its capital – i.e. $\frac{V_0}{k}$, the *average* value of a unit of capital. This is observable – you can get the value of the firm from the stock market capitalization (i.e. market share price times number of shares) and the replacement cost of the firm's capital goods. The theory laid out above, which is explicitly based on optimization, says that investment should depend on q, which is $\frac{\partial V_0}{\partial k}$ – i.e. the *marginal* value of a unit of capital. This is in general not observable. Or is it?

Hayashi (1982) laid out conditions under which *marginal* q and *average* Q are one in the same. In particular, he showed that (i) the production function being homogeneous of degree one and (ii) the total adjustment cost function being homogeneous of degree one (i.e. $k_t\Phi\left(\frac{I_t}{k_t}\right)$) then average and marginal are one in the same.

I eschew a formal proof. But intuitively this result can be seen as follows. Under a Cobb-Douglas production technology, the first order condition for the choice of labor, given the capital stock, wage rate, and level of TFP, is:

$$n_t = \left(\frac{(1-\alpha)a_t}{w_t}\right)^{\frac{1}{\alpha}} k_t \tag{7}$$

This means that n is proportional to k, holding everything else fixed. Put differently, if the firm has twice as much capital, it will want twice as much labor. From the first order condition for investment:

$$q_t = 1 + \Phi'\left(\frac{I_t}{k_t}\right) \tag{8}$$

What this says is that, for a given q_t, if you double k, you also want to double I. Hence, if you give the firm twice as much initial capital, it will double both investment and labor input. Now go the expression for within period profits:

$$\Pi_t = a_t f(k_t, n_t) - w_t n_t - I_t - \Phi\left(\frac{I_t}{k_t}\right) k_t \tag{9}$$

If k doubles, and hence by the arguments above both n and I double, then every term in this doubles. This means that current cash flows are proportional to current k. Intuitively, this means that the value of the firm is therefore proportional to k:

$$V_0 = \varphi k \tag{10}$$

Since value is linear in k, then marginal and average are the same thing. This means that the q that should affect investment is the same thing as Tobin's Q. This has important empirical content, as it means that q is observable and the theory can be tested. Unfortunately, as with most theories, this theory can be rejected in the data. The theory suggests that, once measured, q_t should be a sufficient statistic for investment. This means that, for example, in regressions of investment on measured q_t, the coefficients on *anything else* should be zero. This is similar to tests of the random walk hypothesis for consumption – consumption should be a sufficient statistic for future consumption. People will typically find that things like current cash flow has a significant coefficient when it shouldn't. This is somewhat akin to the "excess sensitivity" finds in the consumption literature that consumption is too sensitive to current income.

3.3 The Phase Diagram

Now let's think about analyzing this in terms of a phase diagram. Let q_t be the jump variable (the variable on the vertical axis) and k_t be the state variable (the variable on the horizontal axis). We want to find two isoclines: one were $q_{t+1} - q_t = 0$ and one where $k_{t+1} - k_t = 0$. Given the accumulation equation, $k_{t+1} - k_t = 0 \Leftrightarrow \frac{I_t}{k_t} = \delta$, which means that $\Phi'(\delta) = 0$ and hence $q_t = 1$. Hence the, $k_{t+1} - k_t = 0$ isocline is a horizontal line at $q_t = 1$.

$$q_t = 1 \tag{11}$$

The $q_{t+1} - q_t$ isocline is a little tougher to figure out. Begin my multiplying both sides by $1 + r$:

$$q_t + rq_t = E_t a_{t+1} f_k(k_{t+1}, n_{t+1}) - E_t \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + E_t \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}} + (1-\delta)q_{t+1}$$

Simplify:

$$q_t = E_t q_{t+1} = 0 \Leftrightarrow rq_t = E_t a_{t+1} f_k(k_{t+1}, n_{t+1}) - E_t \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + E_t \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}} - \delta E_t q_{t+1}$$

Now, so as to make the phase diagram in discrete time more natural, we engage in the abuse of termingology that we've used before, and assume that $q_t \approx q_{t+1}$ and the same for n_t, I_t, and k_t. we can then write the $q_{t+1} - q_t = 0$ isocline as:

$$q_t = \frac{1}{r+\delta}\left(a_t f_k(k_t, n_t) - \Phi\left(\frac{I_t}{k_t}\right) + \Phi'\left(\frac{I_t}{k_t}\right)\frac{I_t}{k_t}\right) \tag{12}$$

This is the $q_{t+1} - q_t$ isocline. What does it look like (q_t, k_t) space? Let's take the derivative with respect to k_t:

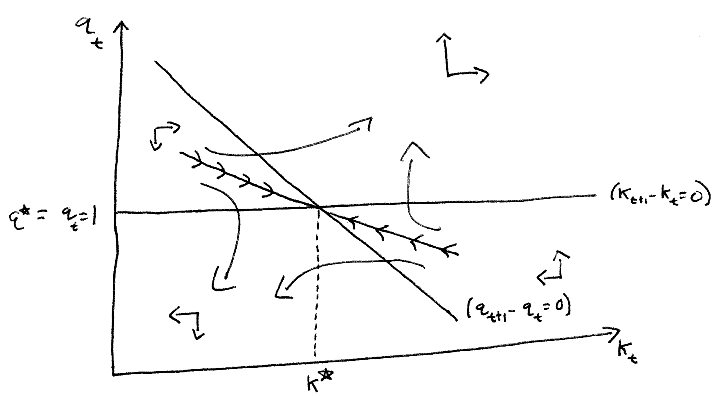
$$\frac{dq_t}{dk_t} = \frac{1}{r+\delta}\left(a_t f_{kk}(k_t, n_t) + \Phi'\left(\frac{I_t}{k_t}\right)\frac{I_t}{k_t^2} - \Phi'\left(\frac{I_t}{k_t}\right)\frac{I_t}{k_t^2} - \Phi''\left(\frac{I_t}{k_t}\right)\frac{I_t}{k_t^3}\right)$$

This of course simplifies to:

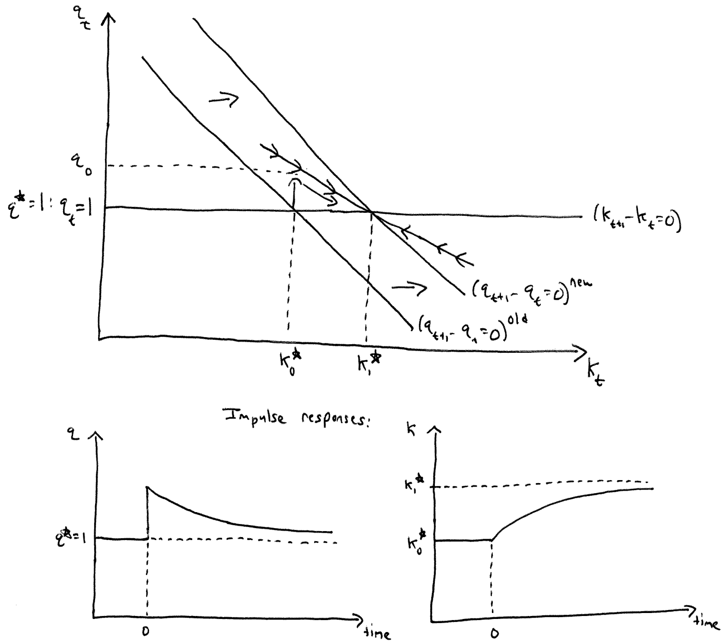
$$\frac{dq_t}{dk_t} = \frac{1}{r+\delta}\left(a_t f_{kk}(k_t, n_t) - \Phi''\left(\frac{I_t}{k_t}\right)\frac{I_t^2}{k_t^3}\right)$$

Under our assumptions – namely that the production function is concave but the adjustment cost function is convex, this derivative must be negative. In other words, the $q_{t+1} - q_t = 0$ isocline slopes down.

The steady state occurs where these two cross. Above the $k_{t+1} - k_t$ locus, $q_t > 1$, and hence capital is growing, so arrows point right. The reverse is true below the $k_{t+1} - k_t = 0$ locus, where the arrows point left. To the right of the $q_{t+1} - q_t = 0$ isocline, k_t is "too big". This means that the current marginal product of capital is small, which means that $q_{t+1} > q_t$, so the arrows point up. To the left of the $q_{t+1} - q_t = 0$ locus the arrows point down. We can see that the saddle path must be downward sloping and flatter than the $q_{t+1} - q_t = 0$ locus. The system is saddle point stable. The phase diagram is shown below.



Consider an unexpected, permanent increase in a. This does not affect the $k_{t+1} - k_t = 0$ locus, which is still horizontal at $q_t = 1$, but it shifts the $q_{t+1} - q_t = 0$ isocline to the right. The new steady state occurs at the same q^*, but a higher k^*. q_t must immediately jump up to the new saddle path, and then ride the dynamics into the new steady state. q_t jumping means that investment will be abnormally high, which is necessary to get to the new steady state. The phase diagram and impulse responses are shown below:



3.4 What if there are no adjustment costs?

From the first order condition above, we see that:

$$q_t = 1 + \Phi' \left(\frac{I_t}{k_t} \right)$$

If there are no adjustment costs, then $\Phi(\cdot) = 0$, and hence $q_t = 1$ always. Think about the specific functional form for the adjustment cost given above: no adjustment costs corresponds to the case where $\phi = 0$, which means that net investment is *infinitely* sensitive to fluctuations in q_t, which means that the "investment demand curve" is perfectly elastic and horizontal at $q = 1$.

In a sense then, investment is "residually determined" in a model without adjustment costs. With perfectly elastic demand (in terms of q_t), the quantity of investment in equilibrium is completely determined by saving supply, which comes from the household side of the model (and which we haven't yet modeled).

4 Convex Adjustment Costs: General Equilibrium

Now let's consider the role of adjustment costs in general equilibrium. It is easiest if we think about this in terms of the household owning the capital stock. The household problem looks very standard, except the fact that it must pay the adjustment cost to change the capital stock. Its problem is:

$$\max_{c_t,n_t,I_t,k_{t+1},b_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + v(1-n_t)\right)$$

s.t.

$$c_t + I_t + b_{t+1} + \Phi\left(\frac{I_t}{k_t}\right) k_t \leq w_t n_t + R_t k_t + (1+r_t)b_t + \Pi_t$$

$$k_{t+1} = I_t + (1-\delta)k_t$$

Basically what this says is that the household can consume, accumulate new capital, accumulate new bonds, and pay adjustment costs associated with adjusting its capital stock with its income, which comes from labor, renting capital, interest income on bonds, and distributed profits. Let's set this up as a current value Lagrangian, with two separate constraints:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg(u(c_t) + v(1-n_t) + \lambda_t \Bigg(w_t n_t + R_t k_t + (1+r_t)b_t + \Pi_t - c_t - I_t - b_{t+1} - \Phi\left(\frac{I_t}{k_t}\right) k_t\Bigg) \ldots$$
$$\ldots \mu_t(I_t + (1-\delta)k_t - k_{t+1})\Bigg)$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow u'(c_t) = \lambda_t \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Leftrightarrow v'(1-n_t) = u'(c_t) w_t \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Leftrightarrow \mu_t = \lambda_t \left(1 + \Phi'\left(\frac{I_t}{k_t}\right)\right) \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow \mu_t = \beta E_t \left(\lambda_{t+1}\left(R_{t+1} - \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}}\right) + (1-\delta)\mu_{t+1}\right) \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1+r_{t+1}) \tag{17}$$

We need to play around with this a little bit. Begin by dividing both sides the first order condition for tomorrow's capital stock by λ_t:

$$\frac{\mu_t}{\lambda_t} = \beta E_t \left(\frac{\lambda_{t+1}}{\lambda_t}\left(R_{t+1} - \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}}\right) + (1-\delta)\frac{\mu_{t+1}}{\lambda_t}\right) \tag{18}$$

Now define $q_t \equiv \frac{\mu_t}{\lambda_t}$. Think about what this says. μ_t is the shadow value of having an extra unit of capital (i.e. the units are loosely "utils per capital"). Dividing this by λ_t (which is equal to the marginal utility of consumption . . . i.e. "utils per consumption") puts this in terms of consumption goods. In other words, q_t is the marginal value of capital measured in terms of consumption goods (i.e. exactly the same concept as in the partial equilibrium firm problem above). Now simplify the above some:

$$q_t = \beta E_t \left(\frac{\lambda_{t+1}}{\lambda_t} \left(R_{t+1} - \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}} \right) + (1-\delta) q_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \right) \tag{19}$$

Using the first order condition for bonds, this becomes:

$$q_t = E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} \left(\left(R_{t+1} - \Phi\left(\frac{I_{t+1}}{k_{t+1}}\right) + \Phi'\left(\frac{I_{t+1}}{k_{t+1}}\right)\frac{I_{t+1}}{k_{t+1}} \right) + (1-\delta) q_{t+1} \right) \right) \tag{20}$$

Note that this is exactly the same as (4) above except that now the interest rate is endogenous and determined by the marginal rate of substitution in consumption. We did all this manipulation so that we could measure the price of capital, q_t, in terms of consumption (which is observable, at least under the Hayashi theorem) as opposed to utils (which are not observable). Now simplify some of the other equations using this definition to get:

$$v'(1-n_t) = u'(c_t) w_t \tag{21}$$

$$q_t = 1 + \Phi'\left(\frac{I_t}{k_t}\right) \tag{22}$$

$$u'(c_t) = \beta E_t \left(u'(c_{t+1})(1 + r_{t+1}) \right) \tag{23}$$

(21) and (23) are entirely standard, and (22) is identical to the condition from the partial equilibrium problem, (3).

The firm problem is standard, where the firms lease the capital from households. The first order conditions are:

$$R_t = a_t f_k(k_t, n_t) \tag{24}$$

$$w_t = a_t f_n(k_t, n_t) \tag{25}$$

Assume that the adjustment cost function takes the following form:

$$\Phi\left(\frac{I_t}{k_t}\right) = \frac{\phi}{2}\left(\frac{I_t}{k_t} - \delta\right)^2 \tag{26}$$

If we use our normal preference specification, the first order conditions characterizing the equilibrium of the model become:

$$\frac{\theta}{1-n_t} = \frac{1}{c_t} w_t \tag{27}$$

$$q_t = 1 + \phi \left(\frac{I_t}{k_t} - \delta \right) \tag{28}$$

$$\frac{1}{c_t} = \beta E_t \left(\frac{1}{c_{t+1}} (1 + r_{t+1}) \right) \tag{29}$$

$$q_t = E_t \left(\beta \frac{c_t}{c_{t+1}} \left(\left(R_{t+1} - \frac{\phi}{2} \left(\frac{I_{t+1}}{k_{t+1}} - \delta \right)^2 + \phi \left(\frac{I_{t+1}}{k_{t+1}} - \delta \right) \frac{I_{t+1}}{k_{t+1}} \right) + (1-\delta) q_{t+1} \right) \right) \tag{30}$$

$$R_t = a_t \alpha k_t^{\alpha-1} n_t^{1-\alpha} \tag{31}$$

$$w_t = a_t (1-\alpha) k_t^{\alpha} n_t^{-\alpha} \tag{32}$$

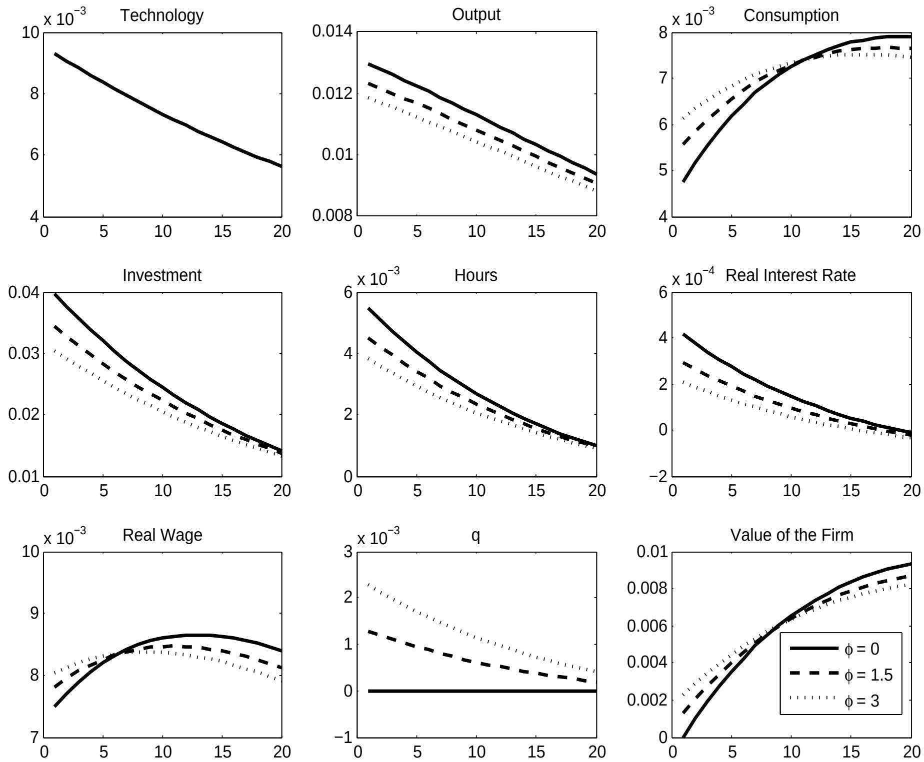
$$k_{t+1} = I_t + (1-\delta) k_t \tag{33}$$

$$y_t = c_t + I_t + \frac{\phi}{2} \left(\frac{I_t}{k_t} - \delta \right)^2 \tag{34}$$

$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha} \tag{35}$$

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_t \tag{36}$$

Below are impulse responses to a technology shock in this model. The parameterization is our standard RBC paramaterization, with three different values of ϕ: 0, 1.5, and 3. The case of $\phi = 0$ corresponds with the case of the standard RBC model without adjustment costs.



As argued above, if there are no adjustment costs ($\phi = 0$), then q is constant at 1 and doesn't respond to the technology shock. The impulse responses of the remaining variables of the model are nevertheless pretty similar for different values of ϕ – the main thing that different values of ϕ buys us is a jumping value of the firm, which does not happen if there are no adjustment costs.

5 Non-Convex Adjustment Costs

The theory of investment studied above (based on quadratic adjustment costs or no adjustment costs at all) has the testable prediction that firms should continuously adjust their capital stock in response to new conditions. Put differently, the theory predicts that investment should be small and continuous.

This is squarely at odds with the micro data, where investment is typically *lumpy* in the sense that there is no investment in most periods with sporadic periods of large adjustment. The theory of non-convex adjustment costs is built to match this feature of the firm-level data. We consider two sources of non-convexity: either fixed costs or kinked costs.

5.1 Fixed Costs

Here we will use the dynamic programming notation that a variable without a superscript denotes a current value, while a variable with a $'$ denotes the next period's value.

Consider the partial equilibrium problem of a firm which discounts future profits by $\frac{1}{1+r}$. For simplicity, abstract from the labor choice decision. Output/revenue is given by $y = af(k)$, where $f'(k) > 0$ and $f''(k) < 0$. The relative price of investment is 1, and the capital accumulation equation is $k' = I + (1-\delta)k$. In addition to paying the direct price of investment (i.e. one unit of investment reduces current profits by 1), the firm must pay a fixed cost of investment, $F \geq 0$, if $I \neq 0$. This means that the firm will either earn current profits: $\Pi = af(k) - (k' - (1-\delta)k) - F$ if $k' \neq (1-\delta)k$ or $\Pi = af(k)$ if $k' = (1-\delta)k$.

The firm faces two decisions: the binary choice of whether to invest or not and the continuous choice of how much to invest if they choose to invest. The state variables are the current capital stock and a. The Bellman equation is given by:

$$V(a,k) = \max\left\{\max_{k'} \ af(k) - (k' - (1-\delta)k) - F + \frac{1}{1+r}EV(a',k'),\ \ af(k) + \frac{1}{1+r}EV(a',(1-\delta)k)\right\}$$

We can think about the policy function for this problem by essentially working backwards. First, suppose that the firm chooses to adjust, so that $k' \neq (1-\delta)k$. If it adjusts, its choice of k' is independent of k. Let's call this k^*: the optimal reset capital stock if you will.

Now that we know what the firm will do *if* it adjusts/does investment, let's now look at the binary choice of *whether* it will invest. If will choose to invest if the value for doing so (the first part of the Bellman equation evaluated at the optimum choice k^* – i.e. this "takes care of" the max operator), is greater than or equal to the value of doing nothing. Put differently, the firm will invest if:

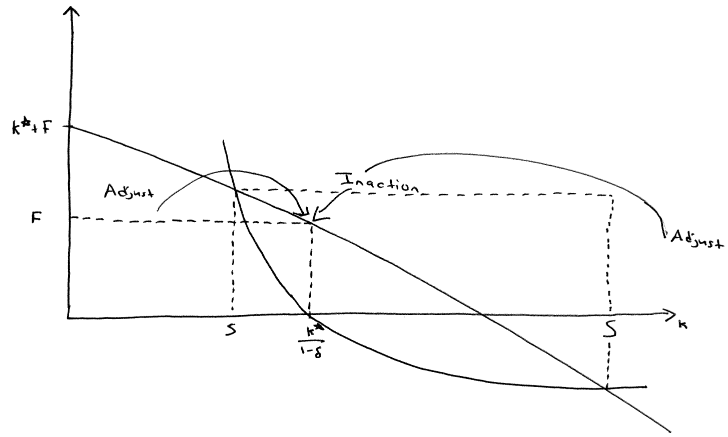
$$af(k) - (k^* - (1-\delta)k) - F + \frac{1}{1+r}EV(a',k^*) \geq af(k) + \frac{1}{1+r}EV(a',(1-\delta)k)$$

This can be simplified to:

$$\frac{1}{1+r}\big(EV(a',k^*) - EV(a',(1-\delta)k)\big) \geq k^* - (1-\delta)k + F \tag{37}$$

This simply says that the firm will invest if the benefit exceeds the cost. The presence of the fixed cost will mean that the firm may choose to do nothing at all. In particular, if k^* is not "too different" from $(1-\delta)k$, then the benefit will be small (this also depends on the amount of curvature in the value function), and the benefit is unlikely to exceed F. We can see that this will result in inaction as follows. Suppose that $(1-\delta)k \approx k^*$. Then the right hand side will be close to F, and the left hand side will be close to zero, and the condition for adjustment will not be satisfied.

Let's think about this result graphically. First, assume that the value function is (i) increasing and (ii) concave, with "Inada" type conditions that the slope of the value function starts out at infinity and ends up at zero. Since k^* is taken to be a constant independent of k, the left hand side of (37) will be decreasing and convex, and will start with a very steep negative slope and will asymptote to a slope of zero. The right hand side is linear in k, with slope $(1-\delta) < 1$. At $k = \frac{k^*}{1-\delta}$ it will be equal to F, and equal to zero at $\frac{k^*+F}{1-\delta}$. At $k = \frac{k^*}{1-\delta}$, the left hand side must be equal to zero. We can plot each side of (37) against k as follows:



Given the assumed Inada type conditions, we can see that the left hand side will cross the right hand side twice – once somewhere below $\frac{k^*}{1-\delta}$, and once somewhere above $\frac{k^*}{1-\delta}$. Call these two points (s, S), with $s < S$. Above or below each of these crossing points, the left hand side of (37) is bigger than the right hand side, so the firm will adjust to k^*. Within those two bounds, however, the firm prefers in action. We call this an (s, S) policy because there is a region of inaction but large adjustment beyond the trigger points defining the region of inaction.

Suppose that there were no adjustment costs (i.e. $F = 0$). The left hand side would still cross the horizontal axis at $\frac{k^*}{1-\delta}$. But with no adjustment costs, we actually know what the slope of the left hand side is. The first order condition of the smooth problem is: $1 + r = \frac{\partial V(a',k')}{\partial k'}$. Thus, if we evaluate the slope of the left hand side at $k^* = (1-\delta)k$, we see that the slope is equal to $(1-\delta)$. This means that the straight line and the curve just "kiss" at this point – at every other point the left hand side is above the right hand side. This means without a fixed cost there is a *point* of inaction if it so happens that the current capital stock will yield the desired capital stock tomorrow, but at every other point there will be adjustment.

It is straightforward to see how the fixed cost will affect the region of inaction. If F goes up, the straight line shifts up, and the region of inaction gets bigger.

5.2 Kinked Adjustment Costs

Here we assume a different kind of non-convexity – a non-symmetric price of capital. In particular, assume that the relative price (in terms of consumption) of accumulating new capital (i.e. $I = k' - (1-\delta)k > 0$) is 1, but that the relative price of *selling* capital (i.e. $I = k' - (1-\delta)k < 0$) is equal to $b < 1$. We can kind of appeal to this on the basis of adverse selection motivation. You can sell your capital for less than you can buy new capital – not because there is any functional difference in the capital (both are equally productive), but because the market worries that you may be selling a "lemon" – think about the market for used cars.

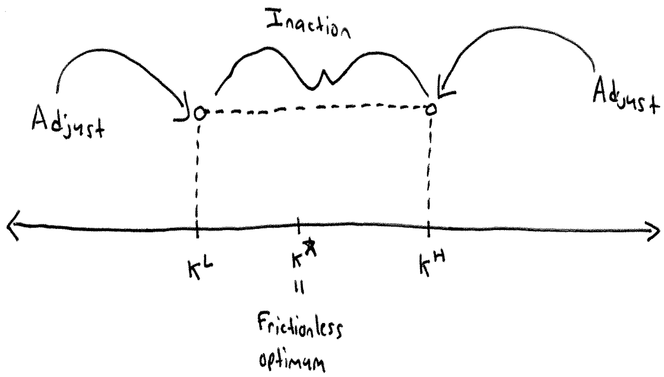
This introduces a "kink" in the adjustment cost function; in particular, the adjustment cost function is non-differentiable at $I = 0$. This means that we can't use something like a Lagrangian to solve the firm's problem. Let's instead set the problem up as a dynamic programming problem. The firm enters the period with capital stock k and takes the level of technology, a, as given. It can do three things: accumulate more capital (price 1), sell some of its capital (at price $b < 1$), or do nothing. The Bellman equation can be written:

$$\begin{aligned} V(a,k) &= \max \ \left\{V^1(a,k), V^2(a,k), V^3(a,k)\right\} \\ V^1(a,k) &= \max_{k'} \ af(k) - (k' - (1-\delta)k) + \frac{1}{1+r} EV(a',k') \\ V^2(a,k) &= \max_{k'} \ af(k) - b(k' - (1-\delta)k) + \frac{1}{1+r} EV(a',k') \\ V^3(a,k) &= \ af(k) + \frac{1}{1+r} EV(a',(1-\delta)k) \end{aligned}$$

The first Bellman equation is the value of doing positive investment; the second is the value of doing negative investment; and the third is the value of doing nothing.

We can characterize the policy function for this specification intuitively. Let k^* denote the k' the firm would choose if there was no kink in the adjustment cost function. Suppose that the firm has an initial capital stock that is well below k^*. It will want to increase its capital stock, but it will *not* go all the way to k^*. Why is this? Because there is essentially an "option value" to have a smaller capital stock – getting stuck with a capital stock that is too large is costly because $b < 1$ and that capital can only be sold off at a loss. Put differently, the firm will "under-adjust" if it has too little capital relative to its frictionless optimum because the downside risk of getting stuck with too much capital is high. In contrast, suppose that the firm starts with too much capital relative to its frictionless optimum. It would want to get rid of some capital, but again this is costly because the sell price of capital is less than the buy price. The firm could get rid of some of that capital over time by letting it depreciate away instead. Hence, the firm will also "under-adjust on the upper end as well.

The policy function will be as follows. If k is sufficiently far below k^*, the firm will adjust to $k^L \leq k^*$. If k is sufficiently far above k^*, the firm will adjust to $k^H \geq k^*$. Hence, $k^H \geq k^L$. This will be a strong inequality so long as $b < 1$. Hence, just as in the fixed cost case, there will be a region of inaction. For values of the current capital stock that will yield next period capital stock's between those two regions – i.e. as long as $\frac{k^L}{1-\delta} \leq k \leq \frac{k^H}{1-\delta}$, the firm will choose no investment and will thus set $k' = (1-\delta)k$. Above either of those triggers, the firm will adjust. Hence, this is kind of like the (s,S) policy described for the fixed cost case. What is different here, however, is that the firm will adjust to the triggers – not to the center of the triggers as in the fixed cost case. Regardless, however, there will be a large region of inaction and resulting lumpiness. See the figure below for a description of the policy function:



These kinked adjustment costs are a kind of *irreversibility.* The extreme case of pure irreversibility occurs when $b = 0$. If you can't sell capital for any positive price, the choice to accumulate new capital is effectively irreversible – you can't get rid of it for any positive price. This is going to induce cautious behavior on the part of firms – they will under-invest when they are below their frictionless optimal level of capital and will also under-adjust when they have too much capital, instead just letting their existing capital depreciate. This "cautiousness" leads to the region of inaction and lumpiness.

5.3 Reprisal

While it is definitely clear that lumpiness is an important feature of the micro data on investment, it is not clear how important it is at an aggregate level. Non-convexities in terms of adjustment costs significantly complicate solving problems once general equilibrium considerations are taken into account – there you have to keep track of the capital stock for *every* firm, which means you quickly run into the curse of dimensionality.

There are a number of papers which essentially argue that models based on convex adjustment costs (i.e. q theory) are sufficient for understanding aggregate investment behavior. This means that ignoring this micro feature is okay from the perspective of understanding aggregate investment (see, for example, Thomas (2002) or House (2009)). To do policy analysis and counterfactuals, however, one needs to know the structural parameters of the model. Even if the convex adjustment costs describe aggregate data well, the parameters of that specification (e.g. ϕ), are reduced form, and policy analysis based on that may be flawed.