Useful Math Facts

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Exponents:

\[
\begin{align*}
    x^1 &= x \\
    x^0 &= 1 \\
    x^{-1} &= \frac{1}{x} \\
    x^a x^b &= x^{a+b} \\
    \frac{x^a}{x^b} &= x^{a-b} \\
    (x^a)^b &= x^{ab}
\end{align*}
\]

Natural Logs:

\[
\begin{align*}
    e &\approx 2.71828 \\
    \ln(e^x) &= x \\
    e(\ln x) &= x \\
    \ln(x^a) &= a \ln x \\
    \ln(xy) &= \ln x + \ln y \\
    \ln\left(\frac{x}{y}\right) &= \ln x - \ln y \\
    \ln 1 &= 0 \\
    \ln 0 &= -\infty
\end{align*}
\]

A useful fact is that the natural log of $1 + x$ is approximately equal to $x$ when $x$ is small (near zero). Verify that for yourself on a calculator.

\[
\ln(1 + x) \approx x
\]

Growth Rates:

The growth rate of a variable is the new value minus the old value divided by the old value:

\[
g_x = \frac{x' - x}{x}
\]

From this we can see that the new value is equal to:
\[ x' = (1 + g_x)x \]

Suppose that the variable keeps growing at this constant rate. Then the value two periods from now will be:

\[ x'' = (1 + g_x)x' = (1 + g_x)^2x \]

If the variable keeps growing at this rate for \( n \) periods, then:

\[ x_{t+n} = (1 + g_x)^nx_t \]

**Summation Notation:**

Suppose that we’re looking at the sum of a variable across time, from period \( t \) to period \( t + n \):

\[ S = x_t + x_{t+1} + x_{t+2} + \ldots + x_{t+n} \]

A convenient and compact way to write this is as follows:

\[ S = \sum_{j=0}^{n} x_{t+j} \]

In words, this says that you start with \( j = 0 \) \((x_t)\) and keep adding up until you get to \( j = n \) \((x_{t+n})\).

**Discounted Sums:**

Often we’re interested discounted sums, where we “discount” future realizations of the variables of interest by some number which decays with time. Let \( \beta < 1 \). A discounted sum would be:

\[ S = x_t + \beta x_{t+1} + \beta^2 x_{t+2} + \ldots + \beta^n x_{t+n} \]

Since \( \beta < 1 \), we place less and less weight on subsequent values. For \( n \) pretty big, \( \beta^n \approx 0 \). Suppose that the \( x \)’s themselves aren’t growing over time, so that \( x_{t+j} = x \) for each possible value of \( j \). Then we have:

\[ S = x + \beta x + \beta^2 x + \ldots + \beta^n x = x(1 + \beta + \beta^2 + \ldots + \beta^n) \]

We can get nice looking solution for this. Multiply both sides by \( \beta \):

\[ S\beta = x(\beta + \beta^2 + \ldots + \beta^{n+1}) \]

Now subtract this from \( S \) and solve for \( S \):

\[ S - S\beta = x(1 - \beta^{n+1}) \]

\[ S = \frac{x(1 - \beta^{n+1})}{1 - \beta} \]
As $n \to \infty$, this reduces to:

$$S = \frac{x}{1-\beta}$$

**Calculus:**

Suppose we have a function denoted by: $y = f(x)$. We denote the first derivative as follows:

$$\frac{dy}{dx} = \frac{d(f(x))}{dx} = f'(x)$$

The second derivative is the derivative of the derivative, and we denote it as follows:

$$\frac{d^2y}{dx^2} = \frac{d^2(f(x))}{dx^2} = f''(x)$$

Suppose that the function is over more than one variable, $y = f(x, z)$. We define the partial derivatives as follows:

$$\frac{\partial y}{\partial x} = \frac{\partial f(x, z)}{\partial x} = f_x(x, z)$$

$$\frac{\partial y}{\partial z} = \frac{\partial f(x, z)}{\partial z} = f_z(x, z)$$

It’s worth noting that a derivative of a function is a function itself. You can get a number by evaluating the derivative at a particular point.

It is also sometimes useful to use “total derivatives”. The total derivative gives you the change in the left hand side variable as a function of the change in the right hand side variables. Let $dy = \Delta y$ denote the change in a variable. Then the total derivative of a univariate function is:

$$dy \approx f'(x)dx$$

In other words, this just says that the total change in $y$ is the derivative of $y$ with respect to $x$ (the change in $y$ for a one unit change in $x$) times the total change in $x$.

The total derivative of a multivariate function is found similarly:

$$dy \approx f_x(x, z)dx + f_z(x, z)dz$$

The basic rules for derivatives which you need to know are as follows:

$$f(x) = a \quad f'(x) = 0$$
$$f(x) = ax \quad f'(x) = a$$
$$f(x) = x^a \quad f'(x) = ax^{a-1}$$
$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$
$$f(x) = h(x) + g(x) \quad f'(x) = h'(x) + g'(x)$$
$$f(x) = h(x)g(x) \quad f'(x) = h'(x)g(x) + h(x)g'(x)$$
$$f(x, z) = h(x)g(z) \quad f_x(x, z) = h'(x)g(z) \quad f_z(x, z) = h(x)g'(z)$$
An example of the last rule for partial derivatives is the Cobb-Douglas production function which we’ll be using in class:

\[ y = k^\alpha n^{1-\alpha} \]

The partial derivatives are:

\[ \frac{\partial y}{\partial k} = \alpha k^{\alpha-1} n^{1-\alpha} \]

\[ \frac{\partial y}{\partial n} = (1 - \alpha) k^\alpha n^{-\alpha} \]