Optimal Monetary Policy in the New Keynesian Model

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Spring 2011

1 Introduction

These notes describe optimal monetary policy in the basic New Keynesian model.

2 Re-writing the Basic Model

The basic NK model can be characterized by two main (log-linear) equations: the Phillips Curve and the Euler/IS equation:

$$\widetilde{\pi}_t = \gamma(\widetilde{y}_t - \widetilde{y}_t^f) + \beta E_t \widetilde{\pi}_{t+1}$$
$$\widetilde{y}_t = E_t \widetilde{y}_{t+1} - \widetilde{r}_t$$

To economize on notation, let's define $\widetilde{x}_t \equiv \widetilde{y}_t - \widetilde{y}_t^f$ as the output gap. Similarly, let's define $\widetilde{r}_t^f$ as the "flexible price real interest rate" (or "natural real interest rate") as that rate that would obtain if prices were fully flexible. We can solve for this by looking at the Euler equation:

$$\widetilde{r}_t^f = E_t \widetilde{y}_{t+1}^f - \widetilde{y}_t^f$$

Because of the (maintained) assumption on preferences, we know from before that the flexible price level of output evolves exogenously in line with the level of technology, with $\widetilde{y}_t^f = \widetilde{a}_t$. If we assume that technology obeys an AR(1), then we can model the flexible price equilibrium level of output as following the same AR(1).

$$\widetilde{y}_t^f = \rho \widetilde{y}_{t-1}^f + \varepsilon_t$$

This means that we can solve for the flexible price real interest rate as:

$$\widetilde{r}_t^f = (\rho - 1)\widetilde{y}_t^f$$

Plugging in the process for $\widetilde{y}_t^f$ and simplifying we get a process for the natural rate of interest:

$$\widetilde{r}_t^f = (\rho - 1)\rho\widetilde{y}_{t-1}^f + (\rho - 1)\varepsilon_t$$
$$\widetilde{r}_t^f = \rho\widetilde{r}_{t-1}^f + (\rho - 1)\varepsilon_t$$

We can then summarize the main equations of the model as follows:

$$\widetilde{\pi}_t = \gamma\widetilde{x}_t + \beta E_t\widetilde{\pi}_{t+1} \tag{1}$$
$$\widetilde{x}_t = E_t\widetilde{x}_{t+1} - (\widetilde{r}_t - \widetilde{r}_t^f) \tag{2}$$
$$\widetilde{r}_t^f = \rho\widetilde{r}_{t-1}^f + (\rho - 1)\varepsilon_t \tag{3}$$

In the background there is also (i) a money demand relationship and (ii) a Fisher relationship. For now, we can think about the central bank effectively being able to choose $\widetilde{r}_t$. Given that, as well as $\widetilde{r}_t^f$ (which is the exogenous driving force), x_t and π_t will be determined.

3 Distortions and Welfare

There are two welfare-reducing distortions in the NK model, one of which is essentially "long run" and the other which is "short run". The "long run" distortion is that the flexible price level of output will be lower than what would obtain in the first best. This is because, in the flexible price version of the model, firms will set price equal to a markup over marginal cost. Hence there will be too little employment. The "short run" distortion is due to price stickiness, and leads to non-optimal fluctuations in relative prices.

We assume that the central bank is concerned with the "short run distortion" and that the "long run distortion" has been taken care of via some kind of Pigouvian tax. This means we can interpret $\widetilde{y}_t^f$ as the optimal equilibrium value of output from the perspective of the central bank. This means that, other things being equal, the central bank would like to eliminate output gaps.

4 Optimal Policy

In addition to disliking output gaps, we also assume that the central bank dislikes inflation. We assume that welfare of the central bank is a present discounted value of a quadratic loss function in inflation and the output gap. Let ω denote the relative weight that the central bank places on the output gap. The central bank would therefore like to minimize the following:

$$\min \quad \frac{1}{2}E_0\left(\sum_{t=0}^{\infty}\beta^t\left(\widetilde{\pi}_t^2 + \omega\widetilde{x}_t^2\right)\right)$$

As noted above, we can think about the central bank as choosing inflation and the output gap, given its choice of $\widetilde{r}_t$. This must be done subject to the constraint of the Phillips Curve, however.

We consider two cases. In the first, called "discretion", the central bank solves the one period problem each period. In the other, called "commitment", the central bank solves the entire problem at the beginning of time and commits to its policy. We start first with the discretion case. The problem can be written:

$$\min_{\widetilde{\pi}_t, \widetilde{x}_t} \quad \frac{1}{2}\left(\widetilde{\pi}_t^2 + \omega \widetilde{x}_t^2\right)$$

$$\text{s.t.}$$

$$\widetilde{\pi}_t = \gamma \widetilde{x}_t + \beta E_t \widetilde{\pi}_{t+1}$$

Set the problem up as a Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\left(\widetilde{\pi}_t^2 + \omega \widetilde{x}_t^2\right) + \lambda\left(\widetilde{\pi}_t - \gamma \widetilde{x}_t - \beta E_t \widetilde{\pi}_{t+1}\right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\pi}_t} = 0 \Leftrightarrow \widetilde{\pi}_t = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{x}_t} = 0 \Leftrightarrow \omega \widetilde{x}_t = -\lambda \gamma$$

Combing FOCs so as to eliminate the Lagrange multiplier, we get:

$$\widetilde{x}_t = -\frac{\gamma}{\omega}\widetilde{\pi}_t \tag{4}$$

Loosely speaking, this first order condition can be interpreted as a "lean against the wind" policy. If the output gap is positive, the Fed will want to pursue a policy in which it lowers inflation (and vice-versa).

One possible solution consistent with this first order condition holding is $\widetilde{x}_t = \widetilde{\pi}_t = 0$. This solution also corresponds with the global minimum of the objective function. This solution is also consistent with the constraints. If $\widetilde{\pi}_t = \widetilde{x}_t = 0$, then it must be the case that $E_t \widetilde{\pi}_{t+1} = 0$ for the Phillips Curve to hold. Since the central bank will expect to optimize in period $t+1$, this would then imply that $E_t \widetilde{x}_{t+1} = 0$ since the first order condition will be expected to hold in $t+1$. $\widetilde{x}_t = E_t \widetilde{x}_{t+1} = 0$ implies that $\widetilde{r}_t = \widetilde{r}_t^f$ from the Euler equation. From the Fisher relationship, $E_t \widetilde{\pi}_{t+1} = 0$ means that $\widetilde{i}_t = \widetilde{r}_t^f$. In other words, if the central banks sets the nominal interest rate equal to the "natural rate" of interest, then it evidently can implement the zero inflation/zero output gap equilibrium.

How would the central bank do this? It can do so by either a money supply rule or an interest rate rule. Suppose that the log-linearized demand for money can be written:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{y}_t - \kappa \widetilde{i}_t$$

Here $\kappa = \frac{1}{v}\left(\frac{1}{i^*} - \frac{1}{1+i^*}\right)$. The solution discussed above means that $\widetilde{y}_t = \widetilde{y}_t^f$ and $\widetilde{i}_t = \widetilde{r}_t^f$. Plug this in:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{y}_t^f - \kappa\widetilde{r}_t^f$$

From above, we have $\widetilde{r}_t^f = (\rho-1)\widetilde{y}_t^f$. Plug this in:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{y}_t^f - \kappa(\rho-1)\widetilde{y}_t^f = \left(\frac{1}{\nu}\left(1-\kappa(\rho-1)\right)\right)\widetilde{y}_t^f$$

Now we know that $\widetilde{y}_t^f = \rho\widetilde{y}_{t-1}^f$. Hence:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{y}_t^f - \kappa(\rho-1)\widetilde{y}_t^f = \left(\frac{1}{\nu}\left(1+\kappa(1-\rho)\right)\right)\rho\widetilde{y}_{t-1}^f + \left(\frac{1}{\nu}\left(1+\kappa(1-\rho)\right)\right)\varepsilon_t$$

Iterate this back one period to eliminate $\widetilde{y}_t^f$:

$$\widetilde{m}_t = \rho\widetilde{m}_{t-1} + \left(\frac{1}{\nu}\left(1+\kappa(1-\rho)\right)\right)\varepsilon_t$$

Now let's write this in terms of the nominal money supply, which the central bank controls, by noting that $\widetilde{m}_t = \widetilde{M}_t - \widetilde{p}_t$, with $\widetilde{\pi}_t = \widetilde{p}_t - \widetilde{p}_{t-1}$:

$$\widetilde{M}_t = \rho\widetilde{M}_{t-1} - \rho\widetilde{\pi}_t - (1-\rho)\widetilde{p}_t + \left(\frac{1}{\nu}\left(1+\kappa(1-\rho)\right)\right)\varepsilon_t \tag{5}$$

In other words, the central bank needs to set the money supply according to an AR(1) in the level, with AR coefficient equal to the AR coefficient on the technology process. As long as $\rho < 1$, it will raise the money supply in response to positive technology shocks. There is also a slight adjustment for lagged inflation and the price level.

The central bank can also accomplish its goal with a modified Taylor rule. In particular, suppose it sets the nominal interest rate according to the following rule.

$$\widetilde{i}_t = \widetilde{r}_t^f + \psi_\pi\widetilde{\pi}_t + \psi_x\widetilde{x}_t \tag{6}$$

In other words the central bank will set the nominal interest rate equal to the natural rate of interest plus coefficients on inflation and the output gap. It is clear that $\widetilde{\pi}_t = \widetilde{x}_t = 0$ will lead to $\widetilde{i}_t = \widetilde{r}_t^f$, which in turn rationalizes $\widetilde{\pi}_t = \widetilde{x}_t = 0$. The only caveat is that ψ_π and ψ_x must be such that the Taylor principle is satisfied, so that the equilibrium is determinate.

Hence, we have seen that the central bank can reach the global minimum of its objective function by setting inflation and the output gap equal to zero each period, which requires setting the nominal rate equal to the natural rate of interest. The solution of the problem under commitment – which means solving the problem at time 0 and committing to the decision – will yield the same solution. Hence, there is no gain from commitment. The resulting solution is sometimes called an inflation targeting rule, because the central bank targets an inflation rate of 0 and achieves it each period.

5 Cost-Push Shocks and an Output-Inflation Tradeoff

Let's modify the Phillips curve to contain an additional term:

$$\widetilde{\pi}_t = \gamma \widetilde{x}_t + \beta E_t \pi_{t+1} + \widetilde{u}_t \tag{7}$$

Here $\widetilde{u}_t$ is referred to as the "cost-push" term. It is a shock which, in a sense, changes the output-inflation tradeoff. It is exogenous and the central bank takes it as given. The structural interpretation of this term is not always very clear; it can be thought of as something which drives a wedge between marginal cost and the output gap. In reality it is a convenient shortcut to make the central bank's problem more interesting. Assume that it follows an AR(1):

$$\widetilde{u}_t = \rho_u \widetilde{u}_{t-1} + \varepsilon_{u,t} \tag{8}$$

The optimization problem of the central bank is identical to above, just subject to this new Phillips Curve with the cost-push term taken as given. Since the central bank takes the cost-push term as given, the first order condition under discretion will be the same: $\widetilde{x}_t = -\frac{\gamma}{\omega}\widetilde{\pi}_t$. This can still be interpreted as a "lean against the wind" strategy. What is different is that, in general, $\widetilde{x}_t = \widetilde{\pi}_t = 0$ will *not* be a viable solution. The reason is precisely because of the presence of the cost-push term: unless $\widetilde{u}_t = 0$, then $\widetilde{x}_t = \widetilde{\pi}_t = 0$ will not be consistent with the Phillips Curve holding.

Let's plug this first order condition into the Phillips Curve and then simplify:

$$\begin{aligned}
\widetilde{\pi}_t &= -\frac{\gamma^2}{\omega}\widetilde{\pi}_t + \beta E_t \pi_{t+1} + \widetilde{u}_t \\
\left(1 + \frac{\gamma^2}{\omega}\right)\widetilde{\pi}_t &= \beta E_t \pi_{t+1} + \widetilde{u}_t \\
\left(\frac{\omega + \gamma^2}{\omega}\right)\widetilde{\pi}_t &= \beta E_t \pi_{t+1} + \widetilde{u}_t \\
\widetilde{\pi}_t = \left(\frac{\beta\omega}{\omega + \gamma^2}\right) E_t \pi_{t+1} &+ \left(\frac{\omega}{\omega + \gamma^2}\right)\widetilde{u}_t
\end{aligned}$$

Now because $\beta\omega < \omega$, we can solve this forward for a non-explosive solution by successively plugging in for $E_t \widetilde{\pi}_{t+1}$.

$$\widetilde{\pi}_t = \left(\frac{\omega}{\omega + \gamma^2}\right) E_t \sum_{j=0}^{\infty} \left(\frac{\beta\omega}{\omega + \gamma^2}\right)^j \widetilde{u}_{t+j}$$

We can simplify this further by noting that $E_t \widetilde{u}_{t+j} = \rho_u^j \widetilde{u}_t$:

$$\widetilde{\pi}_t = \left(\frac{\omega}{\omega + \gamma^2}\right) E_t \sum_{j=0}^{\infty} \left(\frac{\rho_u\beta\omega}{\omega + \gamma^2}\right)^j \widetilde{u}_t$$

This can be simplified:

$$\widetilde{\pi}_t = \left(\frac{\omega}{\omega+\gamma^2}\right)\frac{1}{1-\left(\frac{\rho_u\beta\omega}{\omega+\gamma^2}\right)}\widetilde{u}_t$$

$$\widetilde{\pi}_t = \frac{\omega}{\omega(1-\rho_u\beta)+\gamma^2}\widetilde{u}_t$$

This says that the central bank should let inflation rise if the cost push shock is positive. Plug this into the FOC of the problem to write it in terms of the output gap:

$$\widetilde{x}_t = \frac{-\gamma}{\omega(1-\rho_u\beta)+\gamma^2}\widetilde{u}_t$$

This means that central bank should let the output gap go negative in the face of a positive cost push shock.

Because the first order condition characterizing the solution can be written as an inverse relationship between inflation and the output gap, this kind of policy is again referred to as an inflation-targeting policy. The central bank can implement this kind of rule with an appropriate money supply rule or modified Taylor type rule.

Now let's consider the problem of the central bank under commitment. In the case of no cost-push shocks we saw that there was no welfare gain to commitment. Here there will be. Set the problem up as a current value Lagrangian:

$$\mathcal{L} = E_0\sum_{t=0}^{\infty}\beta^t\left(-\frac{1}{2}\left(\widetilde{\pi}_t^2+\omega\widetilde{x}_t^2\right)+\lambda_t\left(\widetilde{\pi}_t-\gamma\widetilde{x}_t-\beta E_t\widetilde{\pi}_{t+1}-\widetilde{u}_t\right)\right)$$

The time -1 expectation of inflation at time 0, $E_{-1}\widetilde{\pi}_0$, is taken as given. Hence, other that $\widetilde{\pi}_0$, $\widetilde{\pi}_t$ shows up twice in the period t objective, the period t constraint, and the period $t-1$ constraint (i.e. as $E_{t-1}\widetilde{\pi}_t$). Let's work through the FOCs:

$$\frac{\partial\mathcal{L}}{\partial\widetilde{\pi}_0} = 0 \Leftrightarrow \widetilde{\pi}_0 = -\lambda_0$$

$$\frac{\partial\mathcal{L}}{\partial\widetilde{\pi}_t} = 0 \Leftrightarrow -\beta^{t-1}\lambda_{t-1}\beta - E_{t-1}\beta^t\widetilde{\pi}_t + E_{t-1}\beta^t\lambda_t = 0 \Rightarrow E_{t-1}\widetilde{\pi}_t = E_{t-1}\lambda_t - \lambda_{t-1}\ \forall t>0$$

$$\frac{\partial\mathcal{L}}{\partial\widetilde{x}_t} = 0 \Leftrightarrow \widetilde{x}_t = -\frac{\gamma}{\omega}\lambda_t$$

We can combine these first order conditions to get:

$$\widetilde{x}_0 = -\frac{\gamma}{\omega}\widetilde{\pi}_0$$

$$E_t x_{t+1} = x_t - \frac{\gamma}{\omega}E_t\widetilde{\pi}_{t+1}\ \forall t$$

Start this at the beginning of time and substitute forward:

$$
\begin{aligned}
\widetilde{x}_0 &= -\frac{\gamma}{\omega}\widetilde{\pi}_0 \\
E_0\widetilde{x}_1 = x_0 - \frac{\gamma}{\omega}E_0\widetilde{\pi}_1 &= -\frac{\gamma}{\omega}\widetilde{\pi}_0 - \frac{\gamma}{\omega}E_0\widetilde{\pi}_1 \\
E_0\widetilde{x}_0 = E_0\widetilde{x}_1 - \frac{\gamma}{\omega}E_0\widetilde{\pi}_2 &= -\frac{\gamma}{\omega}\widetilde{\pi}_2 - \frac{\gamma}{\omega}E_0\widetilde{\pi}_1 - \frac{\gamma}{\omega}E_0\widetilde{\pi}_2 \\
&\vdots \\
E_0\widetilde{x}_t &= -\frac{\gamma}{\omega}\sum_{j=0}^{t}\widetilde{\pi}_{t-j}
\end{aligned}
$$

Now, what is the sum of the inflation rates between period 0 and period t? It is the (log) price level minus the price level in the period before period 0 . Then the first order condition becomes:

$$E_0\widetilde{x}_t = -\frac{\gamma}{\omega}E_0\left(\widetilde{p}_t - \widetilde{p}_{-1}\right) \tag{9}$$

This looks similar to the first order condition under commitment, but it (i) is based on expectations at the beginning of time and (ii) features the price *level* instead of the inflation rate. As such, this kind of rule is called a price level targeting rule. As $t \to \infty$, $x_t \to 0$, which means that $\lim_{t\to\infty} E_0\widetilde{p}_t = \widetilde{p}_{-1}$.

Thus the problem under commitment and discretion have different implications. In discretion, inflation always returns to target (0), but the price level may wander. In the problem under commitment, the price level is stationary. One way to essentially think about this is that the central bank "corrects" past "mistakes" in terms of inflation (either being too high or too low), whereas under discretion, "bygones are bygones".

6 Implementing Optimal Policy with Cost-Push Shocks

The first order conditions that emerge as the solution to the welfare maximization problem of the central bank in the NK model are, respectively for the case of discretion and commitment:

$$\widetilde{x}_t = -\frac{\gamma}{\omega}\widetilde{\pi}_t \tag{10}$$

$$\widetilde{x}_t = -\frac{\gamma}{\omega}\left(\widetilde{p}_t - \widetilde{p}_{-1}\right) \tag{11}$$

Let's see how the central bank can implement these via interest rate rules. Let's begin with the case of discretion. Plug the FOC into the IS equation:

$$-\frac{\gamma}{\omega}\widetilde{\pi}_t = -\frac{\gamma}{\omega}E_t\widetilde{\pi}_{t+1} - (\widetilde{r}_t - \widetilde{r}_t^f)$$

Simplify, using the Fisher relationship:

$$\widetilde{r}_t = \widetilde{r}_t^f + \frac{\gamma}{\omega}\widetilde{\pi}_t - \frac{\gamma}{\omega}E_t\widetilde{\pi}_{t+1}$$
$$\widetilde{i}_t = \widetilde{r}_t^f + \frac{\gamma}{\omega}\widetilde{\pi}_t + \left(1 - \frac{\gamma}{\omega}\right)E_t\widetilde{\pi}_{t+1}$$

Now, from above we have a relationship between current inflation and the cost-push shock of:

$$\widetilde{\pi}_t = \frac{\omega}{\omega(1-\rho_u\beta)+\gamma^2}\widetilde{u}_t$$

Iterate this forward one period in expectation:

$$E_t\widetilde{\pi}_{t+1} = \frac{\omega}{\omega(1-\rho_u\beta)+\gamma^2}\widetilde{E_t u}_{t+1} = \frac{\omega}{\omega(1-\rho_u\beta)+\gamma^2}\rho_u\widetilde{u}_t$$

Put differently:

$$E_t\widetilde{\pi}_{t+1} = \rho_u\widetilde{\pi}_t$$

Plug this into the equation for the interest rate:

$$\widetilde{i}_t = \widetilde{r}_t^f + \frac{\gamma}{\rho_u\omega}\widetilde{E_t\pi}_{t+1} + \left(1 - \frac{\gamma}{\omega}\right)E_t\widetilde{\pi}_{t+1}$$

Simplify:

$$\widetilde{i}_t = \widetilde{r}_t^f + \left(1 + \frac{\gamma}{\omega}\left(\frac{1-\rho_u}{\rho_u}\right)\right)E_t\widetilde{\pi}_{t+1} \tag{12}$$

Since the coefficient on expectation inflation exceeds 1, this rule satisfies the modified Taylor principle. Hence, the central bank following a rule in which it moves the nominal interest rate one for one with the natural rate of interest rate and more than one for one with expected inflation can implement the discretion equilibrium.

Now consider the problem of the central bank under commitment. Plug the first order condition into the IS equation:

$$-\frac{\gamma}{\omega}\left(\widetilde{p}_t - \widetilde{p}_{-1}\right) = -\frac{\gamma}{\omega}E_t\left(\widetilde{p}_{t+1} - \widetilde{p}_{-1}\right) - \left(\widetilde{r}_t - \widetilde{r}_t^f\right)$$

Simplify:

$$\widetilde{r}_t = \widetilde{r}_t^f - \frac{\gamma}{\omega}E_t\widetilde{\pi}_{t+1}$$

Plug in the Fisher relationship:

$$\widetilde{i}_t = \widetilde{r}_t^f + \left(1 - \frac{\gamma}{\omega}\right)E_t\widetilde{\pi}_{t+1}$$

Note that we have not checked whether this solution is consistent with the Phillips Curve to solve for expectation inflation:

$$E_t\widetilde{\pi}_{t+1} = \frac{1}{\beta}\widetilde{\pi}_t - \frac{\gamma}{\beta}\widetilde{x}_t - \frac{1}{\beta}\widetilde{u}_t$$

Now plug in the first order condition, and normalize $\widetilde{p}_{-1}$ to 1 so as to economize on notation:

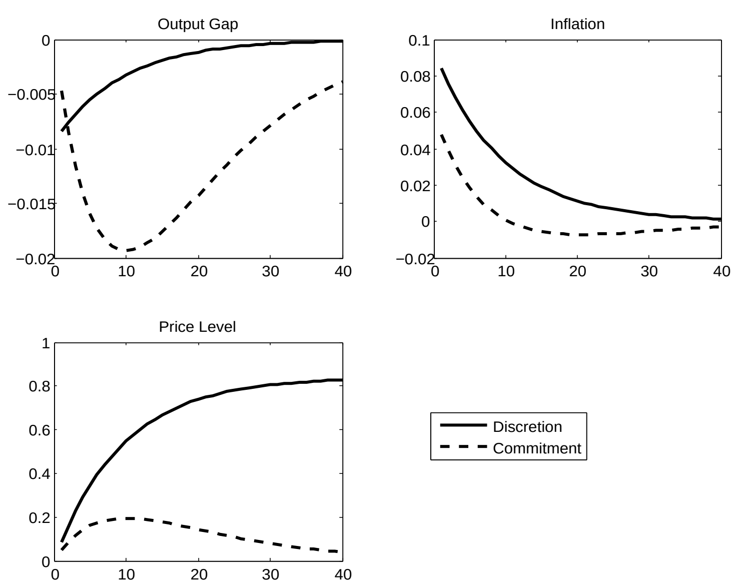
$$E_t\widetilde{\pi}_{t+1} = \frac{1}{\beta}\widetilde{\pi}_t + \frac{\gamma^2}{\beta}\widetilde{p}_t - \frac{1}{\beta}\widetilde{u}_t$$

Now plug this back into the expression for the nominal interest rate:

$$\widetilde{i}_t = \widetilde{r}_t^f + \left(1 - \frac{\gamma}{\omega}\right)\frac{1}{\beta}\widetilde{\pi}_t + \left(1 - \frac{\gamma}{\omega}\right)\frac{\gamma^2}{\beta}\widetilde{p}_t - \left(1 - \frac{\gamma}{\omega}\right)\frac{1}{\beta}\widetilde{u}_t \tag{13}$$

For this interest rate rule to be consistent with a unique solution, we need to impose that $\omega > \gamma$, so that the coefficient on the price level is positive. This will ensure a unique, non-explosive solution. Here the central bank should move the nominal rate one for one with fluctuations in the natural rate, positively with current inflation, positively with the current price level, and opposite the cost-push term.

Comparing and contrasting these two rules, we see that, in both discretion and commitment, the central bank will always want to move the interest rate one for one with the natural rate, which means completely stabilizing inflation and the output gap. The difference arises in how they react in response to the cost-push shock. The following figure shows how various variable react to the cost-push shock under the two cases (given by rules (12) for discretion and (13) for commitment). The main difference is that under commitment the price level returns to its original level, whereas under discretion the price level is permanently affected by the cost-push shock (I do not show the responses to the natural rate shock because these are the same).



7 Issues in Implementation

We have seen that the central bank can implement optimal monetary policies with nominal interest rate rules. Nevertheless, in both the discretion and commitment cases, the central bank needs to move rates one for one with the natural rate of interest, which is not necessarily observable. In the case of commitment it has to react to the cost-push shock directly as well (which is again not necessarily observable).

One can show that a "simple" Taylor rule in which the Fed just reacts to inflation and the output gap (without moving one for one with the natural rate and without reacting directly to the cost-push shock) does fairly well from a welfare perspective and does not require the Fed to observe as much.