Optimal Monetary Policy in the New Keynesian Model

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1 Introduction

These notes describe optimal monetary policy in the basic New Keynesian model.

2 Re-writing the Basic Model

The basic NK model can be characterized by two main (log-linear) equations: the Phillips Curve and the Euler/IS equation. Here I have taken the liberty of setting the elasticity of intertemporal substitution to unity:

\[ \pi_t = \gamma(y_t - \bar{y}_f) + \beta E_t \pi_{t+1} \]
\[ y_t = E_t \bar{y}_{t+1} - \bar{r}_t \]

To economize on notation, let’s define \( \bar{x}_t = \bar{y}_t - \bar{y}_f \) as the output gap. Similarly, let’s define \( \bar{r}_f \) as the “flexible price real interest rate” (or “natural real interest rate”) as that rate that would obtain if prices were fully flexible. We can solve for this by looking at the Euler equation:

\[ \bar{r}_f = E_t \bar{y}_{t+1} - \bar{y}_f \]

Because of the assumption on preferences, in particular that the coefficient of relative risk aversion is 1, we know that labor hours would be constant if prices were flexible, so we know that the flexible price level of output evolves exogenously in line with the level of technology, with \( \bar{y}_f = \bar{a}_t \). If we assume that technology obeys an AR(1), then we can model the flexible price equilibrium level of output as following the same AR(1):

\[ \bar{y}_t = \rho \bar{y}_{t-1} + \epsilon_t \]

This means that we can solve for the flexible price real interest rate as:

\[ \bar{r}_f = (\rho - 1) \bar{y}_f \]
Plugging in the process for $\tilde{y}_t^f$ and simplifying we get a process for the natural rate of interest:

$$\tilde{r}_t^f = (\rho - 1)\tilde{r}_{t-1} + (\rho - 1)\tilde{\varepsilon}_t$$

$$\tilde{r}_t^f = \rho\tilde{r}_{t-1} + (\rho - 1)\tilde{\varepsilon}_t$$

We can then summarize the main equations of the model as follows:

$$\tilde{\pi}_t = \gamma\tilde{x}_t + \beta E_t\tilde{\pi}_{t+1}$$
$$\tilde{x}_t = E_t\tilde{x}_{t+1} - (\tilde{r}_t - \tilde{r}_t^f)$$
$$\tilde{r}_t^f = \rho\tilde{r}_{t-1} + (\rho - 1)\tilde{\varepsilon}_t$$

In the background there is also (i) a money demand relationship and (ii) a Fisher relationship. For now, we can think about the central bank effectively being able to choose $\tilde{r}_t$, given a choice of $\tilde{i}_t$ and an implied path for $E_t\tilde{\pi}_{t+1}$. Given that, as well as $\tilde{r}_t^f$ (which is the exogenous driving force), $x_t$ and $\pi_t$ will be determined.

3 Distortions and Welfare

There are two welfare-reducing distortions in the NK model, one of which is essentially “long run” and the other which is “short run”. The “long run” distortion is that the flexible price level of output will be lower than what would obtain in the first best. This is because, in the flexible price version of the model, firms will set price equal to a markup over marginal cost. Hence there will be too little employment. The “short run” distortion is due to price stickiness, and leads to non-optimal fluctuations in relative prices.

We assume that the central bank is concerned with the “short run distortion” and that the “long run distortion” has been taken care of via some kind of Pigouvian tax. This works out to a subsidy for labor, equal to the inverse price markup. This means we can interpret $\tilde{y}_t^f$ as the optimal equilibrium value of output from the perspective of the central bank. This means that, other things being equal, the central bank would like to eliminate output gaps.

4 Optimal Policy

In addition to disliking output gaps, we also assume that the central bank dislikes inflation. We assume that welfare of the central bank is a present discounted value of a quadratic loss function in inflation and the output gap. This loss function can actually be derived from taking a quadratic approximation to household welfare, while using the linearized equilibrium conditions (see Gali or Woodford’s textbooks for a formal derivation. You may wonder why the central bank cares about inflation over and above the output gap (which, via the logic above, the central bank would like
to eliminate). If you go back to the CES aggregator over intermediate goods, you will note that it is concave, meaning that households (or the final goods firm, if you like) would like to smooth over intermediate inputs. In a flexible price world, all intermediate producers would choose the same price (e.g. they all desire a relative price of 1). If aggregate inflation is different from zero, with price stickiness relative prices at the intermediate firm level get distorted (e.g. there is price dispersion). This leads to a non-smooth allocation of intermediates, which results in a welfare loss.

Let $\omega$ denote the relative weight that the central bank places on the output gap. In the formal derivation, you can show that this is equal to $\frac{2}{\gamma}$, where $\gamma$ is the slope of the Phillips Curve and $\epsilon$ is the price elasticity of demand. The central bank would therefore like to minimize the following:

$$\min \frac{1}{2} E_0 \left( \sum_{t=0}^{\infty} \beta^t (\tilde{\pi}^2_t + \omega \tilde{x}^2_t) \right)$$

The $1/2$ on the outside is just a scaling term that doesn’t affect the optimum but simplifies things a bit. As noted above, we can think about the central bank as choosing inflation and the output gap, given its choice of $\tilde{i}_t$, which then determines $r_t$ given a path of $E_t \pi_{t+1}$. This must be done subject to the constraint of the Phillips Curve, however.

We consider two cases. In the first, called “discretion”, the central bank solves the one period problem each period. In the other, called “commitment”, the central bank solves the entire problem at the beginning of time and commits to its policy. We start first with the discretion case. The problem can be written:

$$\min_{\pi_t, x_t} \frac{1}{2} \left( \pi^2_t + \omega x^2_t \right)$$

s.t.

$$\pi_t = \gamma x_t + \beta E_t \pi_{t+1}$$

Set the problem up as a Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \left( \pi^2_t + \omega x^2_t \right) + \lambda (\pi_t - \gamma x_t - \beta E_t \pi_{t+1})$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \iff \pi_t = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial x_t} = 0 \iff \omega x_t = -\lambda \gamma$$

Combing FOCs so as to eliminate the Lagrange multiplier, we get:

$$x_t = -\frac{\gamma}{\omega} \pi_t$$  \hspace{1cm} (4)
Loosely speaking, this first order condition can be interpreted as a “lean against the wind” policy. If the output gap is positive, the Fed will want to pursue a policy in which it lowers inflation (and vice-versa).

Next, consider the problem under commitment. Here, the objective of the central bank is not just the current objective, but the present discounted value of the flow objective functions. A Lagrangian is:

\[
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left( -\frac{1}{2} (\tilde{\pi}_t^2 + \omega \tilde{x}_t^2) + \lambda_t (\tilde{\pi}_t - \gamma \tilde{x}_t - \beta E_t \tilde{\pi}_{t+1}) \right)
\]

The time -1 expectation of inflation at time 0, \(E_{-1} \tilde{\pi}_0\), is taken as given. Hence, other than \(\tilde{\pi}_0\), \(\tilde{\pi}_t\) shows up twice in the period \(t\) objective, the period \(t\) constraint, and the period \(t-1\) constraint (i.e. as \(E_{t-1} \tilde{\pi}_t\)). Let’s work through the FOCs:

\[
\frac{\partial \mathcal{L}}{\partial \tilde{\pi}_0} = 0 \Leftrightarrow \tilde{\pi}_0 = -\lambda_0
\]

\[
\frac{\partial \mathcal{L}}{\partial \tilde{\pi}_t} = 0 \Leftrightarrow -\beta^{t-1} \lambda_{t-1} \beta - E_{t-1} \beta^t \tilde{\pi}_t + E_{t-1} \beta^t \lambda_t = 0 \Rightarrow E_{t-1} \tilde{\pi}_t = E_{t-1} \lambda_t - \lambda_{t-1} \forall t > 0
\]

\[
\frac{\partial \mathcal{L}}{\partial \tilde{x}_t} = 0 \Leftrightarrow \tilde{x}_t = -\frac{\gamma}{\omega} \lambda_t
\]

We can combine these first order conditions to get:

\[
\tilde{x}_0 = -\frac{\gamma}{\omega} \tilde{\pi}_0
\]

\[
E_t \tilde{x}_{t+1} = x_t - \frac{\gamma}{\omega} E_t \tilde{\pi}_{t+1} \forall t
\]

Start this at the beginning of time and substitute forward:

\[
\tilde{x}_0 = -\frac{\gamma}{\omega} \tilde{\pi}_0
\]

\[
E_0 \tilde{x}_1 = x_0 - \frac{\gamma}{\omega} E_0 \tilde{\pi}_1 = -\frac{\gamma}{\omega} \tilde{\pi}_0 - \frac{\gamma}{\omega} E_0 \tilde{\pi}_1
\]

\[
E_0 \tilde{x}_0 = E_0 \tilde{x}_1 - \frac{\gamma}{\omega} E_0 \tilde{\pi}_2 = -\frac{\gamma}{\omega} \tilde{\pi}_2 - \frac{\gamma}{\omega} E_0 \tilde{\pi}_1 - \frac{\gamma}{\omega} E_0 \tilde{\pi}_2
\]

\[
\vdots
\]

\[
E_0 \tilde{x}_t = -\frac{\gamma}{\omega} \sum_{j=0}^{t} \tilde{\pi}_{t-j}
\]

Now, what is the sum of the inflation rates between period 0 and period \(t\)? It is the (log) price level minus the price level in the period before period 0. Then the first order condition becomes:
\[ E_0 \tilde{x}_t = -\frac{\gamma}{\omega} E_0 (\tilde{p}_t - \tilde{p}_{t-1}) \] (5)

Since this must hold in expectation, and there are no disturbances that show up here either, it must also hold ex-post. This means we can get rid of the expectations operator in the FOC. To simplify matters, we can also normalize the initial price level to 0, so the FOC becomes:

\[ \tilde{x}_t = -\frac{\gamma}{\omega} \tilde{p}_t \] (6)

This looks similar to the first order condition under commitment, but it features the price level as opposed to price inflation. As such, this kind of rule is called a price level targeting rule. As \( t \to \infty \), \( x_t \to 0 \), which means that \( \lim_{t \to \infty} E_0 \tilde{p}_t = \tilde{p}_{t-1} \). This means that the policy under commitment implies that the price level always returns to trend.

Now, let’s give a little thought to the solutions under discretion and commitment. From the FOC for discretion, note that a policy of \( \tilde{x}_t = \tilde{\pi}_t = 0 \) is consistent with the FOC holding. But is it consistent with the other equations of the model holding? It turns out that it is. If you plug these into the Phillips Curve, you can see that the Phillips Curve can hold at all times with \( \tilde{\pi}_t = \tilde{x}_t = 0 \). From the IS equation, this implies that \( \tilde{r}_t = \tilde{r}_t^f \) at all times. With inflation always equal to zero, this implies that the nominal interest rate should track the natural rate of interest at all times, \( \tilde{i}_t = \tilde{r}_t^f \).

Interestingly, this also means that there is no gain from commitment over discretion. If the discretion solution always has both inflation and the gap equal to zero, it achieves the global minimum of the objective function. There is also no relevant tradeoff between real (gap) and nominal (inflation) stabilization: the central bank can achieve both. There are a couple of ways to implement this policy in a Dynare code. The easiest is to simply replace a Taylor rule (or money supply equation) with the first order condition in the list of equilibrium conditions. This will spit out a time path for the nominal interest rate and time paths for inflation and the gap.

Below I show impulse responses for a quantitative version of the model. I set \( \gamma = 0.3 \), \( \beta = 0.99 \), \( \rho = 0.95 \), and \( \epsilon = 10 \), which implies a weight on the output gap of 0.03. For ease of comparison, I compare the optimal discretion/commitment responses (as noted above, these are the same) with the response that would obtain with a Taylor rule of the form \( \tilde{i}_t = \rho_i \tilde{i}_{t-1} + (1 - \rho_i) \phi_{\pi} \tilde{\pi}_t \), where I set \( \rho_i = 0.8 \) and \( \phi_{\pi} = 1.5 \). These are impulse responses to a productivity shock (which manifests itself as a shock to the natural rate of interest):
As predicted via our discussion above, the optimal policy results in no movement of the gap or inflation at any horizon. This also then shows up as a constant price level. In contrast, the Taylor rule results in a negative output gap (output rises by less than the flexible price level), disinflation, and a pretty large fall in the price level. The Taylor rule also results in a larger and more persistent decline in the nominal interest rate than does the optimal policy under discretion.

Would it be possible to implement the optimal discretionary/commitment policy (again, they are the same here) using a rule for the interest rate or money growth? At least in this circumstance, the answer is yes.

First, consider a money rule. Suppose that the demand for real balances is given by:

\[ \tilde{m}_t = \frac{1}{\nu} \tilde{y}_t - \kappa \tilde{i}_t \]

The solution discussed above means that \( \tilde{y}_t = \tilde{y}_f \) and \( \tilde{i}_t = \tilde{r}_f \). Plug this in:

\[ \tilde{m}_t = \frac{1}{\nu} \tilde{y}_f - \kappa \tilde{r}_f \]

From above, we have \( \tilde{r}_f = (\rho - 1) \tilde{y}_f \). Plug this in:

\[ \tilde{m}_t = \frac{1}{\nu} \tilde{y}_f - \kappa (\rho - 1) \tilde{y}_f = \left( \frac{1}{\nu} (1 - \kappa (\rho - 1)) \right) \tilde{y}_f \]

Now we know that \( \tilde{y}_f = \rho \tilde{y}_{f-1} \). Hence:

\[ \tilde{m}_t = \frac{1}{\nu} \tilde{y}_t - \kappa (\rho - 1) \tilde{y}_t = \left( \frac{1}{\nu} (1 + \kappa (1 - \rho)) \right) \tilde{y}_t \]

Iterate this back one period to eliminate \( \tilde{y}_f \):

\[ \tilde{m}_t = \rho \tilde{m}_{t-1} + \left( \frac{1}{\nu} (1 + \kappa (1 - \rho)) \right) \varepsilon_t \]

Now let's write this in terms of the nominal money supply, which the central bank controls, by noting that \( \tilde{m}_t = \tilde{M}_t - \tilde{p}_t \), with \( \pi_t = \tilde{p}_t - \tilde{p}_{t-1} \):
\[ \tilde{M}_t = \rho \tilde{M}_{t-1} - \rho \tilde{\pi}_t - (1 - \rho) \tilde{p}_t + \left( \frac{1}{\nu} \left( 1 + \kappa (1 - \rho) \right) \right) \varepsilon_t \] (7)

In other words, the central bank needs to set the money supply according to an AR(1) in the level, with AR coefficient equal to the AR coefficient on the technology process. As long as \( \rho < 1 \), it will raise the money supply in response to positive technology shocks. There is also a slight adjustment for lagged inflation and the price level.

The central bank can also accomplish its goal with a modified Taylor rule. In particular, suppose it sets the nominal interest rate according to the following rule.

\[ \tilde{i}_t = \tilde{r}_t^{f} + \phi_{\pi} \tilde{\pi}_t + \phi_{x} \tilde{x}_t \] (8)

This Taylor rule looks similar to the Taylor rules we’ve looked at, but with two differences: there is no smoothing parameter, and there is a “stochastic intercept” equal to the natural rate of interest. A policy rule of \( \tilde{i}_t = \tilde{r}_t^{f} \) would result in equilibrium indeterminacy – for an interest rate rule to work, as we have seen, there needs to be a sufficiently strong reaction to endogenous variables like inflation and/or the output gap. What is kind of interesting here, however, is that in equilibrium inflation and the output gap would always be zero with this interest rate rule (provided \( \phi_{\pi} \) and/or \( \phi_{x} \) are sufficiently large). This means that one would observe \( \tilde{i}_t = \tilde{r}_t^{f} \), but if the central bank announced that as the policy rule, it would result in indeterminacy. The central bank in essence has to promise to move interest rates sufficiently in response to inflation and the output gap to prevent those from ever occurring.

5 Cost-Push Shocks and an Output-Inflation Tradeoff

Let’s modify the Phillips curve to contain an additional term:

\[ \tilde{\pi}_t = \gamma \tilde{x}_t + \beta E_t \pi_{t+1} + \tilde{u}_t \] (9)

Here \( \tilde{u}_t \) is referred to as the “cost-push” term. It is a shock which, in a sense, changes the output-inflation tradeoff. It is exogenous and the central bank takes it as given. The structural interpretation of this term is not always very clear – one specific interpretation is time-variation in \( \epsilon \), which means there are time-varying desired markups. More generally, the cost-push term can be thought of as something which drives a wedge between marginal cost and the output gap. In reality it is a convenient shortcut to make the central bank’s problem more interesting. Assume that it follows an AR(1):

\[ \tilde{u}_t = \rho_u \tilde{u}_{t-1} + \varepsilon_{u,t} \] (10)

Why does the inclusion of the cost-push term make the central bank’s problem more interesting? The optimization problem of the central bank is identical to before, and results in the same first order conditions under discretion and commitment:
\[ \tilde{x}_t = -\frac{\gamma}{\omega} \tilde{\pi}_t \]  \hspace{1cm} (11)
\[ \tilde{x}_t = -\frac{\gamma}{\omega} \tilde{p}_t \] \hspace{1cm} (12)

The reason the cost-push shock makes things more interesting is that it is in general not going to be feasible to implement a no gap / no inflation equilibrium. If \( \tilde{u}_t \neq 0 \), then \( \tilde{x}_t = \tilde{\pi}_t \) is not consistent with the Phillips Curve holding. In other words, the inclusion of the cost-push shock (i) generates a non-trivial tradeoff for the central bank, as it cannot achieve a zero inflation / zero gap outcome, and (ii) opens up the door for welfare gains from commitment. Of course, conditional on a productivity shock (so that \( \tilde{u}_t = 0 \)), there is no real tradeoff – so it would be optimal for the central bank to completely stabilize both inflation and the output gap in response to productivity shocks, with no resulting welfare gain from commitment.

Below I compute impulse responses to a cost-push shock under various different monetary policy rules. Given the linearity of the model, the responses to the productivity shock are identical to what I showed above, and under either commitment or discretion both the output gap and inflation are completely stabilized in response to the productivity shock. Regardless of the kind of policy (either of the optimal policies or the Taylor rule), the cost-push shock causes the output gap to decline (output falls) and inflation rises. The first set of plots shows the responses under the baseline Taylor rule as well as those under the optimal policy under discretion. Interestingly, the simple Taylor rule appears to do better in the sense of a small decline in the output gap and a smaller increase in the price level (smaller increase in inflation over most horizons). Under either scenario, there is a permanent increase in the price level following the cost-push shock.

Next, I compare the impulse responses from the optimal policy under discretion with the optimal policy under commitment. Here we see some stark differences. First, the output gap responses is smaller (on impact) under commitment than under discretion. Second, the inflation response is much smaller under commitment and much less persistent, so much so that the price level returns to its original value under commitment, whereas the price level permanently rises under discretion. Third, there is a very different behavior of the nominal interest rate: under commitment the nominal...
rate initially falls, whereas under discretion it rises.

What is the source of the gains from commitment? As you will recall from looking at the first order conditions, the difference between commitment and discretion boils down to an implicit price level (commitment) versus inflation (discretion) target. The price level target has the effect of better anchoring expected inflation, $E_t \tilde{\pi}_{t+1}$, because agents know that the central bank will always enact policy so as to return the price level to its target. Better anchored inflation expectations (e.g. less volatile expected inflation) improves the available tradeoff between current inflation and the output gap that the central bank can achieve. Mechanically from the Phillips Curve, the more $E_t \tilde{\pi}_{t+1}$ moves around, the more either inflation or the output gap will have to move around (or both), either of which reduce welfare. Hence, a central message that comes out of this exercise is that commitment and expectations are of central importance for good monetary policy.