Money in a Real Business Cycle Model

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This document describes how to include money into an otherwise standard real business cycle model. It is not easy to get agents to hold money in equilibrium. The intuition is fairly straightforward – money earns no interest and provides no utility, so there is no direct (utility) or indirect (savings) motivation for holding it. There is a large literature exploring micro foundations of money, wherein the frictions giving rise to money are explicitly modeled (e.g. Kiyotaki and Wright (1989)), typically in a search theoretic framework. It goes without saying that it is extremely difficult to incorporate such a framework into a standard DSGE model. As such, we'll be taking a shortcut (this is quite conventional in the literature).

There are two basic shortcuts for getting agents to hold money – cash in advance and money in the utility function. Both get at the medium of exchange function of money – in the cash in advance framework, money is necessary to conduct exchange, while in the money in the utility function, having more money increases utility, presumably because it makes exchange "easier". An example of money in the utility function is Sidrauski (1967). Examples of cash in advance economies are Lucas (1987) and Cooley and Hansen (1989).

The Budget Constraint

Before going any further we need to talk first about how money enters the households' budget constraint. I will be setting this problem up as a decentralized equilibrium, so I will imagine a world in which households hold nominal bonds issued by firms to get capital, B_t, and/or receive profit distributions from firms. Households have three sources of nominal income in period t: nominal wage income, previous period bond holding plus nominal interest, and previous period money holdings. With this nominal income households can either purchase more goods, more bonds, or hold more money. The constraint is:

$$p_t c_t + B_t + M_t = W_t n_t + p_t \Pi_t + (1 + i_{t-1}) B_{t-1} + M_{t-1}$$

Take note of the dating convention here: B_{t-1} and M_{t-1} denote bond and money holdings held between period $t-1$ and period t, with i_{t-1} the nominal interest rate on bond holdings. We want to write the budget constraint in real terms. Divide everything by p_t:

$$c_t + \frac{B_t}{p_t} + \frac{M_t}{p_t} = \frac{W_t}{p_t} n_t + \Pi_t + (1 + i_{t-1}) \frac{B_{t-1}}{p_t} + \frac{M_{t-1}}{p_t}$$

Define $b_t = \frac{B_t}{p_t}$ as real bond holdings, $m_t = \frac{M_t}{p_t}$ as real money balances, and $w_t = \frac{W_t}{p_t}$ as the real wage. The constraint is then:

$$c_t + b_t + m_t = w_t n_t + \Pi_t + (1 + i_{t-1}) \frac{B_{t-1}}{p_t} + \frac{M_{t-1}}{p_t}$$

We need to play around to get the right hand side in appropriate terms. Define $1 + \pi_t = \frac{p_t}{p_{t-1}}$. Using this, we have:

$$\begin{aligned}
c_t + b_t + m_t &= w_t n_t + \Pi_t + (1+i_{t-1})\frac{B_{t-1}}{p_{t-1}}\frac{p_{t-1}}{p_t} + \frac{M_{t-1}}{p_{t-1}}\frac{p_{t-1}}{p_t} \\
c_t + b_t + m_t &= w_t n_t + \Pi_t + (1+i_{t-1})\frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t}
\end{aligned}$$

The Cash in Advance Constraint

The cash in advance constraint is fairly simple. It says that the household must have enough cash held over from the previous period to finance today's nominal consumption expenditures. Formally:

$$M_{t-1} \geq p_t c_t$$

Writing this out in real terms yields:

$$\begin{aligned}
\frac{M_{t-1}}{p_t} &\geq c_t \\
\frac{M_{t-1}}{p_{t-1}}\frac{p_{t-1}}{p_t} &\geq c_t \\
m_{t-1} &\geq (1+\pi_t)c_t
\end{aligned}$$

The Money Growth Rule

We assume that the quantity of money is set exogenously by some third party (e.g. the Fed). Higher inflation is essentially a tax on the holders of money. For simplicity, we assume that the third party just squanders any revenue it generates by printing money. We'll come back to this issue when studying fiscal policy.

It is clear from the cash in advance constraint that steady state inflation will be equal to the steady state growth rate of money. Since we want to allow for positive steady state inflation, I'm going to write the money rule as an AR(1) in the growth rate (i.e. log first difference) with a positive mean. Formally:

$$\ln M_t - \ln M_{t-1} = (1-\rho_m)\pi^* + \rho_m\left(\ln M_{t-1} - \ln M_{t-2}\right) + \varepsilon_{m,t}$$

Because we want to write everything in terms of real balances (which removes the trend), let's play around with this by adding and subtracting logs of the price level at various lags:

$$\ln M_t - \ln p_t + \ln p_t - \ln p_{t-1} - \ln M_{t-1} + \ln p_{t-1} = (1-\rho_m)\pi^* + ...$$

$$... + \rho_m\left(\ln M_{t-1} - \ln p_{t-1} + \ln p_{t-1} - \ln p_{t-2} - \ln M_{t-2} + \ln p_{t-2}\right) + \varepsilon_{m,t}$$

Nothing that $\ln p_t - \ln p_{t-1} = \pi_t$ (as a first order approximation), we have:

$$m_t - m_{t-1} + \pi_t = (1-\rho_m)\pi^* + \rho_m \pi_{t-1} + \rho_m(m_{t-1} - m_{t-2}) + \varepsilon_{m,t}$$

It is straightforward to verify that this will hold in the steady state with the growth rate of real balances being equal to zero. I can define a new variable to be $dm_t = m_t - m_{t-1}$ so that I can make there be only one lag in the system.

The Rest of the Model

The rest of the model is our standard real business cycle model, plus the additional constraints. I begin with the household problem. The household chooses consumption, labor, real bond holdings, and real money balances to satisfy:

$$\max_{c_t,n_t,b_t,m_t} \qquad E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} + \theta \frac{(1-n_t)^{1-\xi} - 1}{1-\xi} \right)$$

s.t.

$$\begin{aligned} c_t + b_t + m_t &\leq w_t n_t + \Pi_t + (1+i_{t-1}) \frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t} \\ (1+\pi_t) c_t &\leq m_{t-1} \end{aligned}$$

I solve the model using a Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{c_t^{1-\sigma}-1}{1-\sigma} + \theta \frac{(1-n_t)^{1-\xi}-1}{1-\xi} + \lambda_t \left(w_t n_t + \Pi_t + (1+i_{t-1}) \frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t} - c_t - b_t - m_t \right) + ... \\ ... + \mu_t \left(m_{t-1} - (1+\pi_t) c_t \right) \end{array} \right\}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= 0 \Leftrightarrow c_t^{-\sigma} = \lambda_t + \mu_t (1+\pi_t) \\ \frac{\partial \mathcal{L}}{\partial n_t} &= 0 \Leftrightarrow \theta (1-n_t)^{-\xi} = \lambda_t w_t \\ \frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Leftrightarrow \lambda_t = \beta \lambda_{t+1} \frac{1+i_t}{1+\pi_{t+1}} \\ \frac{\partial \mathcal{L}}{\partial m_t} &= 0 \Leftrightarrow \lambda_t = \beta \left(\frac{\lambda_{t+1}}{1+\pi_{t+1}} + \mu_{t+1} \right) \end{aligned}$$

Before proceeding, take a step back to note that these FOCs make sense. First, note that the Fisher relationship is: $1 + r_t = \frac{1+i_t}{1+\pi_{t+1}}$, so the Euler equation looks identical to what we've had before. Second, suppose that the cash in advance constraint weren't there, which would mean that $\mu_t = 0$ at all times. Then the first FOC would look normal, and the third and fourth would be equivalent if and only if $i_t = 0$. If the nominal rate were positive, we'd be in a corner solution (i.e. no one would ever hold money), and so the final FOC wouldn't be necessary for a solution. This makes intuitive sense – the opportunity cost of holding money is the nominal interest rate. If there is no reason to hold money (i.e. no cash in advance constraint), then no one would ever hold money at a positive interest rate.

Now let's go to the firm problem. I assume that the firm owns its capital stock and chooses employment and investment each period to maximize the present discounted value of (real) profits, with a Cobb-Douglas production function, and subject to the capital accumulation equation. The value of the firm can be written as current profits plus the discounted value of future profits:

$$V_t = a_t k_{t-1}^{\alpha} n_t^{1-\alpha} - w_t n_t - I_t + \sum_{j=1}^{\infty} \prod_{k=1}^{j} (1 + r_{t-1+k})^{-1} \left(a_{t+j} k_{t+j-1}^{\alpha} n_{t+j}^{1-\alpha} - w_{t+j} n_{t+j} - I_{t+j} \right)$$

The capital accumulation equation is standard:

$$k_t = I_t + (1 - \delta) k_{t-1}$$

I can solve the problem using a Lagrangian:

$$\begin{aligned}
\mathcal{L} &= a_t k_{t-1}^{\alpha} n_t^{1-\alpha} - w_t n_t - I_t + \sum_{j=1}^{\infty} \prod_{k=1}^{j} (1 + r_{t-1+k})^{-1} \left(a_{t+j} k_{t+j-1}^{\alpha} n_{t+j}^{1-\alpha} - w_{t+j} n_{t+j} - I_{t+j} \right) + ... + \\
&\quad ... + q_t (I_t + (1 - \delta) k_{t-1} - k_t)
\end{aligned}$$

The FOC are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial n_t} &= 0 \Leftrightarrow w_t = (1 - \alpha) a_t k_{t-1}^{\alpha} n_t^{-\alpha} \\
\frac{\partial \mathcal{L}}{\partial I_t} &= 0 \Leftrightarrow q_t = 1 \\
\frac{\partial \mathcal{L}}{\partial k_t} &= 0 \Leftrightarrow -q_t + \frac{1}{1 + r_t} \left(\alpha a_{t+1} k_t^{\alpha-1} n_{t+1}^{1-\alpha} + q_{t+1} (1 - \delta) \right) \\
&\Rightarrow 1 + r_t = \alpha a_{t+1} k_t^{\alpha-1} n_{t+1}^{1-\alpha} + (1 - \delta)
\end{aligned}$$

These are all (or should be) familiar conditions by now.

I assume that technology follows a stationary AR(1) process (i.e. I abstract from trend growth, which doesn't make much of a difference anyway):

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_{a,t}$$

Because the unconditional mean of the log of technology is zero, the steady state of the level of technology is one. Aggregate market-clearing requires:

$$y_t = c_t + I_t$$

The full model can be characterized by the following fourteen equations:

$$c_t^{-\sigma} = \lambda_t + \mu_t (1 + \pi_t) \tag{1}$$

$$\theta(1-n_t)^{-\xi} = \lambda_t w_t \tag{2}$$

$$\lambda_t = \beta \lambda_{t+1} \frac{1+i_t}{1+\pi_{t+1}} \tag{3}$$

$$\lambda_t = \beta \left(\frac{\lambda_{t+1}}{1+\pi_{t+1}} + \mu_{t+1} \right) \tag{4}$$

$$w_t = (1-\alpha) a_t k_{t-1}^{\alpha} n_t^{-\alpha} \tag{5}$$

$$1 + r_t = \alpha a_{t+1} k_t^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta) \tag{6}$$

$$k_{t+1} = I_t + (1-\delta) k_t \tag{7}$$

$$1 + r_t = \frac{1+i_t}{1+\pi_{t+1}} \tag{8}$$

$$y_t = a_t k_{t-1}^{\alpha} n_t^{1-\alpha} \tag{9}$$

$$y_t = c_t + I_t \tag{10}$$

$$dm_t + \pi_t = (1-\rho_m)\pi^* + \rho_m \pi_{t-1} + \rho_m dm_{t-1} + \varepsilon_{m,t} \tag{11}$$

$$dm_t = \ln m_t - \ln m_{t-1} \tag{12}$$

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_{a,t} \tag{13}$$

$$m_{t-1} = (1+\pi_t) c_t \tag{14}$$

Note that, as specified, I do not explicitly include the price level or the nominal money supply in the first order conditions. This is a matter of convenience, as Dynare will not solve the model with non-stationary variables, and both of these variables will be non-stationary. If I desire to reconstruct these variables (as I will in the impulse responses below), I can construct these series after running Dynare by defining them as follows:

$$\ln M_t = \ln m_t + \ln p_t \tag{15}$$

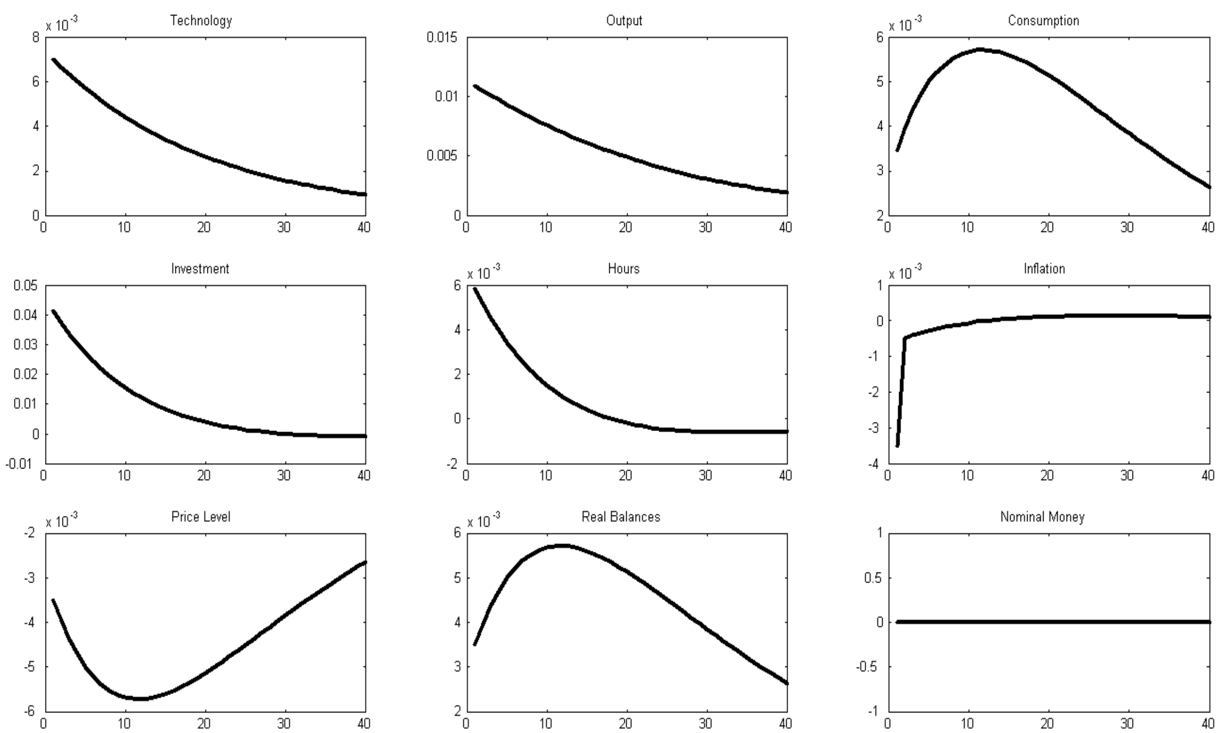
$$\ln p_t = \pi_t + \ln p_{t-1} \tag{16}$$

I calibrate the parameters as follows:

$$
\begin{aligned}
\beta &= 0.98 \\
\alpha &= 0.33 \\
\delta &= 0.03 \\
\rho &= 0.95 \\
\sigma &= 1 \\
\theta &= 3.5 \\
\xi &= 1 \\
\pi^* &= 0.01 \\
\rho_m &= 0.75
\end{aligned}
$$

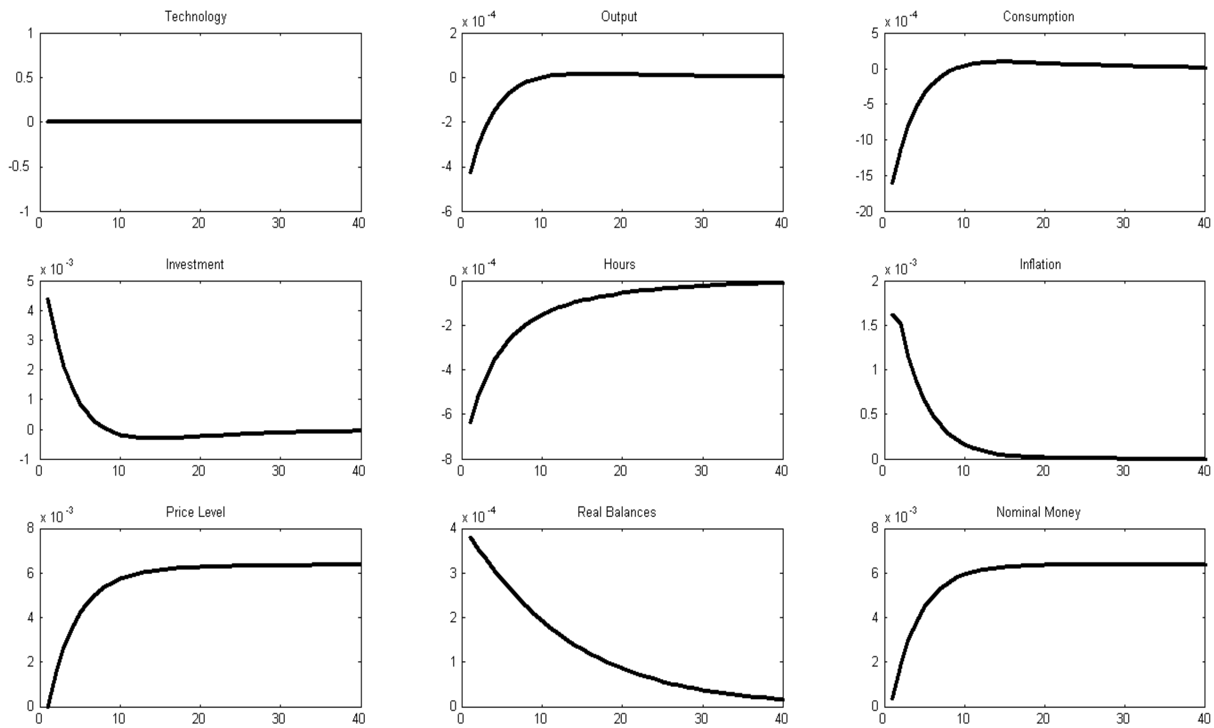
This means that we're in the familiar "log-log" case for preferences and that steady state annual inflation is about 4 percent. In addition, I set the standard deviation of the technology shock to 0.007 and the standard deviation of the money growth shock to 0.002.

I solve the model using a first order log-linear approximation in Dynare. The impulse responses to the two shocks are below. As noted above, I compute the responses of the price level and the nominal money supply outside of the Dynare .mod file.



It is interesting to note that the responses of the real variables of the model are identical in the model here as they would be in an RBC model without money explicitly included. This is comforting – abstracting from money before doesn't really change our conclusions (at least in this framework). We see that the technology shock lowers the price level (i.e. inflation jumps down immediately) and leads to an increase in real balances. Nominal money, by construction, does not respond.

Below are the impulse responses to a money growth shock:



Here we see something very interesting and not very intuitive. Money is not neutral here – it has real effects, albeit these real effects are quite small. Further, we observe that a temporary increase in the growth rate of money (i.e. a permanent change in the level of the nominal money supply) actually lowers output, hours, and consumption, which is not particularly intuitive. We'll come back to this in a moment.

Something else worth playing around with is to change the steady state inflation rate, π^*. Below is a table showing the steady state level of output for different inflation rates:

π^*	y^*
-0.01	0.5443
0.00	0.5400
0.01	0.5359
0.02	0.5317
0.05	0.5196

We see that steady state output is decreasing in steady state inflation. We must impose that $1 + \pi^* > \beta$ for a solution to exist, so it is not possible to have very large steady state deflation. This table (that long run output is decreasing in long run inflation) actually seems intuitive, especially in comparison to the result about that increase in the money supply reduce output in the short run.

The intuition for these results is as follows. Inflation is essentially a tax on the holders of money. The more inflation there is, the more you penalize people who hold money, and thereby the less money people would like to hold, other things being equal (of course in equilibrium they have to hold whatever money the central bank prints). Because money is necessary to consume (i.e the cash in advance constraint), there is a non-neutrality here. When inflation is higher, people want to hold less money. Since in equilibrium they can't hold less money than the central bank prints, they end up substituting away from things which require money (consumption) and into things that don't (leisure). This ends up reducing output in the long run. The temporary increase in the money supply works the same way, albeit only for a while. People try to get away from money and into leisure, so consumption and employment both go down immediately when the growth rate of money jumps up. For these parameters, there is a temporary increase in investment following the increase in money (i.e. the reduction in consumption "dominates" the reduction in hours in terms of what happens to investment). The Dynare code I used to produce these figures is called "money_cia.mod".

Money in the Utility Function

The other popular way of getting people to hold money in equilibrium is to explicitly put the quantity of real balances into the flow utility function. This may seem somewhat ad-hoc, but it gets at the idea that the existence of money makes exchange "easier", thereby increasing utility. Most of the model is identical to above, but preferences are different. The household faces the following problem:

$$\max_{c_t, n_t, b_t, m_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} + \theta \frac{(1-n_t)^{1-\xi} - 1}{1-\xi} + \psi \frac{m_t^{1-\zeta} - 1}{1-\zeta} \right)$$

s.t.

$$c_t + b_t + m_t \leq w_t n_t + \Pi_t + (1 + i_{t-1}) \frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t}$$

Unlike the set up above, there is no cash in advance constraint here. The Lagrangian for the problem can be written:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{c_t^{1-\sigma}-1}{1-\sigma} + \theta \frac{(1-n_t)^{1-\xi}-1}{1-\xi} + \psi \frac{m_t^{1-\zeta}-1}{1-\zeta} + ... \\ ... + \lambda_t \left(w_t n_t + \Pi_t + (1+i_{t-1}) \frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t} - c_t - b_t - m_t \right) \end{array} \right\}$$

The first order conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= 0 \Leftrightarrow c_t^{-\sigma} = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial n_t} &= 0 \Leftrightarrow \theta(1-n_t)^{-\xi} = \lambda_t w_t \\
\frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Leftrightarrow \lambda_t = \beta \lambda_{t+1} \frac{1+i_t}{1+\pi_{t+1}} \\
\frac{\partial \mathcal{L}}{\partial m_t} &= 0 \Leftrightarrow \beta^t \psi m_t^{-\zeta} - \beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \frac{1}{1+\pi_{t+1}} = 0 \\
&\Rightarrow \psi m_t^{-\zeta} = \lambda_t - \beta \lambda_{t+1} \frac{1}{1+\pi_{t+1}}
\end{aligned}
$$

Given the Fisher relationship, the first three first order conditions are identical to what would obtain in a real model without mention of money at all. We can simplify the first order condition for real balances by noting that $\beta\lambda_{t+1}\frac{1}{1+\pi_{t+1}} = \frac{\lambda_t}{1+i_t}$ from the Euler equation. Let's plug this in and simplify:

$$
\begin{aligned}
\psi m_t^{-\zeta} &= \lambda_t - \frac{\lambda_t}{1+i_t} \\
\psi m_t^{-\zeta} &= \lambda_t \left(1 - \frac{1}{1+i_t}\right) \\
\psi m_t^{-\zeta} &= \lambda_t \left(\frac{i_t}{1+i_t}\right) \\
m_t &= \lambda_t^{-\frac{1}{\zeta}} \left(\frac{1}{\psi}\right)^{-\frac{1}{\zeta}} \left(\frac{i_t}{1+i_t}\right)^{-\frac{1}{\zeta}} \\
&\Rightarrow m_t = c_t^{\frac{\sigma}{\zeta}} \psi^{\frac{1}{\zeta}} \left(\frac{1+i_t}{i_t}\right)^{\frac{1}{\zeta}}
\end{aligned}
$$

We see that the demand for real balances is increasing in consumption and decreasing in the nominal interest rate, which makes sense as the nominal interest rate is the opportunity cost of holding money. If we're in the familiar "log-log" case in which $\sigma = \zeta = 1$, then this reduces even further to:

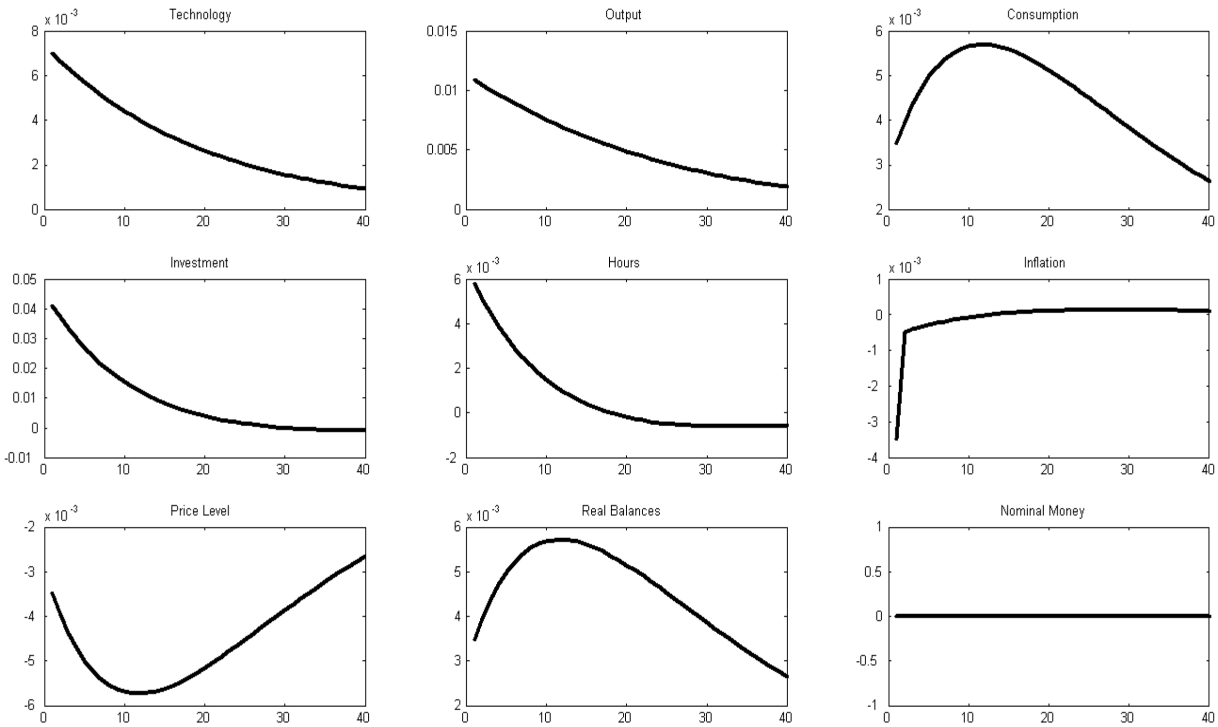
$$
m_t = c_t \psi \left(\frac{1+i_t}{i_t}\right)
$$

This begins to look a lot like the quantity equation (i.e. $MV = PY$), with output replaced by consumption. We can rewrite it:

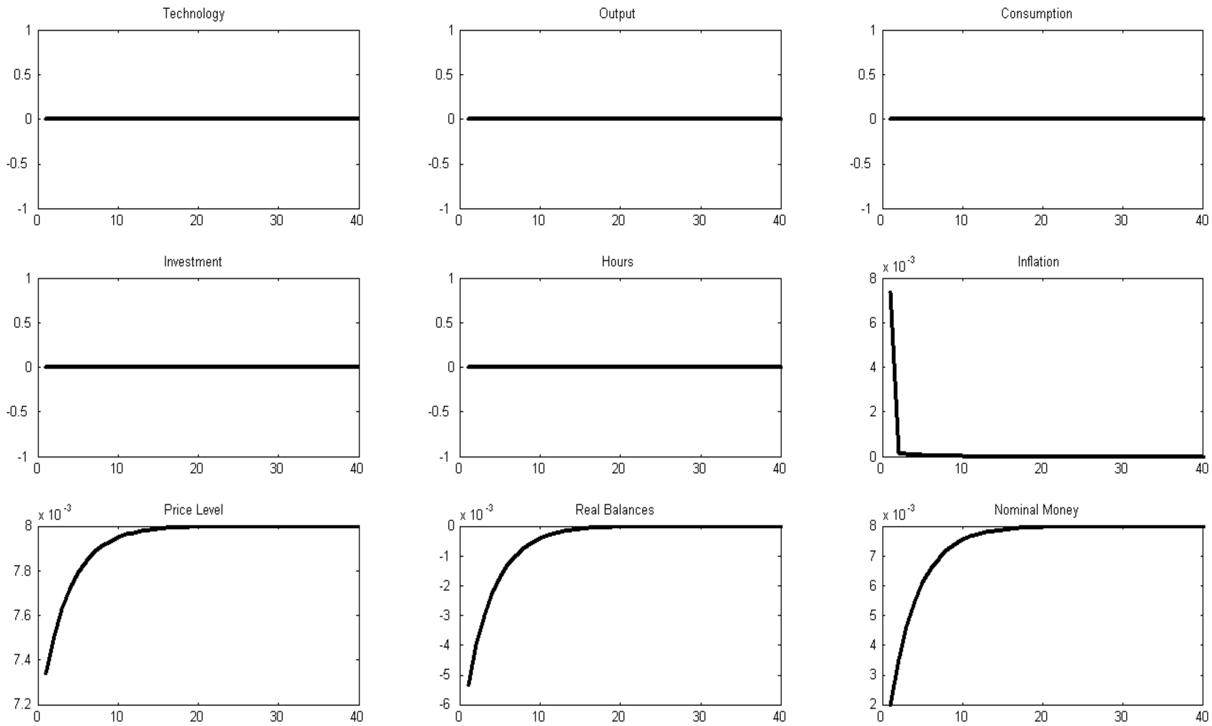
$$
\begin{aligned}
M_t v_t &= p_t c_t \\
v_t &= \frac{i_t}{\psi(1+i_t)}
\end{aligned}
$$

This yields the intuitive result that velocity is increasing in the nominal interest rate (i.e. the more costly it is to hold money, the more times you'd like to spend each individual unit of money).

As noted above, the rest of the model is identical to above, absent the cash in advance constraint. Hence the model has one fewer variable, as there is no Lagrange multiplier on the cash in advance constraint. I calibrate the parameters the same as above, now with $\psi = 1$ and $\zeta = 1$. The impulse responses to both a technology shock and a money growth shock are shown below:



The money growth shock responses are:



The main difference here is in the responses to the money growth shock. In this model, money is completely neutral with respect to real variables (both in the short run and in the long run, as there is no effect of more or less inflation on the steady state values of the real variables). The responses of the real variables to the technology shock are also identical to what would obtain in a real model. The reason money is neutral here is that there is no friction that gives rise to money mattering in the sense of affecting the values of real variables, though it does matter for utility purposes.