1. Math:

- **Useful algebra rules:**
  - $x^a x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$

- **Logs and exponents:**
  - $\ln(1) = 0$
  - $\lim_{x\to 0} \ln x = -\infty$
  - $\ln(xy) = \ln x + \ln y$
  - $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
  - $\ln x^a = a \ln x$
  - $\exp(\ln x) = x$
  - $\ln(1 + a) \approx a$ for $a$ small

- **Growth rates:**
  - Growth rate of a variable between two adjacent periods, $g_t^x = \frac{X_t - X_{t-1}}{X_{t-1}} = \frac{X_t}{X_{t-1}} - 1$
  - The gross growth rate is $1 + g_t^x = \frac{X_t}{X_{t-1}}$
  - If a series grows at a constant rate for $j$ periods, we have $X_{t+j} = (1 + g^x)^j X_t$
  - $\ln X_t - \ln X_{t-1} \approx g_t^x$
  - More generally, the difference of the logs is the approximate percentage difference. This will be a decent approximation as long as the percentage difference isn’t too big

- **Calculus:**
  - Suppose we have a function $y = f(x)$. We say that $f(\cdot)$ is a “mapping” between $x$ and $y$ – given a value of $x$, $f(\cdot)$ determines a value of $y$.
  - The first derivative is $\frac{dy}{dx} = f'(x)$. This is a measure of how $y$ changes as $x$ changes (i.e. slope). The derivative is itself a function: $f'(x)$ is a function of a $x$, $f'(x_0)$ is the function evaluated at a particular point, $x_0$.
  - The second derivative is the derivative of the derivative, $\frac{d^2y}{dx^2} = f''(x)$
  - Derivative rules:
    
    $\begin{align*}
y &= ax, & \frac{dy}{dx} &= a \\
y &= x^a, & \frac{dy}{dx} &= ax^{a-1} \\
y &= \ln x, & \frac{dy}{dx} &= \frac{1}{x}
\end{align*}$
– Other rules:
\[ y = f(x) + g(x), \quad \frac{dy}{dx} = f'(x) + g'(x) \]
\[ y = f(x)g(x), \quad \frac{dy}{dx} = f(x)g'(x) + f'(x)g(x) \]
\[ y = \frac{f(x)}{g(x)}, \quad \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \]
\[ y = f(g(x)), \quad y = f'(g(x))g'(x) \]

– Multivariate derivatives: \( y = f(x, z) \)
– Partial derivatives: derivative of \( y \) with respect to \( x \) (or \( z \)), holding \( z \) (or \( x \)) fixed:
\[ \frac{\partial y}{\partial x} = f_x(x, z). \]
– Use same rules for partial derivatives
– Total derivative: measure of how \( y \) changes as both \( x \) and \( z \) change. Sum of partial derivatives evaluated at an initial point times deviation of \( x \) and \( z \) from that point:
\[ dy = f_x(x_0, z_0)dx + f_z(x_0, z_0)dz \]

- Optimization. Suppose you want to find \( x \) which maximizes \( f(x) \). Write the problem as:
\[ \max_x f(x) \]

- First order condition: \( f'(x) = 0 \). This is necessary for an optimum but not sufficient (could exist many “critical points” where FOC is satisfied)

- Multivariate optimization. Works the same way; first order conditions characterized by setting partial derivatives equal to zero:
\[ \max_{x,z} f(x, z) \]

- FOC are: \( f_x(x, z) = 0 \) and \( f_z(x, z) = 0 \)

- Constraint optimization. Suppose that you want to find optimum of multivariate function subject to some constraint on the choice variables (in this running example, \( x \) and \( z \)). Example problem:
\[ \max_{x,z} f(x, z) \]
\[ \text{s.t.} \]
\[ x + z \leq 1 \]

- Approach: assume constraint “binds” (weak inequality holds with equality), solve for one of the choice variables in terms of the other, and then turn it into an unconstrained problem. So, for example, \( z = 1 - x \). Then problem is:
\[ \max_x f(x, 1 - x) \]

- First order condition is: \( f_x(x, 1 - x) - f_z(x, 1 - x) = 0 \)

- Application: consumer optimization problem

2. Concepts and definitions
• Gross domestic product: current dollar value of final goods and services produced within a geographic boundary (e.g. a country) during a particular period of time (e.g. a year). Sum of prices of goods times quantities
• Production = income = expenditure
• Expenditure accounting identity: $Y_t = C_t + I_t + G_t + X_t - IM_t$
• Consumption is biggest expenditure category, investment the most volatile
• Real vs. nominal:
  – In single good world, think of “real” as meaning denominated in units of goods and “nominal” as denominated in units of money
  – In multi-good world (i.e. the real world) this is clumsy
  – “Real” in NIPA accounts means “constant dollar” (as opposed to current dollar, which is nominal)
  – Constant dollar GDP: pick base year, sum of quantities times base year prices. So real GDP in 2015 might be measured in units of 2005 dollars, for example
  – Chain-weighting: way to deal with arbitrary nature of choice of base year and potentially addresses issues resulting from changing relative prices of goods
• Implicit price deflator: ratio of nominal to real GDP. If prices are rising over time, will be greater than 1 subsequent to base year, equal to 1 in base year (by construction), and less than 1 prior to base year
• Inflation: percentage change in price level (typically period-over-period, but could define it over longer time horizon)

3. Growth

• Average growth rate of real GDP per capita in the US for the last 65 years is about 2 percent per year. Trend dominates the cycle
• If a variable, $X$, grows at a constant rate, $g$, between periods, then $X_{t+1} = (1 + g)X_t$. Over $j$ periods, this is $X_{t+j} = (1 + g)^jX_t$
• Time series stylized facts:
(i) Output per worker (or per capita) grows at an approximately constant rate over long periods of time
(ii) Capital per worker (or per capita) grows at an approximately constant rate over long periods of time
(iii) The capital-output ratio is roughly constant over long periods of time (meaning that capital and output grow at about the same rate)
(iv) The rate of return on capital is roughly constant over long periods of time
(v) The real wage grows at an approximately constant rate over long periods of time, the same rate as output per capita
• Cross-sectional stylized facts:
(i) Large differences in GDP per capita across countries
(ii) Some examples where poor countries catch up to rich countries
(iii) Other examples where large income differences persist
• Capital: something which must itself be produced, which helps you produce other stuff, and which doesn’t completely get used up in the production process
• Capital accumulation: $K_{t+1} = I_t + (1 − δ)K_t$. One period delay between investment and new capital. $0 < δ < 1$ is the depreciation rate. $K_t$ is predetermined within period – it depends on past investment decisions.

• Production function: $Y_t = AF(K_t, N_t)$. $Y_t$ is output, $K_t$ capital, $N_t$ labor. $A$ is a measure of productivity – the higher is $A$, the more efficient you are at transforming inputs into outputs. Function $F$ has the following properties: $F_K(K_t, N_t) > 0$, $F_N(K_t, N_t) > 0$ (positive first partial derivatives, so more input always means more output); $F_{KK}(K_t, N_t) < 0$, $F_{NN}(K_t, N_t) < 0$ (negative second partial derivatives, so marginal products are decreasing in amount of factor); $F(\gamma K_t, \gamma N_t) = \gamma F(K_t, N_t)$ (constant returns to scale; if you scale both inputs by a factor $\gamma$, you scale output by the same factor).

• Profit maximization by firm: $w_t = AF_N(K_t, N_t)$ and $R_t = AF_K(K_t, N_t)$. Firms hire a factor up until point at which marginal product of that factor equals factor price.

• Assumptions on household:
  – Supply one unit of labor inelastically; normalize this to 1. $N_t = 1$
  – Consume a constant fraction of income, $1 − s$. Hence invest/save the other fraction $s$
  – Income = expenditure = production (comes from combining definition of firm profit without household budget constraint)
  – Implies $I_t = sAF(K_t, 1)$. Define $f(K_t) = F(K_t, 1)$ (just a way to keep from having to carry around the 1).

• Central equation of model: $K_{t+1} = sAf(K_t) + (1 − δ)K_t$. Graphically:
• The curve plots $K_{t+1}$ as a function of $K_t$. Given an exogenous value of $K_t$, determine $K_{t+1}$ off the vertical axis. The curve starts off steep (slope greater than 1) and flattens out to a slope of $1 - \delta$ as $K_t$ gets big (this is driven by the assumed concavity of the production function, by which is meant that the second derivative of the production function with respect to $K_t$ is negative).

• There exists one point (other than $K_t = 0$) where $K_{t+1} = K_t$. This is called the steady state, and graphically occurs where the curve crosses a 45 degree line (a line which shows all points where $K_{t+1} = K_t$, and hence has slope of 1).

• To points to the left of the steady state, the curve lies above the 45 degree line, so $K_{t+1} > K_t$ (capital is growing); to points to the right of the steady state, the line lies above the curve, so $K_{t+1} < K_t$ (capital is declining).

• Once you know the dynamics of $K_t$, you can figure out dynamics of every other variable of interest in the model:

\[
\begin{align*}
Y_t &= AF(K_t, N_t) = Af(K_t) \\
C_t &= (1 - s)Y_t = (1 - s)Af(K_t) \\
I_t &= sY_t = sAf(K_t) \\
R_t &= AF_K(K_t, N_t) = Af'(K_t) \\
w_t &= AF_N(K_t, N_t)
\end{align*}
\]

• Impulse response diagram: plots how a single variable evolves in response to a change in an exogenous variable or parameter (e.g. a change in $s$ or $A$). I draw these where the variable initially sits in steady state for several periods prior to a change in an exogenous variable or parameter, which occurs in period $t$, and then trace out the dynamics of the variable subsequent to date $t$. For example, the impulse response of capital, output, and consumption to an increase in $s$: 
Rule of thumb when doing these: $K_t$ does not change in period $t$ unless you are told it changes (due to something like a war or natural disaster). Other variables can change in period $t$, but may not.

- Repeated increase in $s$ not a good candidate to account for sustained long run growth:
  - $s$ cannot go above 1
  - Repeated increases in $s$ would imply continually falling $R_t$, which is inconsistent with the stylized facts
- Repeated increases in $A$ can generated repeated increases in output and other variables which are roughly consistent with stylized facts
- Golden rule: the saving rate which maximizes steady state consumption
- Dynamic inefficiency: situation in which $s > s^{GR}$. Means that we could increase consumption both today and at every subsequent date simply by reducing the saving rate.
- We can explicitly introduce growth into the model by simultaneously allowing population to grow and introducing a new variable called labor augmenting technology. Assume $Z_t = (1 + g_z)Z_{t-1}$ and $N_t = (1 + g_n)N_{t-1}$ for any two adjacent periods, where $g_z$ and $g_n$ are the growth rates of $Z_t$ and $N_t$, respectively.
- Production function is $Y_t = AF(K_t, Z_t N_t)$. Depends on “effective labor input,” not actual labor input.
- Define $\widehat{x}_t = \frac{X_t}{Z_t N_t}$ for generic variable $X_t$. Rest of model identical.
- Modified central equation: $\widehat{k}_{t+1} = \frac{1}{(1+g_z)(1+g_n)} \left[ Af(\widehat{k}_t) + (1-\delta)\widehat{k}_t \right]$, where $f(\widehat{k}_t) = F(\widehat{k}_t, 1)$. Graphically looks similar to diagram above, and there exists a steady state in the per effective worker variables. At $\widehat{k}^*$, we have $\widehat{k}_{t+1} = \widehat{k}_t$.  


• Assume Cobb-Douglas production function: \( Y_t = AK_t^\alpha (Z_tN_t)^{1-\alpha} \). Can write all other variables in terms of \( \hat{k}_t \):

\[
Y_t = \hat{A}\hat{k}_t^\alpha Z_tN_t \\
C_t = (1 - s)A\hat{k}_t^\alpha Z_tN_t \\
I_t = sA\hat{k}_t^\alpha Z_tN_t \\
R_t = \alpha A\hat{k}_t^{\alpha-1} \\
w_t = (1 - \alpha)A\hat{k}_t^\alpha Z_t
\]

• At \( \hat{k}^* \), we have \( \hat{k}_{t+1} = \hat{k}_t \), so:

\[
\frac{Y_{t+1}}{Y_t} = \frac{Z_{t+1}N_{t+1}}{Z_tN_t} = (1 + g_z)(1 + g_n) \\
\frac{K_{t+1}}{K_t} = \frac{Z_{t+1}N_{t+1}}{Z_tN_t} = (1 + g_z)(1 + g_n) \\
\frac{C_{t+1}}{C_t} = \frac{Z_{t+1}N_{t+1}}{Z_tN_t} = (1 + g_z)(1 + g_n) \\
\frac{I_{t+1}}{I_t} = \frac{Z_{t+1}N_{t+1}}{Z_tN_t} = (1 + g_z)(1 + g_n) \\
\frac{R_{t+1}}{R_t} = 1 \\
\frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t} = (1 + g_z) \\
\frac{y_{t+1}}{y_t} = \frac{Z_{t+1}}{Z_t} = (1 + g_z) \\
\frac{k_{t+1}}{k_t} = \frac{Z_{t+1}}{Z_t} = (1 + g_z)
\]

Here \( k_t = \frac{K_t}{N_t} \) and \( y_t = \frac{Y_t}{N_t} \) (per capita variables).

• The expressions for these growth rates at the steady state is consistent with the time series stylized facts.

• There are very large, and in many cases very persistent, differences in GDP per capita across countries.

• Basically, three reasons in model why these differences could exist:
  
  – Some countries are initially poor in that they are endowed with little capital. In other words, they are below their steady state.
  
  – Some countries have low saving rates and therefore little capital
  
  – Some countries have low levels of productivity

• Convergence: if some countries are poor because they start with little capital (i.e. they are below their steady state), Solow model predicts that these countries should grow fast and “catch up” to the richer countries.

• We clearly don’t see this much in the data (with a few possible exceptions): differences in GDP per capita are very persistent for many countries.
• Countries being in their own different steady states with different values of \( s \) is implausible: would need crazy differences in \( s \) to account for observed differences in GDP per capita across countries which we see

• Conclusion: GDP per capita differences must be due to differences in productivity

• Solow model does not explain what productivity is or where it comes from. Reasonable possibilities:
  – Knowledge
  – Climate
  – Geography
  – Institution
  – Laws
  – Infrastructure

• Policy implications:
  – To make countries grow faster over long periods of time, need productivity growth to improve
  – To make poor countries rich, need to get them more productive

4. Consumption

• Household lives for two periods: \( t \) (present) and \( t+1 \) (future)

• Lifetime utility: \( U = u(C_t) + \beta u(C_{t+1}). \) \( C_t \) and \( C_{t+1} \) are period \( t \) and \( t+1 \) consumption; \( u(\cdot) \) is a function mapping consumption into utils, it is assumed to be increasing and concave, \( u'(\cdot) > 0 \) and \( u''(\cdot) < 0 \) (so diminishing marginal utility); \( 0 < \beta < 1 \) is a discount factor, reflects idea of impatience

• Household gets exogenous income of \( Y_t \) and \( Y_{t+1} \)

• Two period budget constraints:
  \[ C_t + S_t = Y_t \]
  \[ C_{t+1} = Y_{t+1} + (1 + r_t)S_t \]

• \( S_t \) is the stock of savings the household takes from \( t \) to \( t+1 \); it pays (real) interest \( r_t \).
  Impose condition that \( S_{t+1} = 0 \) (household “dies” with no savings/debt). \( S_t < 0 \) means borrowing

• Can combine the two into one intertemporal budget constraint. Says present discounted value of stream of consumption equals present discounted value of stream of income:
  \[ C_t + \frac{C_{t+1}}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t} \]

• Can write problem as:
  \[ \max_{C_t, C_{t+1}} U = u(C_t) + \beta u(C_{t+1}) \]
  \[ \text{s.t.} \]
  \[ C_t + \frac{C_{t+1}}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t} \]
• Solve for either $C_t$ or $C_{t+1}$ in terms of the other in intertemporal budget constraint and plug into lifetime utility, transforming problem into unconstrained. First order equation is called the Euler equation

$$u'(C_t) = \beta (1 + r_t) u'(C_{t+1})$$

• Has interpretation as marginal benefit equals marginal cost. Marginal benefit of consuming extra today is $u'(C_t)$; marginal cost is foregoing $1 + r_t$ units of $t + 1$ consumption (because you save one fewer unit today), which reduces utility by $\beta u'(C_{t+1})$. At optimum these must be equal – household is indifferent between saving or consuming a little more

• Euler equation not a consumption function – doesn’t tell you what $C_t$ should be, implicitly defines a relationship between $C_{t+1}$ and $C_t$. For example, with log utility, it works out to:

$$\frac{C_{t+1}}{C_t} = \beta (1 + r_t)$$

• Other things being equal, bigger $\beta$ means growth rate of consumption will be bigger. Make sense because you are more impatient. $r_t$ being bigger also has the same effect on consumption growth – higher $r_t$ means greater incentive to save, which means deferring consumption to the future

• Indifference curve: shows combinations of $C_t$ and $C_{t+1}$ which yield a fixed overall level of utility. Slope of indifference curve is $-\frac{u'(C_t)}{\beta u'(C_{t+1})}$. Given assumption of concavity, indifference curve is steep when $C_t$ is low and $C_{t+1}$ high and flat when $C_t$ is high and $C_{t+1}$ is low. There is an indifference curve associated with each level of lifetime utility. Indifference curves associated with higher levels of lifetime utility are to the “northeast.” Indifference curves cannot cross.

• Budget line: graphical depiction of intertemporal budget constraint. Has slope of $-(1 + r_t)$, and must pass through the endowment point (the point where $C_t = Y_t$ and $C_{t+1} = Y_{t+1}$).

• Graphically, an optimal consumption bundle puts you on the highest possible indifference curve which does not completely lie outside of the budget line. Mathematically, this is a point where an indifference curve and the budget line are tangent (have equal slopes), which is exactly what the Euler equation says. The picture below shows an optimal consumption allocation:
• An increase in $Y_t$ causes the budget line to shift out with no change in its slope. This allows the household to locate on a higher indifference curve. At the new optimum must have higher $C_t$ and $C_{t+1}$. The higher $C_{t+1}$ necessitates that $S_t$ increases in period $t$.

• An increase in $Y_{t+1}$ has a qualitatively similar effect – the budget line shifts out with no change in slope. This allows the household to locate on a higher indifference curve. At the new optimum there must be higher $C_t$ and $C_{t+1}$. The higher $C_t$ necessitates that $S_t$ decreases in period $t$.

• An increase in $r_t$ causes the budget line to become steeper and pivot through the endowment point. There is an ambiguous effect on $C_t$. The substitution effect says to reduce $C_t$ and increase $C_{t+1}$ (i.e. to substitute away from the more expensive good, period $t$ consumption). The income effect is ambiguous. If you are initially saving, an increase in $r_t$ means you have more lifetime income, which means you’d like to increase both $C_t$ and $C_{t+1}$. If you’re initially a borrower, the increase in $r_t$ reduces your lifetime income, which means you’d like to decrease both $C_t$ and $C_{t+1}$. Unless otherwise noted, we assume that the substitution effect always dominates, so that an increase in $r_t$ results in a decrease in $C_t$ (and hence an increase in $S_t$).

• The consumption function is a mapping between current and future income, $Y_t$ and $Y_{t+1}$, as well as the interest rate, $r_t$, and the level of current consumption, $C_t$. $C_t = C(Y_t, Y_{t+1}, r_t)$. From our analysis above, we can see that the first two partials are positive (consumption is increasing in both $Y_t$ and $Y_{t+1}$). By assumption that the substitution effect dominates, consumption is decreasing in $r_t$. The first partial derivative, $\frac{\partial C_t}{\partial Y_t}$, is positive but less than 1. We call this the “marginal propensity to consume” (MPC). For log utility, the consumption function is: $C_t = \frac{1}{1+\beta} \left[ Y_t + \frac{Y_{t+1}}{1+r_t} \right]$. 