1 Basic Neoclassical Growth Model

The economy is populated by a large number of infinitely lived agents. These agents consume, save in physical capital, and supply one unit of labor each period inelastically. Time runs from $t = 0, \ldots, \infty$. The population at each point in time is $N_t = (1 + n)^t N_0$, with $N_0$ given. Households get flow utility from consumption, $C_t$, by an increasing and concave function, $u(C_t)$. They discount the future by $b < 1$.

Firms produce output using capital and labor, where these inputs are turned into outputs through a constant returns to scale production function. The firms are owned by the households. There are two other exogenous inputs to production: $Z_t$, which is called labor augmenting technology; and $A_t$, which is neutral technological progress. Assume that $Z_t = (1 + z)^t Z_0$, $Z_0$ given. Suppose that $A_t$ follows some stationary stochastic process. The production function is:

$$Y_t = A_t f(K_t, Z_t N_t)$$  \hspace{1cm} (1)

Output must be either consumed or used as investment in new capital goods. Hence the aggregate accounting identity is:

$$Y_t = C_t + I_t$$  \hspace{1cm} (2)

Capital accumulates according to the following equation, with $0 < \delta < 1$ the depreciation rate on capital. There is an assumed one period delay between when new capital is accumulated and when it is productive:

$$K_{t+1} = I_t + (1 - \delta)K_t$$  \hspace{1cm} (3)

Equations (1)-(3) can be combined:

$$K_{t+1} = A_t F(K_t, Z_t N_t) - C_t + (1 - \delta)K_t$$  \hspace{1cm} (4)
The equilibrium of the economy can be expressed as the solution to a social planner’s problem:

$$\max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} b^t E_t u(C_t)$$

s.t.

$$K_{t+1} = A_t F(K_t, Z_t N_t) - C_t + (1 - \delta) K_t$$

There are two sources of non-stationarity in the model – population growth, $n$; and technological progress, $z$. We need to make the variables in the model stationary. To do this, define lower-case variables as their large counterparts divided by $Z_t N_t$. Hence:

$$c_t \equiv \frac{C_t}{Z_t N_t}$$
$$y_t \equiv \frac{Y_t}{Z_t N_t}$$
$$k_t \equiv \frac{K_t}{Z_t N_t}$$

Divide both sides of the accounting identity by $Z_t N_t$:

$$\frac{K_{t+1}}{Z_t N_t} = \frac{A_t F(K_t, Z_t N_t)}{Z_t N_t} - \frac{C_t}{Z_t N_t} + (1 - \delta) \frac{K_t}{Z_t N_t}$$

Because $F(K_t, Z_t N_t)$ has constant returns to scale, we can write $\frac{A_t F(K_t, Z_t N_t)}{Z_t N_t} = A_t F\left(\frac{K_t}{Z_t N_t}, \frac{Z_t N_t}{Z_t N_t}\right)$. Define $f(k_t) = F\left(k_t, 1\right)$. Then we can write the constraint as:

$$\frac{K_{t+1}}{Z_t N_t} = A_t f\left(k_t\right) - c_t + (1 - \delta) k_t$$

To get the left hand side in terms of the detrended variables, we need to multiply and divide by $Z_{t+1} N_{t+1}$:

$$\frac{K_{t+1}}{Z_{t+1} N_{t+1}} \frac{Z_{t+1} N_{t+1}}{Z_t N_t} = A_t f\left(k_t\right) - c_t + (1 - \delta) k_t$$

Since $Z_{t+1} = (1 + z) Z_t$ and $N_{t+1} = (1 + n) N_t$, this can simplify to:
\[ \gamma_{k_{t+1}} = A_t f(k_t) - c_t + (1 - \delta)k_t \]

\[ \gamma = (1 + z)(1 + n) \]

Assume that within-period utility takes the following functional form: \( u(C_t) = \frac{c_t^{1-\sigma}}{1-\sigma} \), with \( \sigma > 0 \). Normalizing \( Z_0 = N_0 = 1 \), we can write: \( u(C_t) = (1 + z)(1 + n)\frac{c_t^{1-\sigma}}{1-\sigma} = \gamma^{t(1-\sigma)} \frac{c_t^{1-\sigma}}{1-\sigma} \). Hence, we can re-write the planner’s problem in terms of detrended variables as, defining \( \beta = b\gamma^{1-\sigma} \):

\[
\begin{align*}
\max_{c_t, k_{t+1}} & \sum_{t=0}^{\infty} \beta^t E_t \frac{c_t^{1-\sigma}}{1-\sigma} \\
\text{s.t.} & \\
\gamma_{k_{t+1}} & = A_t f(k_t) - c_t + (1 - \delta)k_t
\end{align*}
\]

2 First Order Conditions

We can find the first order conditions necessary for an interior solution a couple of different ways – either the method of Lagrange multipliers or by expressing the problem as a dynamic program. I’ll begin with the Lagrangian formulation.

I write Lagrangians as current value Lagrangians, which means that the discount factor multiplies the constraints. An alternative formulation is to use a present value Lagrangian, in which case the discount factor does not multiply the constraints. The Lagrangian is:

\[
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t E_t \left( \frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t (A_t f(k_t) - c_t + (1 - \delta)k_t - \gamma_{k_{t+1}}) \right)
\]

The first order conditions are:

\[
\frac{\partial \mathcal{L}}{\partial c_t} = 0 \iff c_t^{-\sigma} = \lambda_t
\]

\[
\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \iff \gamma \lambda_t = \beta E_t \lambda_{t+1} (A_{t+1} f'(k_{t+1}) + (1 - \delta))
\]

\[
\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \iff \gamma k_{t+1} = A_t f(k_t) - c_t + (1 - \delta)k_t
\]

This Lagrangian formulation is called “current value” because the Lagrange multiplier is the marginal utility of consumption at time \( t \). A present value Lagrangian would have the multiplier equal to the marginal utility of consumption at time \( t \) discounted back to the beginning of time.
FOC (5) and (6) can be combined to yield:

\[ \gamma c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma} \left( A_{t+1} f'(k_{t+1}) + (1 - \delta) \right) \] (8)

The transversality condition says that the present discounted value of the capital stock at the “end of time” is zero:

\[ \lim_{t \to \infty} E_0 \beta^t \lambda_t k_{t+1} = 0 \] (9)

An alternative way to find the first order conditions is by setting the problem up as a dynamic programming problem. The state variable here is \( k_t \) (\( A_t \) is also a state, but I’m going to ignore it for now, which is fine since it is exogenous). The choice is \( c_t \). We can express the problem recursively as:

\[
V(k_t) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1}) \\
\text{s.t.} \\
\gamma k_{t+1} = A_t f(k_t) - c_t + (1 - \delta) k_t
\]

We can simplify this by imposing that the constraint holds and eliminating \( c_t \):

\[
V(k_t) = \max_{k_{t+1}} \frac{(A_t f(k_t) + (1 - \delta) k_t - \gamma k_{t+1})^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1})
\]

The first order condition is:

\[ \gamma c_t^{-\sigma} = \beta V'(k_{t+1}) \] (10)

What is \( V'(k_{t+1}) \)? Assuming that \( k_{t+1} \) has been chosen optimally, the dynamic programming problem can be written:

\[
V(k_t) = \frac{(A_t f(k_t) + (1 - \delta) k_t - \gamma k_{t+1})^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1})
\]

The derivative with respect to the argument, \( k_t \), is:
\[ V'(k_t) = c_t^{-\sigma} (A_t f'(k_t) + (1 - \delta)) \]

To get this we ignore terms involving \( \frac{dk_{t+1}}{k_t} \). This is an application of the envelope theorem, which says that, if \( k_{t+1} \) is chosen optimally, the optimal choice isn’t going to change for a small change in \( k_t \). This results from the fact that functions are flat near the optimum. Evaluating this derivative at \( k_{t+1} \), we get:

\[ V'(k_{t+1}) = c_{t+1}^{-\sigma} (A_{t+1} f'(k_{t+1}) + (1 - \delta)) \]

Plugging this back into the first order condition, (10), we get:

\[ \gamma c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma} (A_{t+1} f'(k_{t+1}) + (1 - \delta)) \tag{11} \]

This is identical to (8).

### 3 The Steady State

The solution of the model is a policy function – given the states, what is the optimal choice of the control. Except for very special cases (in particular \( \delta = 1 \)), there is analytical solution for this model. We can analytically characterize the solution for a special case in which the variables of the model are constant, however. This is called the steady state.

Assume that the production function takes the following form:

\[ Y_t = A_t K_t^\alpha (Z_t N_t)^{1-\alpha} \quad 0 < \alpha < 1 \tag{12} \]

It follows that the per effective worker production function is then:

\[ y_t = A_t k_t^\alpha \tag{13} \]

The non-stochastic steady state is defined as a situation in which all variables are constant and where the only source of uncertainty (which in this case is \( A_t \)) is held constant at its unconditional mean. In particular, this requires that \( k_{t+1} = k_t \) and \( c_{t+1} = c_t \). Denote these values \( k^* \) and \( y^* \), respectively. Let \( A^* \) denote the steady state value of \( A_t \), which is equal to its unconditional mean. We can analytically solve for the steady state capital stock from equation (8):
\[ \gamma c^{\ast-\sigma} = \beta c^{\ast-\sigma} \left( \alpha A^*k^{\ast\alpha-1} + (1 - \delta) \right) \]

\[ \frac{\gamma}{\beta} - (1 - \delta) = \alpha A^*k^{\ast\alpha-1} \]

\[ k^{\ast} = \left( \frac{\alpha A^*}{\gamma/\beta - (1 - \delta)} \right)^{\frac{1}{1-\alpha}} \tag{14} \]

Assume that \( \gamma/\beta > (1 - \delta) \) for this to have a well-defined steady state. From the accumulation equation, we know that:

\[ A^{\ast}k^{\ast\alpha} = c^{\ast} + \delta k^{\ast} \]

Hence we can solve for steady state consumption as:

\[ c^{\ast} = A^{\ast} \left( \frac{\alpha A^*}{\gamma/\beta - (1 - \delta)} \right)^{\frac{1}{1-\alpha}} - \delta \left( \frac{\alpha A^*}{\gamma/\beta - (1 - \delta)} \right)^{\frac{1}{1-\alpha}} \tag{15} \]

We can consider a couple of simple comparative statics. If \( A^{\ast} \) increases, then \( k^{\ast} \) and \( c^{\ast} \) also increase. If \( b \) goes up (households become more patient), then \( k^{\ast} \) and \( c^{\ast} \) both go up as well.

4 The Phase Diagram

We can qualitatively characterize the full solution to the model through a phase diagram. In a two dimensional world, the phase diagram typically puts the endogenous state variable on the horizontal axis (in this case \( k_t \)) and the jump variable on the vertical axis (in this case \( c_t \)). Phase diagrams are more natural in continuous time; we will proceed in discrete time with one slight abuse of notation.

We want to find the sets of points where the state (capital) and jump variables are not changing in \((k_t, c_t)\) space. Call these two sets of points the \( \frac{k_{t+1}}{k_t} = 1 \) isocline and the \( \frac{c_{t+1}}{c_t} = 1 \) isocline. There is only one value of \( k_{t+1} \) consistent with \( \frac{c_{t+1}}{c_t} = 1 \), which is the steady state capital stock if \( A^* \) is at its mean. We can see this from the first order condition.

\[ \frac{c_{t+1}}{c_t} = 1 : \quad k_{t+1} = \left( \frac{\alpha A_{t+1}}{\gamma/\beta - (1 - \delta)} \right)^{\frac{1}{1-\alpha}} \tag{16} \]

The \( \frac{k_{t+1}}{k_t} = 1 \) isocline is found by looking at the capital accumulation equation:

\[ k_{t+1} = A_t k_t^{\alpha} - c_t + (1 - \delta)k_t \]
\[ k_{t+1} = k_t \Leftrightarrow c_t = A_t k_t^{\alpha} - \delta k_t \]
Hence the isocline is defined by:

\[
\frac{k_{t+1}}{k_t} = 1 : c_t = A_t k_t^\alpha - \delta k_t
\]  

(17)

A complication arises because \( k_t \) shows up in the \( \frac{k_{t+1}}{k_t} = 1 \) isocline, whereas the \( \frac{c_{t+1}}{c_t} = 1 \) isocline depends on \( k_{t+1} \). This problem would not be present in continuous time, which is why continuous time is more natural for phase diagram. I’m going to simply circumvent this issue by assuming that \( k_{t+1} \approx k_t \), and will treat the \( k_{t+1} \) in the \( \frac{c_{t+1}}{c_t} = 1 \) isocline as \( k_t \).

Given this simplifying assumption, we can graph each of these lines in a plane with \( c_t \) on the vertical axis and \( k_t \) on the horizontal axis. The \( \frac{c_{t+1}}{c_t} = 1 \) isocline is a vertical line at \( k_t = k^* \) (assuming that \( A \) is at its steady state). The \( \frac{k_{t+1}}{k_t} = 1 \) isocline is a bit more complicated. We can see that its slope is \( \frac{dc_t}{dk_t} = \alpha A_t k_t^{\alpha-1} - \delta \). When \( k_t \) is small (i.e. near the origin), then this slope is positive because \( \alpha A_t k_t^{\alpha-1} \) will be large. When \( k_t \) is large (far away from the origin), the slope will be negative and will approach \(-\delta\) (because \( \alpha A_t k_t^{\alpha-1} \) will go to zero). The peak occurs where \( k_t = (\frac{\alpha A_t}{\delta})^{\frac{1}{1-\alpha}} \), which is greater than \( k^* \) when evaluated at the steady state value of \( A^* \). The actual steady state is where the two isoclines cross.

The above picture shows the isoclines and the steady state. It also shows the (i) saddle path and (ii) some unstable dynamic lines. These dynamics can be derived as follows. “Below” the \( \frac{k_{t+1}}{k_t} = 1 \) isocline, \( c_t \) is “too small”, and hence \( k_{t+1} > k_t \), and we draw arrows pointing the “right”, denoting the direction in which \( k_t \) will be expected to travel. “Above” the \( \frac{k_{t+1}}{k_t} = 1 \) isocline, \( c_t \) is “too big”, and \( k_{t+1} < k_t \), and hence we draw arrows pointing left. To get the dynamic arrows relative to the \( \frac{c_{t+1}}{c_t} = 1 \) isocline, we have to engage in a slight abuse of terminology. Technically what governs the evolution of \( c_t \) is where \( k_{t+1} \) is relative to \( k^* \), but what shows up in the diagram is \( k_t \). Let’s ignore this distinction and treat \( k_t \approx k_{t+1} \). To the “right” of the \( \frac{c_{t+1}}{c_t} = 1 \) isocline, \( k_t \) is “too big”; this means that \( \alpha A_{t+1} k_{t+1}^{\alpha-1} \) will be “small”, and consumption will be expected to decline. Hence, to the right of the \( \frac{c_{t+1}}{c_t} = 1 \) isocline we draw arrows pointing down, showing the expected direction of
consumption in that region. To the left of the $\frac{c_{t+1}}{c_t} = 1$ the opposite is true; $k_t$ is “too small”, and consumption will be expected to grow, so we draw arrows pointing up.

Visually inspecting the picture, we see that the arrows point toward the steady state when the system sits to “northeast” and “southwest” of the steady state. In the regions of the picture that are to the “northwest” or “southeast” of the steady state, the arrows point away from the steady state. The idea of the policy function is to pick $c_t$ given $k_t$ such that (i) the first order conditions hold; (ii) the transversality condition holds; and (iii) the solution is feasible with the constraints. Drawing in the dynamics as we have done presumes (i). The transversality condition rules out picking any value of $c_t$ in the “southeast” region – those regions would eventually lead to 0 consumption (so $\lambda_t \to \infty$) and infinite capital (so $k_{t+1} \to \infty$), which leads to a violation of the transversality condition. Picking any value of $c_t$ in the “northwest” region would violate (iii) – we would move towards infinite consumption with zero capital, which is infeasible. Hence, for any given $k_t$ (the state), consumption must start either in the “southwest” or “northeast” regions. Any old value of $c_t$ will not do – there will be a unique value of $c_t$ for each $k_t$ such that we travel towards the steady state. Any other value of $c_t$ (shown by the “explosive” dynamic arrows), would eventually lead to a violation of (ii) or (iii). The unique set of values of $c_t$ consistent with (i) - (iii) being satisfied and holding $A_t$ fixed is given by line with all the arrows pointing toward the steady state – this is the “saddle path”. It is the policy function – for any given current $k_t$, it tells you the value of $c_t$ consistent with optimization.

5 Dynamic Effects of Shocks

In this section I work through two different exercises: (1) an unexpected permanent change in $A^*$; and (2) an unexpected but temporary change in $A_t$. For (2), suppose that $A_t$ increases immediately and is expected to remain at that level until time $T$, at which point is goes back to its initial starting value. For these shifts, always assume that the economy begins in its steady state.

The dynamics of these systems always work as follows. Whenever something exogenous changes, the jump variable (in this case consumption) must jump in such a way that it “rides” the new system dynamics for as long as the change in the exogenous variable is in effect, and in expectation the system must hit a steady state eventually (either returning to the original steady state or going to a new steady state, depending on whether the change in the exogenous variable is permanent or not). The state variable cannot jump immediately, but will follow (in expectation) the dynamics of the system thereafter.

We begin with the permanent increase in $A_t$. This clearly shifts both isoclines. In particular, the $\frac{c_{t+1}}{c_t} = 1$ isocline shifts to the right, and the $\frac{k_{t+1}}{k_t} = 1$ isocline shifts “up and to the right”. The new isoclines are shown as dashed lines; the original isoclines are solid. It is clear that the steady state values of both consumption and the capital stock are higher.

There’s also a new saddle path/policy function. Depending on the slope of the saddle path, this could cross the original value of $k_t$ either above, below, or exactly at the original value of $c$. This means that the initial jump in consumption is ambiguous. I have shown this system with
consumption initially jumping up, which is what will happen under “plausible” parameterizations of $\sigma$. Thereafter the system must ride the new dynamics and approach the new steady state. Since $k_{t+1} > k_t$, investment increases on impact, so that consumption does not increase as much as output. Below the phase diagram I show the impulse responses, which trace out the dynamic responses of consumption and the capital stock to the shock. Again, note that consumption jumps on impact, whereas the capital stock does not, and from thereafter they “ride” the dynamics to a steady state.

Now consider an unexpected but temporary increase in $A$. Here the new isoclines shift in the same way they do in the case of a permanent shift in $A$. I show the new isoclines as dashed lines, and also show the new saddle path as a dashed line. Here, however, consumption cannot jump all the way to the new saddle path as it did in the case of the permanent change in $A$. Consumption must jump and ride the unstable “new” system dynamics until time $T$, when $A$ goes back to its original value, at which point in time it must be back on the original saddle path. From there on it must follow the original saddle path back into the original steady state.
The figure above shows the time path of $c$ with a “scratchy” line. It jumps up, rides the unstable dynamics of the new system, and hits the original saddle path at exactly time $T$. Importantly, we see from the picture that it jumps less than it would if the change in $A$ were permanent (if it were permanent it would jump all the way to the new saddle path). Since $c$ jumps by less on impact but the change in output is the same as if the shock were permanent, investment must jump by more. We can back out the time path of investment from the impulse response for the capital stock, shown in the figure below the phase diagram. These results are consistent with the intuition from the permanent income hypothesis.

This exercise turns out to provide important insight into more complicated problems. In these dynamic systems you can almost never find the analytical solution for a general case, though you can do so for the (i) steady state and (ii) what the solution would look like if the jump variable didn’t jump at all. For the case of temporary but persistent exogenous shocks, the full solution is somewhere in between those two extreme cases (between (i) and (ii)). If the shock is not very persistent, a good approximation to the solution would have the jump variable not changing at all.