1 Introduction

We previously studied equilibrium in a particularly simple world – an endowment economy. This was useful for getting intuition about what determined the real interest rate, but lacked realism.

In this section we endogenize supply. The concept of equilibrium is identical to before, but the set up is more interesting. Now we have firms that produce output using capital and labor, and households that consume, save, and supply labor. In addition to the goods market that must clear, we have another market – the labor market. We have several different exogenous variables and can examine how changes in these variables affect the endogenous variables of the model.

The model we are going to study is sometimes called the Neoclassical model of the business cycle. It alternatively goes by the name of the “Real Business Cycle Model” (RBC) because it emphasizes “real” (as opposed to nominal) supply-side forces as the chief cause of short run economic fluctuations (the principal business cycle driving force in the model is changes in productivity). The model is capable of producing movements in endogenous variables that bear some resemblance to what we observe in the data. The model has the implication that business cycles are the efficient response to changing productivity and as such stabilization policies (either monetary or fiscal) are undesirable. We will later study Keynesian models which do not share this implication.

2 Firm

As in the Solow Model section, we assume that there are many, identical firms. We can therefore normalize the number of firms to be 1, and call this firm the representative firm. The firm will produce output using capital and labor, and an exogenous productivity shifter which will be called $A_t$. Differently than the Solow model, we allow $A_t$ to change over time, and for current $A_t$ to potentially be different from future values of $A$. We abstract from growth, either in population or in the number of effective workers. The firm exists for two periods, $t$ and $t+1$, and wakes up with an exogenously given level of current capital, $K_t$. The firm uses capital and labor to make stuff, and returns its profit to households in the form of dividends.
The production technology is:

\[ Y_t = A_t F(K_t, N_t) \]

The function \( F(\cdot) \) has the usual properties: increasing and concave in both arguments, and constant returns to scale. An example production function satisfying this functional form is Cobb-Douglas:

\[ F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}, \quad 0 < \alpha < 1. \]

We assume that the firm hires labor on a period-by-period basis from the household at wage rate \( w_t \), which both the firm and household take as given.

We depart slightly from our earlier assumption and assume that the firm, rather than the household, owns the capital stock and makes investment decisions. Since the household owns the firm and there are no other frictions, the ownership structure on the capital stock ends up being irrelevant – we would get exactly the same equilibrium whether the firm makes capital accumulation decisions or whether the household makes those decisions. This set-up is perhaps more realistic, however.

The firm’s profit function is given by:

\[ \Pi_t = Y_t - w_t N_t - I_t \]

Here investment in new capital, \( I_t \), comes out of profit. Think of it this way. The firm produces some output and then pays labor. It can either return this net revenue to the household in the form of a dividend or it can re-invest that profit in new capital, which will help the firm produce in the future. Investing today, \( I_t \), reduces current dividends but increases the productive capacity in the future, and hence should increase profit in the future. The capital accumulation equation is standard, subject to one twist to be discussed below, where the firm takes its initial level of capital, \( K_t \), as given:

\[ K_{t+1} = q I_t + (1 - \delta) K_t \]

This expression tells us what capital tomorrow will be given current capital and current investment. The only difference relative to what we saw in the Solow model is the presence of the exogenous variable \( q \), which was implicitly normalized to 1 before. \( q \) is what we will call an “investment shock.” An increase in \( q \) makes the economy more efficient at transforming investment into new capital goods (likewise a decrease in \( q \) means the economy is less good at transforming investment into capital). One thing that \( q \) could represent is the health of the financial system. The financial system exists to transform investment into capital. When the system isn’t functioning well, \( q \) will be low. By not putting a time subscript on it, I am implicitly assuming that \( q \) takes on the same value in periods \( t \) and \( t + 1 \). In principle this accumulation equation can be iterated forward one period:

\[ K_{t+2} = q I_{t+1} + (1 - \delta) K_{t+1} \]

Recall that the “world ends” after \( t + 1 \), so the firm will have no interest in leaving any capital
over for period $t + 2$. As long as it can do “negative” investment, it will want to set $K_{t+2} = 0$, or $I_{t+1} = -\frac{(1-\delta)K_{t+1}}{q}$. Think about negative investment in the second period as the firm liquidating itself – it produces output in $t + 1$ using $K_{t+1}$, and after that production there is $(1 - \delta)K_{t+1}$ of capital left over, which the firm returns to the household in the form of a dividend. This presumes that the household can consume the capital. This may seem a little hokey, but is standard and we wouldn’t have to make such a strict assumption in a multi-period version of the model.

The firm’s objective is to maximize the present value of the dividends that it returns to households. The firm discounts future profit/dividend by $(1 + r_t)^{-1}$, which serves as the discount factor for goods. The discounted value of profit/dividends can be interpreted as the value of the firm. The maximization problem is dynamic because investing in new capital today reduces the current dividend but increases future dividends. The firm can choose labor input and investment in both periods. The firm wants to maximize:

$$\max_{N_t, N_{t+1}, I_t, I_{t+1}} V_t = \Pi_t + \frac{1}{1 + r_t} \Pi_{t+1}$$

s.t.

$$K_{t+1} = qI_t + (1 - \delta)K_t$$
$$K_{t+2} = qI_{t+1} + (1 - \delta)K_{t+1}$$

Plugging in the profit function, we get:

$$\max_{N_t, N_{t+1}, I_t, I_{t+1}} V_t = A_t F(K_t, N_t) - w_tN_t - I_t + \frac{1}{1 + r_t} (A_{t+1} F(K_{t+1}, N_{t+1}) - w_{t+1}N_{t+1} - I_{t+1})$$

s.t.

$$K_{t+1} = qI_t + (1 - \delta)K_t$$
$$K_{t+2} = qI_{t+1} + (1 - \delta)K_{t+1}$$

This is a constrained problem. As before, we can take care of the constraints by substituting them into the objective function. In particular, impose the terminal condition discussed above that $I_{t+1} = -\frac{1}{q}(1 - \delta)K_{t+1}$, and eliminate $I_t = \frac{1}{q}(K_{t+1} - (1 - \delta)K_t)$. This allows us to write the problem as one of choosing $K_{t+1}$ instead of investment (similarly to how we wrote the household problem as one of choosing future consumption instead of current saving):

$$\max_{N_t, N_{t+1}, K_{t+1}} V_t = A_t F(K_t, N_t) - w_tN_t - \frac{1}{q}(K_{t+1} - (1 - \delta)K_t) + \frac{1}{1 + r_t} \left( A_{t+1} F(K_{t+1}, N_{t+1}) - w_{t+1}N_{t+1} + \frac{(1 - \delta)K_{t+1}}{q} \right)$$

1I am using the terms “dividend” and “profit” interchangeably. In practice the firm may not return profit each period, but households get compensated indirectly through capital gains (share price appreciation). As long as there are no credit market imperfections the distinction between a dividend received and a capital gain is immaterial.
We can characterize the optimum by taking the partial derivatives with respect to the choice variables and setting those partials equal to zero:

\[
\frac{\partial V_t}{\partial N_t} = 0 \Leftrightarrow A_t F_N(K_t, N_t) = w_t
\]

\[
\frac{\partial V_t}{\partial N_{t+1}} = 0 \Leftrightarrow A_{t+1} F_N(K_{t+1}, N_{t+1}) = w_{t+1}
\]

\[
\frac{\partial V_t}{\partial K_{t+1}} = 0 \Leftrightarrow 1 = \frac{q}{1 + r_t} A_{t+1} F_K(K_{t+1}, N_{t+1}) + \frac{1 - \delta}{1 + r_t}
\]

Two of these first order conditions should look familiar; all have the interpretation as marginal benefit = marginal cost conditions. The first condition simply says that the firm should hire labor up until the point at which the marginal product of labor (marginal benefit) equals the marginal cost of an additional unit of labor (the real wage). The second condition says the same thing for labor in period \( t + 1 \). Even though profits in period \( t + 1 \) are discounted by \( \frac{1}{1 + r_t} \) in the objective function, because both benefit and cost incur in the same period \( (t + 1) \), then no discounting shows up in that first order condition.

The marginal product of labor being equal to the real wage condition (which must hold in the same form in both periods, and is hence sometimes called a static condition) implicitly defines a demand curve for labor. When \( w_t \) goes up, the firm will need to increase the marginal product of labor, which, holding everything else fixed, requires reducing \( N_t \) given the assumed concavity of the production function. The labor demand curve would shift right if \( A_t \) were to increase, while it would shift left if current \( K_t \) were to exogenously decline (say, due to a natural disaster). The labor demand curve, \( N^d_t = N(w_t, A_t, K_t) \), therefore looks as follows:
The final optimality condition for the choice of future capital stock looks a little different than what we’ve seen before, but has an intuitive interpretation. It says to equate the marginal cost of investment (foregone current dividends/profits of 1) with the marginal benefit of an extra unit of investment, which is the (i) extra output produced in $t+1$ (the marginal product of future capital, $A_{t+1}F_K(K_{t+1}, N_{t+1})$) and (ii) the salvage or liquidation value of the marginal unit of capital, which is $\frac{1}{q}(1 - \delta)$. Because 1 unit of investment generates $q$ units of future capital, the right hand side is multiplied by $q$. Since all of the benefit of extra investment is received in the future, but the cost is born in the present, the future benefit gets discounted by $\frac{1}{1+r}$. This condition can be re-arranged to yield:

$$r_t + \delta = qA_{t+1}F_K(K_{t+1}, N_{t+1})$$

For given values of $N_{t+1}$ and $A_{t+1}$, this condition says that the desired period $t+1$ capital stock, $K_{t+1}$, is a decreasing function of the real interest rate, $r_t$. This is because of the assumption of concavity – when $r_t$ goes up, the right hand side must go up, which requires making $K_{t+1}$ smaller. Reducing $K_{t+1}$, given a current value of $K_t$, means reducing current investment. Put differently, this first order condition implicitly defines an investment demand curve, $I_t = I(r_t, A_{t+1}, q, K_t)$. As we just discussed, investment will be decreasing in the real interest rate. It will be increasing in future $A_{t+1}$. This is an important point – investment is fundamentally forward-looking – investment today increases productive capacity in the future, so the more productive you expect to be in the future, the more investment you’d like to do in the present. Investment demand will also be increasing in $q$: the bigger is $q$, the more efficient you are at transforming investment into productive capital. The investment demand curve also depends on current capital, which is effectively exogenous. The condition above implicitly defines an optimal choice of $K_{t+1}$ which is independent of current $K_t$. But if current $K_t$ is, say, low, then investment would have to be high to achieve a given target level of $K_{t+1}$. Hence, desired investment is decreasing in $K_t$, holding everything else fixed.

We can draw an investment demand curve below. It is downward-sloping in $r_t$, and shifts right if $A_{t+1}$ or $q$ increase or if $K_t$ decreases exogenously (say, due to a hurricane or other natural disaster):

---

\[2\text{You may counter that the first order condition says that you want higher } K_{t+1} \text{ if } q \text{ is higher (holding everything else fixed), but this does not necessarily imply higher } I_t \text{ because a given amount of investment yields more future capital when } q \text{ goes up. This would be a good point, but for most reasonable production functions (e.g. Cobb-Douglas) one can in fact show that desired } I_t \text{ is increasing in } q.\]

\[3\text{Technically, the level of } N_{t+1} \text{ is also something that should show up in the investment demand function – the higher } N_{t+1} \text{, the higher the marginal product of future capital, and hence the more future capital a firm would want, given an interest rate. But since the firm can choose } N_{t+1}, \text{ I omit it as an argument.}\]
Note that the real interest rate here serves not as an explicit cost of investment, but rather as an implicit cost of investment. As I’ve written the problem, the firm is not borrowing to finance investment. Rather, investment comes out of current dividends, and since the benefit of investment (more profit) is only received in the future, the benefit gets discounted. So \( r_t \) has the interpretation here as the opportunity cost of investment – rather than accumulating more capital, the firm could have returned more current profit to households. I could have also written down the model where I force the firm to finance new capital accumulation through borrowing. It turns out to not matter (under assumptions we have made) how the firm finances itself – essentially here equity (reducing current dividend to yield more in the future) or debt.

3 Household

The representative household lives for two periods, consuming and working in both periods. The household problem is similar to what we had before, with the twist that it gets to choose how much to work.

Let the household’s endowment of time each period be normalized to 1. Let \( N_t \) denote how much labor it supplies. Therefore its leisure is \( 1 - N_t \). We assume that leisure is an argument in the utility function, with more leisure leading to more utility (in other words, households don’t like to work). Let \( v(1 - N_t) \) be a function that maps leisure into utility; we assume that it is increasing, \( v'(\cdot) > 0 \), and concave, \( v''(\cdot) < 0 \). An example function satisfying this property is the natural log: \( v(1 - N_t) = \ln(1 - N_t) \). Quick note to avoid confusion: if utility is increasing and concave in leisure, then it is decreasing and convex in labor input (first derivative with respect to \( N_t \) is < 0, second derivative with respect to \( N_t \) is > 0).

We assume that “period utility” is separable in consumption and leisure; this means that utilis
in period $t$ are given by $u(C_t) + v(1 - N_t)$, where $u(\cdot)$ has the same properties as before. Lifetime utility is just the weighted sum of period utility in the two periods, with $\beta$ the weight on future utils:

$$U = u(C_t) + v(1 - N_t) + \beta(u(C_{t+1}) + v(1 - N_{t+1}))$$

The household faces a sequence of two within period budget constraints. Conceptually they are the same as we have seen before, but we are no longer in an endowment economy, so income is not exogenous. The household has two sources of income: wage income, $w_tN_t$, and distributed profits from firms, $\Pi_t$, which the household takes as given. The household may also have to pay taxes to a government, $T_t$ and $T_{t+1}$. The household can use its first period income to consume or to save in bonds, $S_t$. In the second period it will want to consume all of its income, where its total income comes from labor income, $w_{t+1}N_{t+1}$, distributed profits from firms, $\Pi_{t+1}$, and interest on first period savings, $(1 + r_t)S_t$. The two within period constraints are:

$$C_t + S_t = w_tN_t - T_t + \Pi_t$$

$$C_{t+1} = w_{t+1}N_{t+1} - T_{t+1} + \Pi_{t+1} + (1 + r_t)S_t$$

As before, $S_t$ shows up in both period constraints, and so we can eliminate it and combine into one intertemporal budget constraint, which has the same flavor as before but has endogenous total income:

$$C_t + \frac{C_{t+1}}{1 + r_t} = w_tN_t - T_t + \Pi_t + \frac{w_{t+1}N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}$$

We can write the household problem as one of choosing, at time $t$, a sequence of consumption, $(C_t, C_{t+1})$, and labor, $(N_t, N_{t+1})$, to maximize its lifetime utility subject to the intertemporal budget constraint. In actuality the household really chooses saving and solves the problem period-by-period, but it works out easier (and identically) to think about them solving the problem this way. As before, the household is a price-taker, and so takes the real interest rate, $r_t$, and the real wage rate, $w_t$, as given. It also takes distributed profits as given.

$$\max_{C_t, C_{t+1}, N_t, N_{t+1}} U = u(C_t) + v(1 - N_t) + \beta(u(C_{t+1}) + v(1 - N_{t+1}))$$

s.t.

$$C_t + \frac{C_{t+1}}{1 + r_t} = w_tN_t - T_t + \Pi_t + \frac{w_{t+1}N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}$$

This is a constrained, multivariate optimization problem. To handle it, we will solve for $C_{t+1}$ from the constraint, plug that in, and thereby transform the problem to an unconstrained problem in only three choice variables. Solving for $C_{t+1}$, we get:

$$C_{t+1} = (1 + r_t)(w_tN_t - T_t + \Pi_t - C_t) + w_{t+1}N_{t+1} - T_{t+1} + \Pi_{t+1}$$
Plug into the objective function:

\[
\max_{C_t, N_t, N_{t+1}} U = u(C_t) + v(1 - N_t) + \beta \left( u \left( (1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} \right) + v(1 - N_{t+1}) \right)
\]

Let’s take the derivatives with respect to the choice variables to find the conditions characterizing optimal behavior:

\[
\frac{\partial U}{\partial C_t} = 0 \iff u'(C_t) + \beta u' \left( (1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} \right) (-1 + r_t) = 0
\]

\[
\frac{\partial U}{\partial N_t} = 0 \iff -v'(1 - N_t) + \beta u' \left( (1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} \right) (1 + r_t) w_t = 0
\]

\[
\frac{\partial U}{\partial N_{t+1}} = 0 \iff \beta u' \left( (1 + r_t)(w_t N_t - T_t + \Pi_t - C_t) + w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1} \right) w_{t+1} - \beta v' \left( 1 - N_{t+1} \right) = 0
\]

These conditions can be simplified and made more interpretable by substituting back in for \(C_{t+1}\).

Doing so:

\[
u'(C_t) = \beta (1 + r_t) u'(C_{t+1})
\]

\[
v'(1 - N_t) = w_t \beta (1 + r_t) u'(C_{t+1})
\]

\[
v'(1 - N_{t+1}) = w_{t+1} u'(C_{t+1})
\]

We can actually simplify the second condition by nothing that \(\beta (1 + r_t) u'(C_{t+1}) = u'(C_t)\). Then we can write the first order conditions as:

\[
u'(C_t) = \beta (1 + r_t) u'(C_{t+1})
\]

\[
v'(1 - N_t) = w_t u'(C_t)
\]

\[
v'(1 - N_{t+1}) = w_{t+1} u'(C_{t+1})
\]

You will note that the first condition is just the consumption Euler equation that we’ve seen before. This says to equate the marginal benefit of consuming a little more today, \(u'(C_t)\), with the marginal cost, which takes the form of foregone future consumption which reduces future utility, \(\beta(1 + r_t)u'(C_{t+1})\). The second two conditions take identical forms for each period, \(t\) and \(t + 1\). These also have the interpretation as marginal benefit equals marginal cost conditions. Suppose that the household chooses to consume one additional unit of leisure in period \(t\). The marginal benefit of this change is the marginal utility of leisure, or \(v'(1 - N_t)\). The marginal cost of this is foregone consumption – working less means you have \(w_t\) fewer units of income to consume, which reduces utility by \(w_t u'(C_t)\). Hence, the right hand side is the marginal cost of an additional unit of leisure today. The period \(t + 1\) condition for labor is identical to the period \(t\) condition in an
analogous way to how the labor demand conditions for the firm are identical in \( t \) and \( t+1 \).

We can analyze these first order conditions graphically with an indifference curve-budget line diagram. Because income is now endogenous, we have to hold everything else fixed outside of the graph. The first indifference curve-budget line diagram is the familiar one with current consumption, \( C_t \), on the horizontal axis and future consumption, \( C_{t+1} \), on the vertical axis. The budget line can be derived from the intertemporal budget constraint, holding \( N_t \) and \( N_{t+1} \) fixed. The slope is \(-(1 + r_t)\), and the optimality condition is where the indifference curve and the budget line are tangent.

We can also draw an indifference curve-budget line diagram for the choice of leisure/labor and consumption. We will do the picture with \( C_t \) on the vertical axis and \( 1 - N_t \) on the horizontal axis. The plot is complicated by the fact that there is an upper bound on leisure – it cannot be in excess of 1, which occurs when \( N_t = 0 \). This is going to introduce a kink into the budget constraint. Solve for \( C_t \) from the budget constraint in terms of everything else:

\[
C_t = \frac{-C_{t+1}}{1 + r_t} + w_t N_t - T_t + \Pi_t + \frac{w_{t+1} N_{t+1} - T_{t+1} + \Pi_{t+1}}{1 + r_t}
\]

The derivative of \( C_t \) with respect to \( N_t \) is: \( \frac{\partial C_t}{\partial N_t} = w_t \). Now this is itself complicated by the fact that we don’t have \( N_t \) on the horizontal axis in the budget line diagram we want to draw – we have \( 1 - N_t \). This does not complicate things much, since \( \frac{\partial C_t}{\partial (1 - N_t)} = -\frac{\partial C_t}{\partial N_t} \). Hence, the slope of the budget line is \(-w_t\), which is the price of leisure in terms of foregone consumption – if you consume an additional unit of leisure, you earn \( w_t \) fewer units, and thus have \( w_t \) fewer units available to consume.

The budget line ends up having a kink, at \( 1 - N_t = 1 \), or \( N_t = 0 \). This means that the household could locate at the kink, in which case the first order condition derived above is no longer necessary. Basically, if the household were sufficiently willing to substitute leisure across time (essentially \( v(\cdot) \) close to linear) and there were a big differential in wages across time, the household may choose to work all in one period and not at all in the other. We are going to rule that out, and assume that we are always at an interior solution away from the kink – the household always works some. The graphical optimality condition is shown below:
What happens when there is an increase in the real wage, $w_t$? This would lead the budget line to become steeper, but the kink occurs at the same point. Similarly to the effect of an increase in the real interest rate in the current/future consumption diagram, there turn out to be competing income and substitution effects at work. The substitution effect of a higher real wage says to work more (consume less leisure). But the income effect says to work less – a higher real wage $w_t$ makes you feel richer, which makes you want to consume more goods and more leisure (work less). Hence, there are competing income and substitution effects, and the effect of a change in the real wage on labor income is ambiguous. This means that the labor supply curve could be upward or downward sloping in principle.

The figure below shows the effects of an increase in the wage in the indifference curve budget line diagram making use of a decomposition into income and substitution effects similar to what we did in the two period consumption model to study the effects of a change in the interest rate. The black lines depict the initial situation. The blue budget line is the budget line with a higher slope due to an increase in the real wage. The new bundle is on the blue indifference curve labeled $U = U_1$ (note that the household has to be at least weakly better off with a higher wage). The green line is a hypothetical budget line with slope given by the new wage, but where some exogenous income has been taken away from the household in such a way that it would locate on the original indifference curve and therefore have the same initial level of utility. This thought experiment isolates the substitution effect – the substitution effect of an increase in $w_t$ is to consume less leisure and more consumption (since leisure is effectively more expensive); consuming less leisure means working...
more. We can see this clearly in the budget line diagram, where we would hypothetically move from the initial bundle (a) to a new bundle on the same indifference curve, (b). The income effect is the difference between the actual new bundle and the hypothetical bundle on the original indifference curve at the new wage. The income effect is to consume more consumption and more leisure (i.e. work less). Since the income and substitution effects of a higher \( w_t \) work in opposite directions on \( N_t \), it is ambiguous how labor responds to an increase in the wage, though consumption must increase.

![Diagram](image)

In plotting the indifference curve - budget line diagram above, I assumed that the substitution effect dominates so that \( N_t \) increases (equivalently, leisure goes down). *Like we did for the effect of the real interest rate on consumption, we will always assume that the substitution effect dominates, so that \( N_t \) increases when \( w_t \) goes up.* We can see this for a particular utility function, logarithmic over both consumption and leisure. The optimality condition would be:

\[
\frac{1}{1 - N_t} = \frac{1}{C_t} w_t
\]

Now, suppose that there is an increase in the real wage, \( w_t \). For a given level of labor input, this would represent an increase in income of \( w_t \) units. We know from earlier that the marginal propensity to consume out of current income is less than one. This means that, if \( w_t \) increases, \( C_t \) would increase for a given real interest rate, but less than one for one. Put differently, when \( w_t \) goes up, the right hand side gets bigger since \( C_t \) goes up by less \( w_t \) if there were no change in \( N_t \). If
the right hand side goes up, the left hand side must go up for the optimality condition to hold. To get the left hand side to go up, we’d need $1 - N_t$ to get smaller, which requires $N_t$ getting bigger. Hence, $N_t$ would rise when $w_t$ goes up, holding everything else fixed.

We plot the labor supply curve as upward-sloping in $W_t$, with $N_t$ on the horizontal axis. In principle, the curve will shift with anything other than $w_t$ which affects $C_t$. This could include future income, government spending/taxes, or the real interest rate. We are going to assume all of this away. Instead, we will call $H_t$ a variable other than $w_t$ which affects labor supply, and we will treat it as exogenous. We can think of $H_t$ as representing changing preferences for leisure or policy-induced things like taxes or unemployment benefits which might make working more or less desirable. Assuming away all other endogenous sources of labor supply shifts is not technically correct, but makes our lives far easier and does not really cost us anything in terms of how the model functions.

Graphically, therefore, the labor supply curve is just an upward-sloping plot in a graph with $N_t$ on the horizontal axis and $w_t$ on the vertical. An increase in $H_t$ causes the labor supply curve to shift to the right, and a decrease in $H_t$ induces the labor supply curve to shift to the left.

---

4 A useful observation to make here is that the effect of a change in the wage on labor supply ought to depend on how persistent the wage change is. If, for example, the current and future real wages both went up simultaneously, we would expect a bigger increase in $C_t$ via the logic we studied before, and therefore a smaller increase in $N_t$ (or perhaps no increase in $N_t$ at all, or even a decrease).

5 Note: in past iterations of this course we allowed labor supply to shift with things like the interest rate. This is interesting in its own right but complicates the model without providing much new insight. In particular, it introduces lots of ambiguities in terms of how $N_t$ is affected by changes in exogenous variables that do not appear when we make the simplifying assumption that labor supply only depends on the real wage. I’m trying to make your lives (and mine) easier this way.
4 The Government

The government problem is simple and is identical to what we’ve already had. The government exogenously picks a sequence of spending, \( G_t \) and \( G_{t+1} \), and taxes must adjust to make the government budget constraint hold.

The government faces two within period constraints:

\[
G_t + S_t^G = T_t \\
G_{t+1} = T_{t+1} + (1 + r_t)S_t^G
\]

These can be combined into one intertemporal budget constraint:

\[
G_t + \frac{G_{t+1}}{1 + r_t} = T_t + \frac{T_{t+1}}{1 + r_t}
\]

Since the household knows that the government budget constraint must hold, the household intertemporal budget constraint can be written:

\[
C_t + \frac{C_{t+1}}{1 + r_t} = w_t N_t - G_t + \Pi_t + \frac{w_{t+1}N_{t+1} - G_{t+1} + \Pi_{t+1}}{1 + r_t}
\]

In other words, the household will behave as though the government balances its budget period-by-period, just as in the earlier equilibrium notes. That is, Ricardian equivalence continues to hold. This means that changes in taxes that are not matched by changes in the time path of government spending will have no impact on the economy.

We therefore have exactly the same consumption function we did before, where the household behaves as though the government balances its budget each period, and consumption depends on current and future perceived net income as well as the real interest rate:

\[
C_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)
\]

We can derive the aggregate resource constraint by combining the household’s period \( t \) budget constraint with the firm definition of profit:

\[
C_t + S_t = w_t N_t - T_t + \Pi_t \\
C_t + S_t = w_t N_t - T_t + Y_t - w_t N_t - I_t
\]

Bond market-clearing requires that \(-S_t^G = S_t\) (i.e. government borrowing, the negative gov-

\footnote{The astute student may not there is a bit of a slight of hand here. We would be able to derive something like this consumption function by plugging the Euler equation into the intertemporal budget constraint and solving for \( C_t \), just as we did in the consumption notes. That would work out fine, but the slight of hand here occurs in writing the arguments of the consumption function as aggregate income, \( Y_t \), rather than income the household earns (which is endogenous). This would be a reasonable observation, but it turns out to not matter much. Writing the consumption function this way facilitates comparison with what we have done earlier and is standard fare.}
ernment saving, must equal household saving). Since \( S_t^G = T_t - G_t \), this condition implies that \( S_t = G_t - T_t \). Plugging this in above yields the aggregate resource constraint of:

\[
Y_t = C_t + I_t + G_t
\]

5 The \( Y^s \) and \( Y^d \) Curves and Equilibrium

Now we are going to take the decision problems of the household and firm and combine them to derive output demand and supply curves. This exercise is conceptually the same as what we did for studying equilibrium in an endowment economy. The \( Y^d \) and \( Y^s \) curves will even look the same as in the endowment economy. But how we get to these curves is a little more complicated because of the presence of endogenous production.

The full set of equations characterizing the equilibrium are given below:

\[
N_t = N^d(w_t, A_t, K_t)
\]

\[
N_t = N^s(w_t, H_t)
\]

\[
Y_t = A_t F(K_t, N_t)
\]

\[
C_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)
\]

\[
I_t = I(r_t, A_{t+1}, q, K_t)
\]

\[
Y_t = C_t + I_t + G_t
\]

This is six equations in 6 endogenous variables – \( N_t, w_t, Y_t, C_t, I_t, \) and \( r_t \). The first equation is labor demand and the second is labor supply, which can be motivated from the first order conditions for the firm and household problems. The third equation is the production function. The fourth is the consumption function, which, as noted above and discussed further in the preceding footnote, can be derived by taking the consumption Euler equation and combining that with the intertemporal budget constraint. The fifth equation is the investment demand curve, implicitly defined by the first order condition for investment discussed above. The final equation is the aggregate resource constraint, which comes from combining the definition of firm profit with the household’s first period budget constraint. The exogenous variables are \( A_t, A_{t+1}, G_t, G_{t+1}, q, \) and \( H_t \). We want to graphically analyze these equations and use those graphs to determine the equilibrium and analyze how the equilibrium changes when one of the exogenous variables changes.

In the endowment economy, the \( Y^s \) curve was just a vertical line at the exogenous level of the current endowment. It’s still going to be vertical in this setup, but we have to be a bit careful. Formally, we define the \( Y^s \) curve as the set of \((r_t, Y_t)\) pairs consistent with household and firm optimization, the firm’s production function, and the labor market being in equilibrium. In terms of the equations above, the \( Y^s \) curve therefore summarizes the labor demand curve, the labor supply curve, and the production function. The labor market being in equilibrium (or the labor market
“clearing” if you prefer) requires that the quantity of labor demand equals the quantity supplied. In other words, we must be at the intersection of the labor demanded and supply curves:

We derive the $Y^s$ curve as follows. Start with a value of $r_t$. Determine the value of $N_t$ consistent with the labor market-clearing given this $r_t$. Then take that value of $N_t$, combined with the exogenous value of current $K_t$, to determine the value of $Y_t$ from the production function. This is an $(r_t, Y_t)$ pair. Then consider a higher or lower value of $r_t$. See how this affects the equilibrium value of $N_t$. Plug this value of $N_t$ into the production function and see what happens to $Y_t$. This gives you a new $(r_t, Y_t)$ pair. Then simply connect the dots.

We can do this formally with a four part graph. In the upper left quadrant we have the labor market, plotting labor demand and supply as functions of $w_t$. Right below that we plot the production function, with $Y_t$ as a function of $N_t$. In plotting this we are holding the values of $A_t$ and $K_t$ fixed. In the lower right quadrant we just have a 45 degree line, showing all points where $Y_t = Y_t$. This is simply a graphical device to “reflect” the vertical axis onto the horizontal axis. In the upper right quadrant we have $r_t$ on the vertical axis and $Y_t$ on the horizontal axis. Since $r_t$ affects neither the labor demand nor the labor supply curves, higher or lower values of $r_t$ don’t affect the equilibrium value of $N_t$. Since $r_t$ also doesn’t appear directly in the production function, this means that the value of $Y_t$ consistent with the production function and labor market clearing does not depend on the value of $r_t$. In other words, the $Y^s$ curve is vertical, just as it was in the endowment economy.\footnote{As referenced above, in past versions of the course, we assumed that higher values of $r_t$ induced households to supply more labor, which resulted in the $Y^s$ curve being upward sloping. Empirically, the effect of the real interest rate on labor supply is likely small, so our assumptions which give rise to a vertical $Y^s$ curve are not too far off from reality.}
What would cause the $Y^s$ curve to shift? This is slightly more complicated than in the endowment economy framework because production is now endogenous. The primary thing which neoclassical models emphasize is a change in $A_t$, current productivity. Suppose that $A_t$ increases from $A^0_t$ to $A^1_t$. In this graphical setup, there are two effects of an increase in $A_t$. First, an increase in $A_t$ shifts the labor demand curve to the right. This raises the level of $N_t$ consistent with the labor market being in equilibrium. Second, an increase in $A_t$ shifts the production function up – you get more $Y_t$ for a given level of $N_t$. Both of these effects are shown with the blue lines in the graph below. If you combine a higher value of $N_t$ with a higher value of $A_t$, you get a bigger value of $Y_t$ for any value of $r_t$. In other words, the $Y^s$ curve shifts right from $Y^s$ to $Y'^s$. It would shift left has we considered a decrease in $A_t$. 
As noted above, while we assume that the labor supply curve doesn’t shift endogenously in response to anything else in the model, we can entertain exogenous shifts in the labor supply curve. This is captured by the exogenous variable $H_t$. In the real world, such shifts could be driven by changing demographics (e.g. an aging population, so fewer people wanting to work, so the labor supply curve shifting in), changing preferences for work (e.g. people find work less taxing, so labor supply shifts out), or changing policies which affect the incentives to work (e.g. changes in taxes or unemployment benefits). In the blue lines in the figure below, I consider the effects of an exogenous outward shift in the labor supply curve due to an increase in $H_t$ from $H_0^t$ to $H_1^t$. The labor supply curve shifts out. This results in a higher level of $N_t$ consistent with the labor market clearing. The production function does not shift. Higher $N_t$ means more $Y_t$, so the $Y^*$ curve shifts to the right.
The $Y^d$ curve is conceptually identical to what we had before, but is a bit more complicated because of the presence of investment. We again define it as the set of $(r_t, Y_t)$ pairs where income, $Y_t$, equals desired expenditure, $Y^d_t$, where desired expenditure is defined as the sum of desired expenditure by households on consumption, firms on investment in new capital, and the government, where these spending decisions by the various actors are chosen optimally. In other words, total desired expenditure equals $Y^d_t = C_t + I_t + G_t$. Plugging in the optimal consumption function and investment demand function, we have $Y^d_t = C(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t) + I(r_t, A_{t+1}, q, K_t) + G_t$. This is identical to what we had before in the endowment economy model except now we have investment. As before, desired expenditure is a function of income because consumption is a function of income through the consumption function. The marginal propensity to consume is less than one, and other desired components of expenditure are still positive if $Y_t = 0$. This means that graphically $Y^d_t$ as a function of $Y_t$ has a positive intercept when $Y_t = 0$ and is upward-sloping with slope less than one. The point where $Y^d_t = Y_t$ graphically occurs where the desired expenditure line crosses a 45 degree line showing points where $Y_t = Y^d_t$. 

\[ Y_t = A_t(K_t, N_t) \]
We derive the $Y^d$ curve as we did before. The vertical position of desired expenditure as a function of income depends on $r_t$. It does so now through two components, consumption and investment, both of which are smaller for larger values of $r_t$. We start with a value of $r_t$, say $r_t^0$. This determines the vertical position of the desired expenditure line. We find the $Y_t$ where $Y^d_t = Y_t$, and this gives a $(r_t, Y_t)$ pair. Consider a lower value of $r_t$, say $r_t^1$. This increases both desired consumption and investment for every level of $Y_t$, so the expenditure line shifts up, and the $Y_t$ where $Y^d_t = Y_t$ is bigger. A higher value of the real interest rate, say $r_t^2$, has the opposite effect. Connecting the dots, we get a downward sloping $Y^d$ curve. In terms of the equations presented above, the $Y^d$ curve summarizes the consumption function, the investment demand function, and the aggregate resource constraint.

The $Y^d$ curve will shift if desired expenditure changes due to a change in anything other than $r_t$ or $Y_t$. For example, consider an increase in $A_{t+1}$. This makes firms want to do more investment (and households will want to do more consumption, because $Y_{t+1}$ will be bigger). This shifts the expenditure line up, which increases the $Y_t$ where $Y^d_t = Y_t$ for any value of $r_t$. In other words, the $Y^d$ curve would shift right. An increase in $q$ would have a similar effect by leading to higher
desired investment. An increase in $G_t$ would also shift the $Y^d$ curve to the right – the direct effect of higher $G_t$ on desired expenditure is one, and the indirect effect coming in via the consumption function is negative but less than one, so total desired expenditure increases, and therefore the $Y^d$ curve shifts right. A decrease in $G_{t+1}$ would make households want to consume more in the present (because they would effectively pay fewer taxes in $t + 1$, meaning more net income in $t + 1$). This would increase desired expenditure for every level of $Y_t$ and $r_t$ and therefore would shift the $Y^d$ curve to the right. A reduction in $K_t$ (due to something like a natural disaster) would make firms want to do more investment. This would also result in the $Y^d$ curve shifting to the right. We can see this in the graph below:

Equilibrium occurs where we are on both the $Y^d$ and $Y^s$ curves. Being on the $Y^s$ curve means that the labor market clears; being on the $Y^d$ curve means that the “goods market clears” in the sense that desired expenditure equals actual income. Being on both curves means that the economy is in general equilibrium – all markets simultaneously clear. We can see this graphically below. For ease of exposition I omit the derivation of the $Y^d$ curve with the plot of desired expenditure as a function of income, but show the labor market since we will be interested in how the real wage and
labor input vary in response to changes in exogenous variables.

\[ Y_t = A_t(K_t, N_t) \]

6 Equilibrium Effects of Changes in Exogenous Variables

Given this setup, we can graphically analyze how changes in exogenous variables affect the values of endogenous variables. The exogenous variables are \( A_t, A_{t+1}, q, G_t, G_{t+1}, K_t, \) and \( H_t \). I will not present or focus on results of exogenous reductions in \( K_t \); this gets messy because both the \( Y^s \) and \( Y^d \) curves will shift simultaneously. Graphically, we will be able to determine values of \( Y_t, N_t, r_t, \) and \( w_t \). To determine how \( C_t \) and \( I_t \) are impacted by a change in exogenous variable we will need to look more closely at the consumption function and investment demand function.

There is one point to mention before proceeding. In particular, we will assume that fluctuations in current \( I_t \) have no discernible effect on future \( Y_{t+1} \). Why is this important? Suppose that some exogenous variable changes, which causes \( I_t \) to rise. \( I_t \) being higher means that \( K_{t+1} \) will be higher. \( K_{t+1} \) being higher means that \( Y_{t+1} \) will be higher. But \( Y_{t+1} \) being higher would influence current desired consumption, which would then cause the \( Y^d \) curve to shift, and so on. We will ignore
effects like this and it is safe to do so – the stock of capital is very large relative to the flow of investment, and a small change in the future capital stock will have only small effects on future output. In other words, we can effectively think of \( Y_{t+1} \) as being exogenous, with the exception of reactions to increases in expected future productivity, \( A_{t+1} \), which will increase expected future income, \( Y_{t+1} \), and will therefore make households want to consume more in the present. Changes in \( G_t, G_{t+1}, H_t, A_t, \) and \( q \) will be assumed to have no effect on \( Y_{t+1} \).

Let’s start by analyzing the consequences of an exogenous increase in \( A_t \), say from \( A_t^0 \) to \( A_t^1 \). As we saw above, this causes the the labor demand curve to shift to the right and the production function to shift up. Taken together, this means that the \( Y^s \) curve shifts right. There is no effect of a change in \( A_t \) on the \( Y^d \) curve. Graphically, we can therefore determine that in the new equilibrium (where the \( Y^d \) and \( Y^s \) curves intersect), \( Y_t \) is higher, \( r_t \) is lower, \( w_t \) is higher, and \( N_t \) is higher. We can see this graphically below:

We have to look at the equations underlying the consumption and investment demand functions to determine how these variables change. Since \( Y_t \) is higher and \( r_t \) is lower, with nothing else affected, we know that \( C_t \) will be higher. Since \( r_t \) is lower and there is no change in \( A_{t+1}, q, \) or \( K_t \),
we also know that $I_t$ will be higher. Hence, an increase in $A_t$ results in increases in $Y_t$, $C_t$, $N_t$, $w_t$, and $I_t$, and a decrease in $r_t$.

Next, consider an exogenous increase in $H_t$ from $H^0_t$ to $H^1_t$. This causes the labor supply curve to shift to the right. This results in a higher value of $N_t$ consistent with the labor market being in equilibrium. There is no shift in the production function. But $N_t$ being higher entails more $Y_t$ for every $r_t$. In other words, the $Y^s$ curve shifts to the right. This entails a higher value of $Y_t$ and a lower value of $r_t$ in the new equilibrium. $N_t$ is higher and $w_t$ is lower. We can see this in the graph below:

How do $C_t$ and $I_t$ change? Because $r_t$ is lower and $Y_t$ higher, we can determine that $C_t$ will increase. Because $r_t$ is lower, we can determine that $I_t$ is higher. Hence, qualitatively the effects of a labor supply shock are similar to the effects of a productivity shock, with the exception of the real wage, which increases when $A_t$ increases but decreases when $H_t$ increases.

\footnote{The warning two paragraphs above is relevant here. Higher $I_t$ would mean higher $K_{t+1}$, which would mean higher $Y_{t+1}$. This would mean even higher $C_t$, but would also entail the $Y^d$ curve shifting right, which would work to raise $r_t$ and therefore lower $I_t$, other things being equal. Taking these effects into account wouldn’t actually change any of the conclusions from our analysis but it is much cleaner to ignore them.}
Next, let’s consider a change in an exogenous variable which causes the $Y^d$ curve to shift to the right. This could be caused by an increase in expected future productivity, $A_{t+1}$, an increase in the investment shock, $q_t$, an increase in current government spending, $G_t$, or a decrease in expected future government spending, $G_{t+1}$.

For the purposes of the main graph, it does not matter which of these exogenous variables are changing (but it will matter for determining how consumption and investment are affected). The $Y^d$ curve shifting to the right results in a higher real interest rate and no change in output. There is no impact on the labor market. Hence, the only effect of an increase in demand in the neoclassical model is an increase in $r_t$ – shifts in $Y^d$ do not impact output.

A decrease in the current capital stock, $K_t$, would also have this effect, but would also be associated with a shift in the $Y^s$ curve.

There is a tradition, dating back to Keynes, which emphasizes the role of “animal spirits” in driving economic fluctuations. The basic gist is that “animal spirits” represents optimism or pessimism that makes households and firms want to spend more in the present. One could think about changes in expectations of $A_{t+1}$ as representing waves of optimism or pessimism in this model – it is a change in an exogenous variable unrelated to current “fundamentals” which induces households and firms to want to spend more or less in the present. In the neoclassical model exogenous changes in $A_{t+1}$ will not affect output and therefore cannot be a driver of the business cycle, but in Keynesian models these changes can affect $Y_t$. 
or labor market variables in this model. In other words, in the neoclassical model only supply shocks (changes in productivity or exogenous labor supply shifts) can affect output in equilibrium.

Now, how consumption and investment are affected will depend on which exogenous variable is changing. Let’s start with an increase in $A_{t+1}$. Holding the interest rate fixed, this would mean that households would want to do more consumption (an increase in $A_{t+1}$ would lead to higher $Y_{t+1}$, because when we get to period $t+1$ an increase in $A_{t+1}$ is just like an increase in $A_t$ in period $t$) and firms would want to do more investment. But the interest rate rises, and we know that there is no change in $Y_t$. Since there is no change in $G_t$ (it’s exogenous), we therefore know that $dC_t + dI_t = 0$ – in other words, the sum of the changes in consumption and investment must be zero, so if one of these rises, the other must fall. We cannot determine with certainty the signs of these effects – the higher interest rate works in the opposite direction of the higher $A_{t+1}$ for both variables, so it is unclear how each react. Thus, the effects of higher $A_{t+1}$ on both $C_t$ and $I_t$ are ambiguous.

What about an increase in $q$? This makes firms want to do more investment for a given interest rate. But in equilibrium the real interest rate rises. This works against the direct effect of higher $q$ on $I_t$, and would apparently make the total effect ambiguous. But this is not so. How do we know? Like above, we know that neither output nor government spending change. Therefore, the sum of the changes in consumption and investment must be zero, i.e. $dC_t + dI_t = 0$. The higher real interest rate, combined with no change in current $Y_t$, means that consumption will fall. But if consumption falls, investment must rise for there to be no change in $Y_t$. Hence, investment rises (even though $r_t$ is higher) and consumption falls.

Now, consider an increase in $G_t$. This results in a higher $r_t$ and no change in $Y_t$. The higher $r_t$ means that investment will fall. The higher $r_t$ plus the direct effect of the increase in $G_t$ mean that $C_t$ will also fall. In other words, private expenditure, $C_t + I_t$, gets completely “crowded out” after an increase in government spending, meaning that $dG_t = -(dC_t + dI_t)$, or that change in government spending must equal the negative sum of changes in consumption and investment. Since investment falls, consumption will fall less than one-for-one with the increase in $G_t$, which is different than what we saw in the endowment economy where there was no investment.

Finally, let’s consider the effects of an increase in $G_{t+1}$ known to agents in period $t$. This results in an inward shift of the $Y^d$ curve and therefore a lower real interest rate. The direct effect of an increase in $G_{t+1}$ is for consumption to fall in period $t$, but the lower real interest rate seemingly counteracts that. It turns out that we know that consumption must fall. Why is this? The lower $r_t$ causes $I_t$ to rise. Since current output is unaffected, and current $G_t$ isn’t changing, we again have the condition that $dC_t + dI_t = 0$. If $I_t$ rises then $C_t$ must fall.

\[11\] If we entertained changes in some of these variables impacting labor supply we would get changes in $Y_t$ and labor market variables, but these would be small, and so it is safe to ignore them.
The table above summarizes the qualitative effects of changes in the exogenous variables on the endogenous variables. + markers indicate that the variable in question increases, − markers that it decreases, and 0 indicates no change. The only exogenous variables which can impact output, labor hours, or the real wage are $A_t$ and $H_t$.

7 Empirical Evidence

The neoclassical model predicts that the only exogenous variables which can affect output are changes in productivity, $A_t$, or labor supply shocks, picked up by the exogenous variable $H_t$. We know that in the data output and labor market variables move around over the business cycle. Can the neoclassical model provide a realistic description of these movements?

We begin with a quick discussion of business cycle data. Our model makes predictions for the following economic series: GDP/output, consumption, investment, the real interest rate, hours worked, and the real wage. The model is sometimes called the “real business cycle” model because all of these series are real. We can download these series from a variety of different websites; don’t worry about the exact definitions of the series I use here.\(^\text{12}\)

As noted early on in the course, many economic time series trend up. For looking at the business cycle, we want to focus on deviations about the trend (whereas growth theory studies the behavior of the trend itself). There are different ways of de-trending series. Perhaps the most obvious way to do so is with a linear trend: basically fit a straight line to a series that is trending up, and then calculate the de-trended or filtered series as the deviation of the actual series about the trend. I do something that is slightly more sophisticated than straight linear de-trending. In particular, I de-trend the natural logs of series using the Hodrick-Prescott (HP) filter. The HP filter calculates a trend by essentially calculating a two-sided moving average of the series. Below is a plot of log.

---

\(^\text{12}\)The series for GDP, consumption, and investment are real series taken from the National Income and Product Accounts accounts and are the broadest definitions of consumption and investment. For the real interest rate I take the three month Treasury Bill rate minus one-period ahead inflation; this is an approximation to the fact that the real interest rate is based on expected inflation. Total hours worked are hours worked in the non-farm business sector divided by the population aged 16 and over. The real wage series is real hourly compensation in the non-farm business sector. The hours and wage series are available from the Bureau of Labor Statistics; the data on GDP, consumption, investment, and the price level are from the Bureau of Economic Analysis; I downloaded the three month Treasury Bill rate from the St. Louis Fed FRED data base, and computed the real interest rate calculating inflation from the GDP price deflator, which is available from the BEA.
real GDP, the HP trend, and the deviations of actual GDP from the HP trend. The left scale is for the deviations from trend (the de-trended or filtered series), and has as units percent deviations from trend. The right scale is a log scale.

One thing that immediately pops out is that the trend and the actual series are pretty close to one another; put differently, deviations from the trend are quite small. I also took logs of the other series I mentioned and detrended them using the HP filter. The tradition within business cycle analysis is to focus on “second moments” of the data, or variances and co-variances (correlations are re-scaled covariances bound between -1 and 1). Correlations are therefore a measure of how series co-move. Positive correlations mean that one series being above trend typically coincides with the other series being above trend, and vice-versa. Since we will not be doing any serious quantitative analysis, we will focus on correlations and not volatilities. Below is a table with the correlations of the HP filtered series with HP filtered GDP:

<table>
<thead>
<tr>
<th>Series</th>
<th>Correlation with GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.78</td>
</tr>
<tr>
<td>Investment</td>
<td>0.85</td>
</tr>
<tr>
<td>Hours</td>
<td>0.87</td>
</tr>
<tr>
<td>Real Wage</td>
<td>0.14</td>
</tr>
<tr>
<td>Real Interest Rate</td>
<td>-0.05</td>
</tr>
</tbody>
</table>

We can see that GDP, its components, and total hours worked are very strongly correlated with one another – the pairwise correlations between these series are all greater than 0.75. Movements in GDP are positively correlated with movements in the real wage, though this correlation is

\[14\] An exception here is that I do not take logs of the interest rate, since its units are already in percentage units.
weaker (though are some issues related to real wage measurement that are beyond the scope of this course).\footnote{In particular, there is an issue related to what is often called the “composition bias.” In our model everyone is all the same and there is no heterogeneity in workers’ productivities or wage rates. In the real world this is not so – there are high wage workers and low wage workers, and the spread between the two is large. In the data, we typically see that most job loss during recessions is concentrated among lower wage workers. Since the aggregate real wage series is essentially a measure of the average real wage, disproportionate job loss among low wage workers tends to mechanically make the average wage appear higher during a recession. This is the composition bias – the composition of workers tends to shift towards higher wage workers in downturns and towards lower wage workers in expansions. Because the aggregate real wage series is an average, this compositional change tends to make the real wage look less procyclical than it likely really is.} We say that these series are all \textit{procyclical}. A series is said to be procyclical if it is positively correlated with GDP – in words, when output is above trend, the series is above trend on average. The real interest rate is \textit{countercyclical} in the sense that it is negatively correlated with GDP, though this correlation, like the correlation of the real wage with GDP, is closer to zero than to one.

Can our model make sense of these facts? The only two exogenous variables in the model that can account for movements in output are $A_t$ and $H_t$. An increase in $A_t$ causes output, consumption, hours, investment, and the real wage to all increase, while the real interest rate decreases. A decrease in $A_t$ has the opposite effects. These movements are qualitatively consistent with what we observe in the data. An increase in $H_t$ causes output, consumption, investment, and hours to all increase together, while the real wage and the real interest rate decrease. The movement of the real wage, which is countercyclical driven by labor supply shocks, is inconsistent with the qualitative comovements we observe in the data. Therefore, the model can potentially account for these movements if it is predominantly driven by changes in productivity, $A_t$.\footnote{Note that we are not claiming that there are no movements in the other exogenous variables. These almost certainly move, but since they don’t affect $Y_t$, they can’t be responsible for movements in $Y_t$ if this is the right model. Movements in these other variables do impact $r_t$, however. These “demand shocks” which move $r_t$, but which are not associated with changes in $Y_t$, can help the model fit the data better in the sense of driving down the correlation between the real interest rate and output closer to what we observe in the data (which is negative but nearly zero). If the only exogenous variable moving were $A_t$ the correlation between the real interest rate and output would be expected to be very close to -1.}

If the model can potentially provide a decent account of the qualitative comovements we observe in the data if it is predominately driven by changes in $A_t$, this begs the question of whether or not there are in fact changes in $A_t$ in the data that align with observed changes in $Y_t$. In principle, as an exogenous variable $A_t$ is not directly observed, but we can make use of some assumptions to try to back out a measure of it in the data. In particular, let’s assume that the production function is Cobb-Douglas. This means that:

\[
Y_t = A_tK_t^\alpha N_t^{1-\alpha}
\]

Take logs of this:

\[
\ln Y_t = \ln A_t + \alpha \ln K_t + (1 - \alpha) \ln N_t
\]

Now, $N_t$ and $Y_t$ are things which we can observe in the data. What about $K_t$? It turns out
that we can get measurements of this as well in the data. What about $\alpha$? As discussed earlier, the model has the implication that $1 - \alpha = \frac{w N_t}{Y_t}$, which we can measure in the data. On average this is roughly $2/3$, implying that $\alpha = 1/3$. Given all this measurement, we can therefore back out an empirical measure of $A_t$, which is often called “total factor productivity” (TFP) (or sometimes the “Solow residual”). This is simply:

$$\ln A_t = \ln Y_t - \alpha \ln K_t - (1 - \alpha) \ln N_t$$

Empirically, this series is often referred to as “total factor productivity” because it is a measure of productivity – it is the surplus output you get over and above what comes from measured inputs. It is also often called a “residual” for the same reason – it is output that cannot be explained via observable inputs.

Below is a time series plot of detrended total factor productivity and detrended real GDP.

We can see that these two series look very similar. All post-war US recessions are associated with drops in measured $A_t$ that line up nicely with drops in $Y_t$. In other words, empirically there is some evidence that there are in fact large and frequent changes in $A_t$ that line up well with movements in $Y_t$. Indeed, the correlation between de-trended TFP and de-trended output is 0.79. In other words, not only is the model capable of qualitatively matching the data when driven by changes in $A_t$, there is some relatively “model-free” evidence that there are changes in $A_t$ that line up well with observed changes in $Y_t$. This provides some empirical support for using the model to think seriously about business cycles.

---

16 Measurement of the capital stock is somewhat tricky, and relies on direct measurements of physical plant and equipment as well as making use of the capital accumulation equation to infer values of the capital stock given observed movements in investment.

17 For the purposes of calculating $A_t$, I do not do any detrending. I present detrended (or “filtered”) series here because both $Y_t$ and measured TFP have upward trends. The upward trend in measured TFP is essentially coming from increase in $Z_t$ using terminology from the Solow model, which we want to “get rid of” when thinking about the business cycle.
8 Policy Implications of the Neoclassical Model

We have thus far established that (i) our neoclassical or “real business cycle model” is capable of qualitatively matching actual business cycle data when driven by changes in $A_t$, and (ii) there is some evidence consistent with there actually being changes in $A_t$ that line up well with the actual business cycle. This means that there is some empirical evidence to suggest that the model is a pretty good model. Given that we think the model fits the data fairly well, we want to ask what kinds of policy implications the model has.

We want to analyze the problem of a fictitious social planner who acts to benevolently maximize household utility subject to the resource constraints that the economy as a whole faces. There will be no explicit prices (e.g. the real interest rate and the real wage) in the planner’s problem, since prices serve to coordinate actions in a decentralized equilibrium. Think about the planner coordinating actions by choosing quantities. Our objective is to find the solution to the social planner’s problem and then to compare it to the competitive equilibrium outcome. To the extent to which the two solutions do not coincide there may be welfare gains to be had from government intervention of one sort or the other. This is exactly analogous to the planner’s problem we considered in the endowment economy problems with heterogeneous agents. We are just applying the same concept to a model with no heterogeneity and production. The idea is the same as before – the planner picks quantities and ignores prices. We want to see how the planner’s solution compares to the competitive equilibrium solution.

The planner faces the resource constraint that consumption plus investment plus government spending (which is taken as an exogenous value) be equal to total production. I am not going to model the choices of $G_t$ and $G_{t+1}$: I take those as exogenously given and beyond the planner’s control. If I wanted to, I could assume that the household received some utility flow from government spending and model that choice. There is one resource constraint for each period:

$$C_t + \frac{K_{t+1}}{q} - \frac{(1 - \delta)K_t}{q} + G_t = A_t F(K_t, N_t)$$

$$C_{t+1} - \frac{(1 - \delta)K_{t+1}}{q} + G_{t+1} = A_{t+1} F(K_{t+1}, N_{t+1})$$

In writing these constraints I have eliminated investment using the capital accumulation equation, so that $I_t = K_{t+1} - (1 - \delta)K_t$, and have imposed the previously discussed terminal condition that $I_{t+1} = -(1 - \delta)K_{t+1}$. The planner’s objective is to maximize household lifetime utility subject to these two constraints:

$$\max_{C_t, C_{t+1}, N_t, N_{t+1}, K_{t+1}} U = u(C_t) + v(1 - N_t) + \beta u(C_{t+1}) + \beta v(1 - N_{t+1})$$

s.t.
\[
C_t + \frac{K_{t+1}}{q} - \frac{(1-\delta)K_t}{q} + G_t = A_t F(K_t, N_t)
\]
\[
C_{t+1} - \frac{(1-\delta)K_{t+1}}{q} + G_{t+1} = A_{t+1} F(K_{t+1}, N_{t+1})
\]

This is a constrained problem with two constraints. To deal with the two constraints I am going to eliminate both \(C_t\) and \(C_{t+1}\) as choice variables by solving for them from the constraints and substituting back into the objective function. The re-written problem is:

\[
\max_{N_t, N_{t+1}, K_{t+1}} U = u \left( A_t F(K_t, N_t) - \frac{K_{t+1}}{q} + \frac{(1-\delta)K_t}{q} - G_t \right) + v(1 - N_t) + \ldots + \beta u \left( A_{t+1} F(K_{t+1}, N_{t+1}) + \frac{(1-\delta)K_{t+1}}{q} - G_{t+1} \right) + \beta v(1 - N_{t+1})
\]

This is now an unconstrained problem. To characterize the solution take the derivatives with respect to the choice variables and set them equal to zero:

\[
\frac{\partial U}{\partial N_t} = 0 \Leftrightarrow u' \left( A_t F(K_t, N_t) - \frac{K_{t+1}}{q} + \frac{(1-\delta)K_t}{q} - G_t \right) A_t F_N(K_t, N_t) - v'(1 - N_t) = 0
\]
\[
\frac{\partial U}{\partial N_{t+1}} = 0 \Leftrightarrow \beta u' \left( A_{t+1} F(K_{t+1}, N_{t+1}) + \frac{(1-\delta)K_{t+1}}{q} - G_{t+1} \right) A_{t+1} F_N(K_{t+1}, N_{t+1}) - \beta v'(1 - N_{t+1}) = 0
\]
\[
\frac{\partial U}{\partial K_{t+1}} = 0 \Leftrightarrow -\frac{1}{q} u' \left( A_t F(K_t, N_t) - \frac{K_{t+1}}{q} + \frac{(1-\delta)K_t}{q} - G_t \right) + \ldots + \beta u' \left( A_{t+1} F(K_{t+1}, N_{t+1}) + \frac{(1-\delta)K_{t+1}}{q} - G_{t+1} \right) \left( A_{t+1} F_K(K_{t+1}, N_{t+1}) + \frac{(1-\delta)}{q} \right) = 0
\]

These can be re-arranged and re-written by substituting \(C_t\) and \(C_{t+1}\) back in to obtain:

\[
v'(1 - N_t) = u'(C_t) A_t F_N(K_t, N_t)
\]
\[
v'(1 - N_{t+1}) = u'(C_{t+1}) A_{t+1} F_N(K_{t+1}, N_{t+1})
\]
\[
u'(C_t) = \beta u'(C_{t+1}) (q A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta))
\]

These three conditions characterize the solution to the social planner’s problem.

Now, let’s take a step back and recall what were the equilibrium conditions of the competitive equilibrium. On the household side, we had:

\[
v'(1 - N_t) = u'(C_t) w_t
\]
\[
v'(1 - N_{t+1}) = u'(C_{t+1}) w_{t+1}
\]
\[
u'(C_t) = \beta (1 + r_t) u'(C_{t+1})
\]
On the firm side, we had:

\[ w_t = A_tF_N(K_t, N_t) \]
\[ w_{t+1} = A_{t+1}F_N(K_{t+1}, N_{t+1}) \]
\[ 1 + r_t = qA_{t+1}F_K(K_{t+1}, N_{t+1}) + (1 - \delta) \]

If you combine the firm first order conditions with the household first order conditions in such a way as to eliminate prices, you get:

\[ v'(1 - N_t) = u'(C_t)A_tF_N(K_t, N_t) \]
\[ v'(1 - N_{t+1}) = u'(C_{t+1})A_{t+1}F_N(K_{t+1}, N_{t+1}) \]
\[ u'(C_t) = \beta u'(C_{t+1})(qA_{t+1}F_K(K_{t+1}, N_{t+1}) + (1 - \delta)) \]

These are exactly the same first order conditions that obtain in the planner’s solution. Put differently – the competitive equilibrium outcome and the solution to the planner’s problem are one in the same. This means that a planner can do no better than the private economy left to its own devices. In some sense this is a formalization of Adam Smith’s *laissez faire* idea – a private economy left to its own devices achieves a Pareto optimal allocation, by which I mean that it would not be possible to improve upon the allocations. Sometimes we say that the equilibrium is “efficient.” In modern economics this result is formalized by the first fundamental welfare theorem, which states that, under some conditions, a competitive equilibrium is optimal from the planner’s perspective. Those conditions include things like price-taking behavior and no distortionary taxation and are satisfied in the model we have written down to this point.

This result has a very important implication. The business cycle model we’ve written down leads to an equilibrium which is efficient and cannot be improved upon by a planner. Driven by exogenous productivity shocks (exogenous movements in \( A_t \)), it can produce movements in output and its components that share qualitative features with actual data, and an empirical measure of shocks to \( A_t \) (total factor productivity or the Solow residual) are strongly correlated with movements in \( Y_t \). Put differently, there is some suggestive evidence that our business cycle model “fits” actual aggregate data pretty well, which means that it is plausible that a model like that generated the actual data we observe. If that is the case, then it means that business cycles are “efficient” and that there is no welfare justification for activist government policies to try and “smooth out” business cycles via countercyclical policies.

This was (and is) a controversial idea, and the basic real business cycle framework resulted in a Nobel Prize for its two main developers, Edward Prescott and Finn Kydland. Prior to their work in the late 1970s and early 1980s, economists essentially assumed that movements in output about trend were necessarily bad from a welfare perspective, and that policy (both fiscal and monetary) should try to prevent such movements. The real business cycle framework offers an alternative
interpretation: fluctuations in output about trend may be the efficient response of the economy to changes in $A_t$ (which the model does not seek to explain). This means that there is no justification for activist short run stabilization policies. Rather, policies should focus on raising the level of $A_t$ in the long run without regard to what happens in the short run (as in the Solow model). Furthermore, in the background there may be some income and consumption inequality among agents that a planner would like to address via transfers, as we saw earlier. But there should be no attempt to counteract fluctuations in $Y_t$ in the short run.

9 Criticisms of the Neoclassical Model

Real business cycle theory has met with substantial criticism since its inception. Some of this criticism is empirical, questioning the idea that what show up as changes in $A_t$ in the data really represent changes in $A_t$ in the model. Some other criticisms are more theoretical, arguing that the model is not realistic because it fails to capture several aspects of the real world that the critics feel are relevant.

A non-exhaustive list of criticisms of the real business cycle model are below:

1. In the data output and hours worked are about as volatile as one another. The model does not generate large enough movements in hours worked relative to the data. A potential way for the model to generate more movements in hours is to also have volatile labor supply shocks through $H_t$. This would also tend to lower the model’s correlation between $w_t$ and $Y_t$. However, it seems pretty unrealistic that there are large swings in preferences over work period-to-period.

2. The model has no role for active monetary policy and features monetary non-neutrality. In the real world people certainly think that the Fed matters. Empirically, measures of the money supply are positively correlated with economic activity. This could represent reverse causality (a central bank adjusting the money supply in response to economic activity), but a lot of empirical research suggests that exogenous changes in the stance of monetary policy do impact economic activity.

3. The model does not allow for demand shocks to impact output, which seems counter to what we observe in the data.

4. If $A_t$ is a measure of productivity, what does it mean for productivity to decline? Did we forget how to produce? Is it plausible that that $A_t$ moves around a lot over short horizons?

5. There have been lots of people that have questioned the measurement of $A_t$. In particular, we may not have good measures of $N_t$ and $K_t$. In particular, there could be unobserved utilization – for example, during recessions workers sit at work idle, so actual labor input is declining, but measured labor input does not decline by much, and so too much of the movement in output gets attributed to changes in $A_t$. 

33
6. The model does not have any heterogeneity. Real world recessions are costly not because everyone suffers income that is slightly lower, but rather because some people lose all of their income whereas others suffer no income loss. This means that we want to be careful about taking the model’s policy and welfare implications too seriously. See the previous comments about inequality and some of our previous work on inequality.

Each of these criticisms has merit. I am least sympathetic to point (4); properly understood, $A_t$ is more than just “technology,” and so there is no compelling reason to insist that it can’t decline or that it can’t move around a lot over short horizons.

My own view is that real business cycle theory is a decent first pass at understanding economic fluctuations, though it is admittedly too simple and one probably needs to be weary about taking its policy and welfare implications too seriously. Nevertheless, I think it’s a good benchmark. What was (and is) revolutionary about the real business cycle theory is that it suggests that there may be no justification for activist policies because fluctuations in economic aggregates may be the efficient reaction to changing conditions like $A_t$. This does not mean that in all circumstances there is no need for activist economic policy; what it does mean is that proponents of activist policies need to articulate the friction or market failure that moves us beyond the real business cycle world and into a world in which policy can improve outcomes by smoothing out the cycle.

References


[4] Williamson, Macroeconomics, Ch. 10, 4th edition (though note that he makes assumptions which allow the $Y^s$ curve to be upward-sloping but not vertical because he assumes that the real interest rate affects labor supply, which we ignore)