Graduate Macro Theory II: Notes on New Keynesian Model

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1 Introduction

This note describes the simplest version of the New Keynesian model. The NK model takes a real business cycle model as its backbone and adds to it sticky prices. The sticky prices give rise to non-trivial monetary neutralities and also give rise to a serious role for active economic policy.

2 Households

The household side of the model ends up looking very standard. Households consume goods, supply labor, and hold money. We assume that they receive utility from holding money. They can save through bonds, which pay nominal interest rate i. The household problem can be written:

$$\max_{c_t, n_t, B_{t+1}, M_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln c_t + \theta \frac{(1-n_t)^{1-\xi} - 1}{1-\xi} + \frac{\left(\frac{M_{t+1}}{p_t}\right)^{1-\nu} - 1}{1-\nu} \right)$$

s.t.

$$c_t + \frac{B_{t+1} - B_t}{p_t} + \frac{M_{t+1} - M_t}{p_t} \leq w_t n_t + \Pi_t + i_t \frac{B_t}{p_t}$$

We can set the problem up using a Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln c_t + \theta \frac{(1-n_t)^{1-\xi} - 1}{1-\xi} + \frac{\left(\frac{M_{t+1}}{p_t}\right)^{1-\nu} - 1}{1-\nu} + \dots \right.$$
$$\left. \dots + \lambda_t \left(w_t n_t + \Pi_t + (1+i_t)\frac{B_t}{p_t} - c_t - \frac{B_{t+1}}{p_t} - \frac{M_{t+1} - M_t}{p_t} \right) \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow \frac{1}{c_t} = \lambda_t \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Leftrightarrow \theta(1-n_t)^{-\xi} = \lambda_t w_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \left(\lambda_{t+1}(1+i_{t+1})\frac{p_t}{p_{t+1}} \right) \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial M_{t+1}} = 0 \Leftrightarrow \left(\frac{M_{t+1}}{p_t} \right)^{-\nu} = \lambda_t - \beta E_t \left(\lambda_{t+1}\frac{p_t}{p_{t+1}} \right) \tag{4}$$

$$TV: \lim_{T\to\infty} \beta^T E_t \lambda_{t+T} \frac{B_{t+T+1}}{p_{t+T}} = 0 \tag{5}$$

We can use (1) to simplify this to three first order conditions (plus the transversality condition):

$$\theta(1-n_t)^{-\xi} = \frac{1}{c_t} w_t \tag{6}$$

$$\frac{1}{c_t} = \beta E_t \left(\frac{1}{c_{t+1}}(1+i_{t+1})\frac{p_t}{p_{t+1}} \right) \tag{7}$$

$$\left(\frac{M_{t+1}}{p_t} \right)^{-\nu} = \left(\frac{i_{t+1}}{1+i_{t+1}} \right) \frac{1}{c_t} \tag{8}$$

(6) is the standard static labor supply condition; (7) is the dynamic consumption Euler equation, with $1 + r_{t+1} = E_t(1+i_{t+1})\frac{p_t}{p_{t+1}}$; and (8) implicitly defines a demand for real balances: $m_t \equiv \frac{M_{t+1}}{p_t}$.

3 Firms

As in our earlier discussion of monopolostic competition, we break production up into two sectors: final goods and intermediate goods. We discuss each in turn.

3.1 Final Goods Firm

There is a single final goods firm which bundles intermediate goods into a final good available for consumption. There are a continuum of intermediate goods producers populating the unit interval; index these firms by j. The final good is a CES aggregate of these intermediate goods:

$$y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

The profit maximization problem is:

$$\max_{y_{j,t}} \quad p_t \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 p_{j,t} y_{j,t}$$

The first order condition is:

$$p_t \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}-1} = p_{j,t}$$

This can be simplified to:

$$\left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{1}{\varepsilon-1}} y_{j,t}^{-\frac{1}{\varepsilon}} = \frac{p_{j,t}}{p_t}$$

Simplifying further, and noting the definition of the final good, we get the demand function for each intermediate good j:

$$y_{j,t} = \left(\frac{p_{j,t}}{p_t} \right)^{-\varepsilon} y_t \tag{9}$$

The nominal value of the final good is the sum of prices times quantities of intermediates:

$$p_t y_t = \int_0^1 p_{j,t} y_{j,t} dj$$

Plug in the demand curve for intermediate goods and simplify:

$$p_t y_t = \int_0^1 p_{j,t}^{1-\varepsilon} p_t^{\varepsilon} y_t dj$$

$$p_t^{1-\varepsilon} = \int_0^1 p_{j,t}^{1-\varepsilon} dj$$

$$p_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} \tag{10}$$

3.2 Intermediate Goods Firms

Intermediate goods use labor as the only input of production. They are affected by aggregated TFP. The production function is linear in labor input:

$$y_{j,t} = a_t n_{j,t} \tag{11}$$

Given the downward sloping demand curve, firms have some market power and can set prices. It is assumed that they are not freely able to adjust prices in any period, however. In particular, firms face a constant hazard, $1-\phi$, of being able to adjust their price in any period. With probability ϕ then $p_{j,t} = p_{j,t-1}$. This is a form of price stickiness due to Calvo (1983).

Firms are nevertheless able to freely choose how much labor to use each period, whether they can adjust their price or not. Hence, let's first consider the problem of optimal labor choice for a given price. Since the firm can freely hire labor each period, we can write this as a static problem. Let's write it as the problem of cost-minimization. The reason for this is that the firm will always choose labor to minimize costs regardless of the price chosen, but they are not able to choose price to maximize profits:

$$\min_{n_{j,t}} \quad W_t n_{j,t}$$
$$\text{s.t.}$$
$$a_t n_{j,t} \geq \left(\frac{p_{j,t}}{p_t}\right)^{-\varepsilon} y_t$$

The problem is to minimize nominal costs subject to the restriction that the firm produces at least as much as is demanded at the given price. W_t is the nominal wage common to all firms. Since we take the price as given, the constraint is not a choice variable. Set the problem up as a Lagrangian:

$$\mathcal{L} = -W_t n_{j,t} + \varphi_{j,t}\left(a_t n_{j,t} - \left(\frac{p_{j,t}}{p_t}\right)^{-\varepsilon} y_t\right)$$

The first order condition is:

$$\frac{\partial \mathcal{L}}{\partial n_{j,t}} = 0 \Leftrightarrow W_t = \varphi_{j,t} a_t$$

$\varphi_{j,t}$ has the interpretation as (nominal) marginal cost – how much do costs go up if the firm has to produce one more unit of output. Hence the first order condition says to hire labor up until the point where the wage equals marginal cost times the marginal product of labor.

Now let's consider the pricing decision. Recall that a firm can change its price in any period only with probability $1-\phi$. In expectation it will be "stuck" with that price going forward into the future. Hence, the pricing problem of a firm that gets to update its price in period t is dynamic – the price it picks today will, in expectation, affect the profits it earns both today and in the future.

The firm discounts future profits flows by the stochastic discount factor, $M_t = \beta^t u'(c_t)$. In addition, it will discount future profit flows by ϕ^t. Profits as measured are in nominal terms; divide by p_{t+s} to put them into real terms (since that is what the households care about). Conditional on choosing its price today, there is a ϕ probability of having that price in effect the next period, ϕ^2 for the period after that, and so on. This profit maximization problem is conditional on (i) the demand function, (ii) the production function, and (iii) the first order condition for labor choice holding:

$$\max_{p_{j,t}} \quad E_t \sum_{s=0}^{\infty} M_{t+s} \frac{1}{p_{t+s}} \phi^s \left(p_{j,t} y_{j,t+s} - W_{t+s} n_{j,t+s}\right)$$
$$\text{s.t.}$$
$$y_{j,t+s} = \left(\frac{p_{j,t}}{p_{t+s}}\right)^{-\varepsilon} y_{t+s}$$
$$y_{j,t+s} = a_t n_{j,t+s}$$
$$W_{t+s} = \varphi_{j,t+s} a_{t+s}$$

We can make this an unconstrained problem by substituting the constraints in to the objective function:

$$\max_{p_{j,t}} \quad E_t \sum_{s=0}^{\infty} (\beta\phi)^s \, u'(c_{t+s}) \frac{1}{p_{t+s}} \left(p_{j,t} \left(\frac{p_{j,t}}{p_{t+s}} \right)^{-\varepsilon} y_{t+s} - \varphi_{j,t+s} \left(\frac{p_{j,t}}{p_{t+s}} \right)^{-\varepsilon} y_{t+s} \right)$$

This follows because $n_{j,t+s} = \frac{y_{j,t+s}}{a_{t+s}}$, and multiplying by W_{t+s} eliminates the a_{t+s}. Let's take the derivative with respect to $p_{j,t}$:

$$E_t \sum_{s=0}^{\infty} (\beta\phi)^s \, u'(c_{t+s}) \left((1-\varepsilon) p_{j,t}^{-\varepsilon} p_{t+s}^{\varepsilon-1} y_{t+s} + \varepsilon p_{j,t}^{-\varepsilon-1} \varphi_{j,t+s} p_{t+s}^{\varepsilon-1} y_{t+s} \right) = 0$$

We can simplify this by noting that we can pull the $p_{j,t}$ out of the sum since it does not depend on s:

$$(\varepsilon-1) p_{j,t}^{-\varepsilon} E_t \sum_{s=0}^{\infty} (\beta\phi)^s \, u'(c_{t+s}) \left(p_{t+s}^{\varepsilon-1} y_{t+s} \right) = \varepsilon p_{j,t}^{-\varepsilon-1} E_t \sum_{s=0}^{\infty} (\beta\phi)^s \, u'(c_{t+s}) \left(\varphi_{j,t+s} p_{t+s}^{\varepsilon-1} y_{t+s} \right)$$

Simplifying further:

$$p_t^{\#} = \frac{\varepsilon}{\varepsilon-1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi)^s \, u'(c_{t+s}) \left(\varphi_{t+s} p_{t+s}^{\varepsilon-1} y_{t+s} \right)}{E_t \sum_{s=0}^{\infty} (\beta\phi)^s \, u'(c_{t+s}) \left(p_{t+s}^{\varepsilon-1} y_{t+s} \right)}$$

I do two things in the final step. First, I remove the j subscript from $\varphi_{j,t+s}$, which has the interpretation of marginal cost. From the first order condition for optimal labor choice given above, marginal cost is just the ratio of the wage to a_t. Since this ratio does not depend on j, marginal cost is the same for all firms. Hence, looking at the above, all firms able to update their price will choose the same price. I denote this with a # and call it the "optimal reset price".

This formula essentially says that the optimal reset price is a markup over expected future marginal costs. It is easy to see that, if prices are not sticky, then $\phi = 0$ and this reduces to a condition we've seen before: $p_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\varphi_t$. With $\phi > 0$, the pricing decision will be forward-looking.

4 Aggregate Conditions and Market-Clearing

We assume the existence of a central bank that sets the money supply according to a money growth rule similar to what we've seen before:

$$\ln M_{t+1} - \ln M_t = (1-\rho_m)\pi^* + \rho_m \left(\ln M_t - \ln M_{t-1} \right) + \varepsilon_{m,t}$$

We want to write this in terms of real money balances (which will be stationary). Define $m_t \equiv \frac{M_{t+1}}{p_t}$. Add and subtract $\ln p_t$ terms where necessary:

$$\ln M_{t+1} - \ln p_t + \ln p_t - \ln p_{t-1} - \ln M_t + \ln p_{t-1} = (1-\rho_m)\pi^* + \ldots$$
$$\cdots + \rho_m \left(\ln M_t - \ln p_{t-1} + \ln p_{t-1} - \ln p_{t-2} - \ln M_{t-1} + \ln p_{t-2}\right) + \varepsilon_{m,t}$$

Define $\pi_t = \ln p_t - \ln p_{t-1}$ and $\Delta \ln m_t = \ln m_t - \ln m_{t-1}$. Then we can write this as:

$$\Delta \ln m_t = (1-\rho_m)\pi^* - \pi_t + \rho_m \pi_{t-1} + \rho_m \Delta \ln m_{t-1} + \varepsilon_{m,t}$$

Assume that aggregate technology follows an AR(1) in the log:

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_{a,t}$$

In equilibrium, the household and all firms must be maximizing, all output must be consumed, bond-holding must be zero, the money market must clear, and the labor market must clear. These conditions can be summarized as follows:

$$\begin{aligned} c_t = y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \\ b_t = 0 \\ n_t = \int_0^1 n_{j,t} \\ m_t^{-\nu} = \left(\frac{i_{t+1}}{1+i_{t+1}}\right)\frac{1}{c_t} \end{aligned}$$

The m_t must be equal to the m_t supplied by the central bank, and the n_t must be consistent with the first order condition for labor supply. The aggregate price level is defined by:

$$p_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$$

At time t, the fraction $1-\phi$ of firms will adjust their price. They will all adjust to $p_t^{\#}$. The other fraction ϕ of firms will be stuck with the price they charged the previous period:

$$p_t = \left(\int_0^{1-\phi} p_t^{\#1-\varepsilon} dj + \int_{1-\phi}^1 p_{j,t-1}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}} = \left((1-\phi)p_t^{\#1-\varepsilon} + \int_{1-\phi}^1 p_{j,t-1}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$$

Because the firms chosen to update their prices are chosen completely randomly (and because there are a continuum of firms), the average price charged by non-updaters in period t will be equal to the previous period's aggregate price:

$$\int_{1-\phi}^1 p_{j,t-1}^{1-\varepsilon} dj = \int_{1-\phi}^1 p_{t-1}^{1-\varepsilon} dj = \phi p_{t-1}^{1-\varepsilon}$$

This means the aggregate price level can be written:

$$p_t = \left((1-\phi)p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$$

The full set of equilibrium conditions for this model are given below:

$$\theta(1-n_t)^{-\xi} = \frac{1}{c_t}w_t \tag{12}$$

$$\frac{1}{c_t} = \beta E_t\left(\frac{1}{c_{t+1}}(1+i_{t+1})\frac{p_t}{p_{t+1}}\right) \tag{13}$$

$$m_t^{-\nu} = \left(\frac{i_{t+1}}{1+i_{t+1}}\right)\frac{1}{c_t} \tag{14}$$

$$p_t = \left((1-\phi)p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \tag{15}$$

$$y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{16}$$

$$y_{j,t} = a_t n_{j,t} \tag{17}$$

$$n_t = \int_0^1 n_{j,t} \tag{18}$$

$$w_t = \frac{W_t}{p_t} \tag{19}$$

$$\frac{\varphi_t}{p_t} = \frac{w_t}{a_t} \tag{20}$$

$$p_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\frac{E_t\sum_{s=0}^{\infty}(\beta\phi)^s u'(c_{t+s})\left(\varphi_{t+s}p_{t+s}^{\varepsilon-1}y_{t+s}\right)}{E_t\sum_{s=0}^{\infty}(\beta\phi)^s u'(c_{t+s})\left(p_{t+s}^{\varepsilon-1}y_{t+s}\right)} \tag{21}$$

$$y_t = c_t \tag{22}$$

$$\Delta\ln m_t = (1-\rho_m)\pi^* - \pi_t + \rho_m\pi_{t-1} + \rho_m\Delta\ln m_{t-1} + \varepsilon_{m,t} \tag{23}$$

$$\ln a_t = \rho\ln a_{t-1} + \varepsilon_{a,t} \tag{24}$$

$$1+\pi_t = \frac{p_t}{p_{t-1}} \tag{25}$$

5 Log-Linearization

Let's log-linearize this model. We want to do this in terms of stationary variables. Hence let's write the aggregate pricing equation in terms of (gross) inflation by dividing both sides of (15) by p_{t-1}:

$$\frac{p_t}{p_{t-1}} = \frac{1}{p_{t-1}}\left((1-\phi)p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$$

This can be written as:

$$1+\pi_t = \left(p_{t-1}^{-(1-\varepsilon)}(1-\phi)p_t^{\#1-\varepsilon} + \phi p_{t-1}^{-(1-\varepsilon)}p_{t-1}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$$
$$1+\pi_t = \left((1-\phi)\left(\frac{p_t^{\#}}{p_{t-1}}\right)^{1-\varepsilon} + \phi\right)^{\frac{1}{1-\varepsilon}}$$

Define $1+\pi_t^{\#} = \frac{p_t^{\#}}{p_{t-1}}$. We can write this as:

$$1+\pi_t = \left((1-\phi)(1+\pi_t^{\#})^{1-\varepsilon} + \phi\right)^{\frac{1}{1-\varepsilon}}$$

Let's log-linearize this about the steady state:

$$\ln(1+\pi_t) = \frac{1}{1-\varepsilon}\ln\left((1-\phi)(1+\pi_t^{\#})^{1-\varepsilon} + \phi\right)$$
$$\frac{\pi_t-\pi^*}{1+\pi^*} = \frac{1}{1-\varepsilon}\frac{\left((1-\varepsilon)(1-\phi)(1+\pi^{\#*})^{-\varepsilon}(\pi_t^{\#}-\pi_t^{\#*})\right)}{1+\pi^*}$$

Let's assume that $\pi^* = 0$ (maintain this assumption from here on out . . . this is standard in this literature). Defining $\widetilde{\pi}_t = \pi_t - \pi^*$ and similarly for reset price inflation. This then becomes:

$$\widetilde{\pi}_t = (1-\phi)\widetilde{\pi}_t^{\#}$$

We can write the reset price expression somewhat more simply as follows:

$$p_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\frac{A_t}{B_t}$$
$$A_t = E_t\sum_{s=0}^{\infty}(\beta\phi)^s u'(c_{t+s})\left(\varphi_{t+s}p_{t+s}^{\varepsilon-1}y_{t+s}\right) = \varphi_t p_t^{\varepsilon-1} + \phi\beta E_t A_{t+1}$$
$$B_t = E_t\sum_{s=0}^{\infty}(\beta\phi)^s u'(c_{t+s})\left(p_{t+s}^{\varepsilon-1}y_{t+s}\right) = p_t^{\varepsilon-1} + \phi\beta E_t B_{t+1}$$

The last parts of these expression follow from the fact that $u'(c_t) = \frac{1}{c_t}$ and $c_t = y_t$. Now define:

$$\widehat{a}_t = \frac{A_t}{p_t^{\varepsilon-1}}$$
$$\widehat{b}_t = \frac{B_t}{p_t^{\varepsilon-1}}$$

Then we can write:

$$\widehat{a}_t = \varphi_t + \phi\beta\frac{E_tA_{t+1}}{p_t^{\varepsilon-1}} = \varphi_t + \phi\beta\left(\frac{E_tp_{t+1}^{\varepsilon-1}}{p_t^{\varepsilon-1}}\right)E_t\widehat{a}_{t+1} = \varphi_t + \phi\beta(1+E_t\pi_{t+1})^{\varepsilon-1}E_t\widehat{a}_{t+1}$$

$$\widehat{b}_t = 1 + \phi\beta\frac{E_tB_{t+1}}{p_t^{\varepsilon-1}} = 1 + \phi\beta\left(\frac{E_tp_{t+1}^{\varepsilon-1}}{p_t^{\varepsilon-1}}\right)E_t\widehat{b}_{t+1} = 1 + \phi\beta(1+E_t\pi_{t+1})^{\varepsilon-1}E_t\widehat{b}_{t+1}$$

Given these definitions, we can then derive an expression for reset price inflation as:

$$\frac{p_t^{\#}}{p_{t-1}} = \frac{\varepsilon}{\varepsilon-1}\frac{\widehat{a}_t/p_{t-1}}{\widehat{b}_t}$$

Define: $\widehat{\widehat{a}}_t = \frac{\widehat{a}_t}{p_{t-1}}$. This can be written:

$$\widehat{\widehat{a}}_t = \frac{\varphi_t}{p_{t-1}} + (1+E_t\pi_{t+1})^{\varepsilon-1}\phi\beta\frac{E_t\widehat{a}_{t+1}}{p_{t-1}} = (1+\pi_t)\left(mc_t + (1+E_t\pi_{t+1})^{\varepsilon-1}\phi\beta E_t\widehat{\widehat{a}}_{t+1}\right)$$

Here I have defined $mc_t = \frac{\varphi_t}{p_t}$, or real marginal cost. This means that we have:

$$1 + \pi_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\frac{\widehat{\widehat{a}}_t}{\widehat{b}_t}$$

Let's log-linearize this:

$$\ln(1+\pi_t^{\#}) = \ln\varepsilon - \ln(\varepsilon-1) + \ln\widehat{\widehat{a}}_t - \ln\widehat{b}_t$$

$$\frac{\pi_t^{\#} - \pi^{\#*}}{1+\pi^{\#*}} = \frac{\widehat{\widehat{a}}_t - \widehat{\widehat{a}}^*}{\widehat{\widehat{a}}^*} - \frac{\widehat{b}_t - \widehat{b}^*}{\widehat{b}^*} \Rightarrow \widetilde{\pi}_t^{\#} = \widetilde{\widehat{\widehat{a}}}_t - \widetilde{\widehat{b}}_t$$

So now we need to log-linearize the auxiliary variables, $\widehat{\widehat{a}}_t$ and $\widehat{b}_t$:

$$\ln\widehat{\widehat{a}}_t = \ln(1+\pi_t) + \ln\left(mc_t + \phi\beta(1+E_t\pi_{t+1})^{\varepsilon-1}E_t\widehat{\widehat{a}}_{t+1}\right)$$

$$\frac{\widehat{\widehat{a}}_t - \widehat{\widehat{a}}^*}{\widehat{\widehat{a}}^*} = \frac{\pi_t - \pi^*}{\pi^*} + \frac{1+\pi^*}{\widehat{\widehat{a}}^*}\left((mc_t - mc^*) + (\varepsilon-1)\phi\beta(1+\pi^*)^{\varepsilon-2}\widehat{\widehat{a}}^*(E_t\pi_{t+1} - \pi^*) + \phi\beta(1+\pi^*)^{\varepsilon-1}(\widehat{\widehat{a}}_t - \widehat{\widehat{a}}^*)\right)$$

Using the fact that we have assumed $\pi^* = 0$, this becomes:

$$\widetilde{\widehat{\widehat{a}}}_t = \widetilde{\pi}_t + \frac{mc^*}{\widehat{\widehat{a}}^*}\widetilde{mc}_t + (\varepsilon-1)\phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\widetilde{\widehat{\widehat{a}}}_{t+1}$$

Given the definition of $\widehat{\widehat{a}}$, we know that $\frac{mc^*}{\widehat{\widehat{a}}^*} = (1-\phi\beta)$. Hence we have:

$$\widetilde{\widehat{a}}_t = \widetilde{\pi}_t + (1-\phi\beta)\widetilde{mc}_t + \varepsilon\phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\widetilde{\widehat{a}}_{t+1}$$

Now log-linearize the expression for $\widetilde{b}_t$:

$$\ln \widehat{b}_t = \ln(1 + \phi\beta(1+E_t\pi_{t+1})^{\varepsilon-1}E_t\widehat{b}_{t+1})$$

$$\frac{\widehat{b}_t - \widehat{b}^*}{\widehat{b}^*} = \frac{1}{\widehat{b}^*}\left((\varepsilon-1)\phi\beta(1+\pi^*)^{\varepsilon-2}\widehat{b}^*(E_t\pi_{t+1}-\pi^*) + \phi\beta(1+\pi^*)^{\varepsilon-1}(E_t\widehat{b}_{t+1}-\widehat{b}^*)\right)$$

Simplifying with the fact that $\pi^* = 0$:

$$\widetilde{\widehat{b}}_t = (\varepsilon-1)\phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\widetilde{\widehat{b}}_{t+1}$$

Now use these expressions to find an expression for reset price inflation:

$$\widetilde{\pi}_t^{\#} = \widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t = \widetilde{\pi}_t + (1-\phi\beta)\widetilde{mc}_t + \phi\beta E_t\left(\widetilde{\widehat{a}}_{t+1} - \widetilde{\widehat{b}}_{t+1}\right)$$

Now use the fact that $\widetilde{\pi}_t = (1-\phi)\widetilde{\pi}_t^{\#} = (1-\phi)(\widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t)$. This means:

$$\widetilde{\pi}_t = (1-\phi)\widetilde{\pi}_t + (1-\phi)(1-\phi\beta)\widetilde{mc}_t + (1-\phi)\phi\beta\left(\frac{E_t\widetilde{\pi}_{t+1}}{1-\phi}\right)$$

Solving for current inflation, we get:

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\widetilde{mc}_t + \beta E_t\widetilde{\pi}_{t+1} \tag{26}$$

This is called the New Keynesian Phillips Curve. It relates current inflation to current real marginal cost and expected future inflation. If you solve it forward (imposing that inflation not explode), then one can write current inflation as a present discounted value of real marginal costs:

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\sum_{s=0}^{\infty}\beta^s E_t\widetilde{mc}_{t+s} \tag{27}$$

Log-linearizing the remainder of the equations of the model is much more straightforward. Let's start with the household FOC and go ahead an impose the equilibrium condition that $c_t = y_t$:

$$\frac{1}{y_t} = \beta\left(\frac{1}{y_{t+1}}(1+r_{t+1})\right)$$
$$-\ln y_t = \ln\beta - E_t\ln y_{t+1} + r_{t+1}$$

$$\widetilde{y}_t = E_t\widetilde{y}_{t+1} - \widetilde{r}_{t+1} \tag{28}$$

This is just the log-linearized consumption Euler equation. It sometimes gets called the "New

Keynesian IS Curve" because it shows a negative relationship between current spending and the real interest rate. This is kind of bad terminology, as there is no investment here. Also, unlike old Keynesian IS curves, this is explicitly forward-looking. Next go to the labor supply condition (let's assume that $\xi = 1$ so that preferences over leisure are log):

$$\frac{\theta}{1-n_t} = \frac{1}{y_t} w_t$$

$$\ln\theta - \ln(1-n_t) = -\ln y_t + \ln w_t$$

$$\left(\frac{n^*}{1-n^*}\right)\widetilde{n}_t = -\widetilde{y}_t + \widetilde{w}_t \tag{29}$$

Next, let's look at the aggregate production function:

$$y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}}\,dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\ln y_t = \frac{\varepsilon}{\varepsilon-1}\ln\left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)$$

Note that, since inflation is zero in steady state, all firms will charge the same prices and will hence produce the same amount in steady state. Hence:

$$y^* = y_j^* \quad \forall j$$

Applying this fact above, we get:

$$\frac{y_t - y^*}{y^*} = \frac{1}{\int_0^1 y^{*\frac{\varepsilon-1}{\varepsilon}}\,dj}\left(y^{*\frac{\varepsilon-1}{\varepsilon}-1}\right)\int_0^1 (y_{j,t} - y^*)$$

$$\frac{y_t - y^*}{y^*} = \frac{\left(y^{*\frac{\varepsilon-1}{\varepsilon}-1}\right)}{y^{*\frac{\varepsilon-1}{\varepsilon}}}\int_0^1 (y_{j,t} - y^*)$$

$$\frac{y_t - y^*}{y^*} = \frac{\int_0^1 (y_{j,t} - y^*)}{y^*}$$

$$\widetilde{y}_t = \int_0^1 \widetilde{y}_{j,t} \tag{30}$$

Now log-linearize the intermediate goods firm production functions:

$$\ln y_{j,t} = \ln a_t + \ln n_{j,t}$$

$$\widetilde{y}_{j,t} = \widetilde{a}_t + \widetilde{n}_{j,t} \tag{31}$$

Now use this in the aggregate production function:

$$\widetilde{y}_t = \int_0^1 (\widetilde{a}_t + \widetilde{n}_{j,t})\, dj = \widetilde{a}_t + \int_0^1 \widetilde{n}_{j,t} dj$$

Now use the market-clearing condition for labor:

$$n_t = \int_0^1 n_{j,t} dj$$
$$\ln n_t = \ln \left(\int_0^1 n_{j,t} dj \right)$$

$$\widetilde{n}_t = \int_0^1 \widetilde{n}_{j,t} dj \tag{32}$$

This means that the log-linearized aggregate production function is identical to the log-linearized intermediate goods production function:

$$\widetilde{y}_t = \widetilde{a}_t + \widetilde{n}_t \tag{33}$$

The linearized firm first order condition for labor demand does not depend upon j subscripts:

$$w_t = mc_t a_t$$
$$\ln w_t = \ln mc_t + \ln a_t$$

$$\widetilde{w}_t = \widetilde{mc}_t + \widetilde{a}_t \tag{34}$$

Next we need to linearize the money demand function. The money supply function is already log-linear.

$$m_t^{-\nu} = \left(\frac{i_{t+1}}{1 + i_{t+1}} \right) \frac{1}{y_t}$$
$$-\nu \ln m_t = \ln i_{t+1} - \ln(1 + i_{t+1}) - \ln y_t$$

$$-\nu \widetilde{m}_t = -\widetilde{y}_t + \left(\frac{1}{i^*(1 + i^*)} \right) \widetilde{i}_{t+1} \tag{35}$$

The log-linearized Fisher relationship relates nominal to real interest rates:

$$\widetilde{i}_{t+1} = \widetilde{r}_{t+1} + E_t \widetilde{\pi}_{t+1} \tag{36}$$

It is sometimes more common to see the Phillips Curve expressed in terms of of an output gap instead of real marginal cost. With flexible prices, real marginal cost is constant. Why? Because it's just equal to the inverse of the markup, which is constant if prices are flexible. Hence $\widetilde{mc}_t^f = 0$.

The superscript f denotes "flexible price equilibrium". Using this fact, we would have:

$$\widetilde{w}_t^f = \widetilde{a}_t$$

Plug this into the FOC for labor supply:

$$\left(\frac{n^*}{1-n^*}\right)\widetilde{n}_t^f = -y_t^f + \widetilde{a}_t$$

Now using the log-linearized production function, we have:

$$\left(\frac{n^*}{1-n^*}\right)\widetilde{n}_t^f = -\widetilde{a}_t - \widetilde{n}_t^f + \widetilde{a}_t$$

The only possible solution is for $\widetilde{n}_t^f = 0$, which means that $\widetilde{y}_t^f = \widetilde{a}_t$. Now look at the log-linearized FOC for labor supply (outside of the flexible price equilibrium) imposing that the wage be given by the firm first order condition:

$$\left(\frac{n^*}{1-n^*}\right)\widetilde{n}_t = -\widetilde{y}_t + \widetilde{mc}_t + \widetilde{w}_t$$

Now use the log-linearized production function, $\widetilde{y}_t = \widetilde{a}_t + \widetilde{n}_t$ to eliminate $\widetilde{n}_t$:

$$\left(\frac{n^*}{1-n^*}\right)(\widetilde{y}_t - \widetilde{a}_t) = -\widetilde{y}_t + \widetilde{mc}_t + \widetilde{w}_t$$

$$\widetilde{mc}_t = \left(1 + \frac{n^*}{1-n^*}\right)(\widetilde{y}_t - \widetilde{a}_t)$$

But because $\widetilde{y}_t^f = \widetilde{a}_t$, we can therefore write current marginal cost as a deviation of output from its flexible price equilibrium (i.e. as an output gap):

$$\widetilde{mc}_t = \left(1 + \frac{n^*}{1-n^*}\right)(\widetilde{y}_t - \widetilde{y}_t^f)$$

Plug this back into the Phillips Curve, defining $\kappa = 1 + \frac{n^*}{1-n^*}$, to get:

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\kappa\left(\widetilde{y}_t - \widetilde{y}_t^f\right) + \beta E_t\widetilde{\pi}_{t+1} \tag{37}$$

The full set of log-linearized equilibrium conditions can be characterized as follows:

$$\widetilde{y}_t = \widetilde{a}_t + \widetilde{n}_t \tag{38}$$

$$\widetilde{i}_{t+1} = \widetilde{r}_{t+1} + E_t\widetilde{\pi}_{t+1} \tag{39}$$

$$-\nu\widetilde{m}_t = -\widetilde{y}_t + \left(\frac{1}{i^*(1+i^*)}\right)\widetilde{i}_{t+1} \tag{40}$$

$$\widetilde{y}_t^f = \widetilde{a}_t \tag{41}$$

$$\widetilde{y}_t = E_t\widetilde{y}_{t+1} - \widetilde{r}_{t+1} \tag{42}$$

$$\widetilde{w}_t = \widetilde{mc}_t + \widetilde{a}_t \tag{43}$$

$$\left(\frac{n^*}{1-n^*}\right)\widetilde{n}_t = -\widetilde{y}_t + \widetilde{w}_t \tag{44}$$

$$\widetilde{mc}_t = \left(1 + \frac{n^*}{1-n^*}\right)(\widetilde{y}_t - \widetilde{y}_t^f) \tag{45}$$

$$\widetilde{a}_t = \rho\widetilde{a}_{t-1} + \varepsilon_{a,t} \tag{46}$$

$$\Delta\widetilde{m}_t = (1-\rho_m)\pi^* - \widetilde{\pi}_t + \rho_m\widetilde{\pi}_{t-1} + \rho_m\Delta\widetilde{m}_{t-1} + \varepsilon_{m,t} \tag{47}$$

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\kappa\left(\widetilde{y}_t - \widetilde{y}_t^f\right) + \beta E_t\widetilde{\pi}_{t+1} \tag{48}$$

This is 11 equation in 11 variables. It is common to significantly reduce the system into a smaller number of equations. In particular, we can reduce this to four equations: a "demand" equation, (42), the so-called "New Keynesian IS curve"; a "supply" equation, (48), the Phillips Curve; an exogenous supply shifter, which comes from combining (41) with (46); a money demand equation, (40); and a money supply equation, (47). This reduced system can be written:

$$\widetilde{y}_t = E_t\widetilde{y}_{t+1} - \widetilde{r}_{t+1} \tag{49}$$

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\kappa\left(\widetilde{y}_t - \widetilde{y}_t^f\right) + \beta E_t\widetilde{\pi}_{t+1} \tag{50}$$

$$\widetilde{y}_t^f = \rho\widetilde{a}_{t-1}^f + \varepsilon_{a,t} \tag{51}$$

$$-\nu\widetilde{m}_t = -\widetilde{y}_t + \left(\frac{1}{i^*(1+i^*)}\right)(\widetilde{r}_{t+1} + E_t\widetilde{\pi}_{t+1}) \tag{52}$$

$$\Delta\widetilde{m}_t = (1-\rho_m)\pi^* - \widetilde{\pi}_t + \rho_m\widetilde{\pi}_{t-1} + \rho_m\Delta\widetilde{m}_{t-1} + \varepsilon_{m,t} \tag{53}$$

$$\Delta\widetilde{m}_t = \widetilde{m}_t - \widetilde{m}_{t-1} \tag{54}$$

This is 6 equations in 6 variables (output, inflation, the interest rate, flexible price output, real money balances, and real money balance growth). We can analyze impulse responses to two shocks – the flexible price shock (which is equivalent to a technology shock here) and the money supply shock. The model is already log-linearized, so we can solve for the policy functions in the usual way.

I parameterize the model as follows (this should not be thought of as a serious calibration exercise). I set $\beta = 0.99$, $n^* = \frac{1}{3}$, $\rho = 0.9$, $\sigma_{e,a} = 0.01$, $\rho_m = 0.5$, and $\sigma_{e,m} = 0.01$. It remains to parameterize ϕ. We can loosely calibrate this to the data to match the average duration between price changes. Suppose that you change a price today. The probability of getting to change it tomorrow is $1-\phi$, so the probability associated with a duration of one period is $1-\phi$. There is a ϕ probability of not getting to change your probability next period, which means there is a ϕ probability you enter two periods from now with the same price. The probability of getting to change that price is $(1-\phi)$, so the probability of a two period duration is $(1-\phi)\phi$. You can go on like this . . . the probability of a j period duration is $(1-\phi)\phi^{j-1}$. Hence, the expected duration condition on a price change today is:

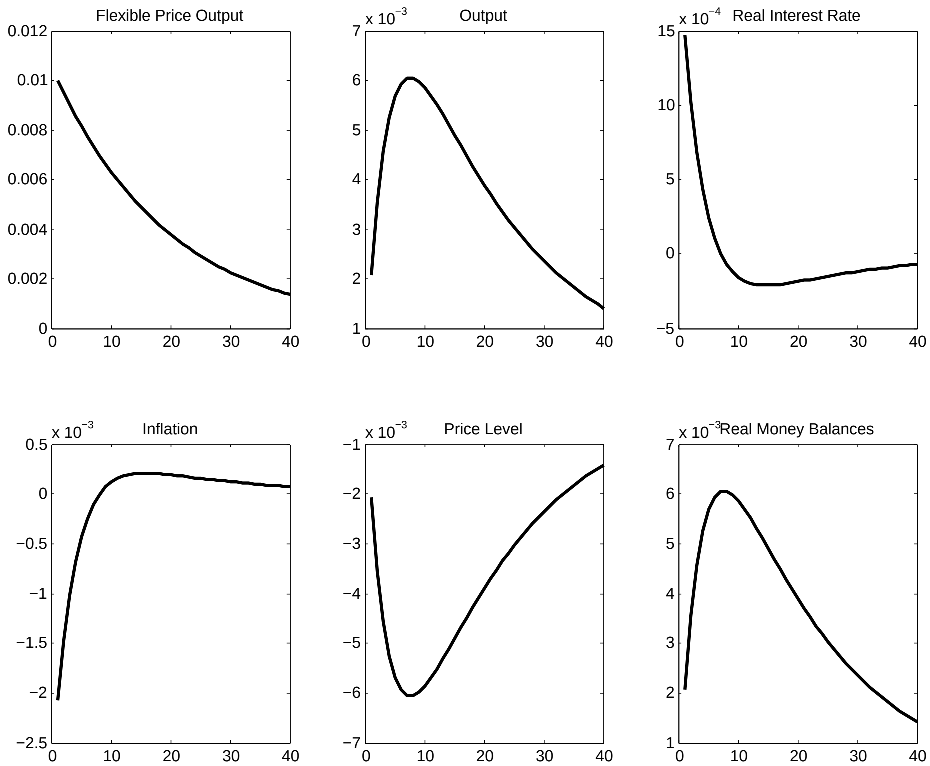
$$\text{Expected Duration} = (1-\phi)\sum_{j=1}^{\infty}\phi^{j-1}j$$

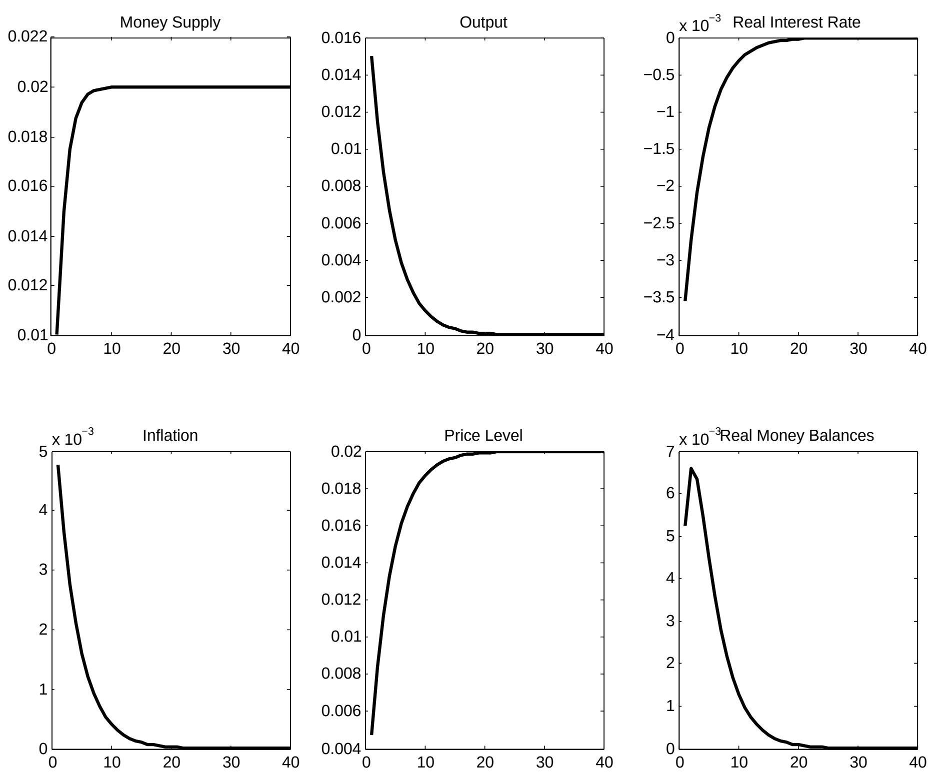
We can analytically work out the part inside the summation:

$$\begin{aligned}
S &= 1 + 2\phi + 3\phi^2 + 4\phi^3 + \ldots \\
S\phi &= \phi + 2\phi^2 + 3\phi^3 + 4\phi^4 + \ldots \\
(1-\phi)S &= 1 + \phi + \phi^2 + \phi^3 + \phi^4 + \ldots \\
\phi(1-\phi)S &= \phi + \phi^2 + \phi^3 + \phi^4 + \ldots \\
(1-\phi)S - \phi(1-\phi)S &= 1 \\
S &= \frac{1}{(1-\phi)^2}
\end{aligned}$$

Hence the overall expected duration between price changes is $\frac{1}{1-\phi}$. If it is observed that prices change on average once every five quarters, $\phi = 0.8$, for example. Let's go with that for now.

The impulse responses to a technology shock and a money supply shock are shown below:


Flexible Price Output
Output
Real Interest Rate
Inflation
Price Level
Real Money Balances


Money Supply
Output
Real Interest Rate
Inflation
Price Level
Real Money Balances

Let's first analyze the responses to the technology shock. We observe that output rises by much less on impact than does $\widetilde{y}_t^f$. This means that hours must decline in response to the positive technology shock. Because output is expected to rise, the real interest rate must initially rise. Because the output gap has declined, inflation falls. Real money balances rise as a result of falling inflation (even though the nominal money supply is constant). I impute the price level responses given the response of inflation. Since real money balances are stationary and nominal money hasn't changed, we know that the price level must return to its starting position.

Now let's look at what happens following an unexpected increase in money supply. Here we see that an increase in the money supply leads to an immediate reduction in the real interest rate and an increase in output. Inflation rises and so too does the price level. Here the price level response is permanent because the nominal money response is permanent (even though real balances are stationary). This looks kind of like how we think a monetary shock should work.

What is driving these results is $\phi > 0$ (i.e. price stickiness). From the Phillips Curve, if $\phi = 0$, then the coefficient on the output gap would be infinity, which would require the output gap to always be equal to zero (or $\widetilde{y}_t = \widetilde{y}_t^f$.). This would mean output would react by the same amount as the flexible price level to a technology shock (and hours would be constant), while output would not react to the monetary shock.

That's kind of a mechanical explanation. What's really going on? Recall the definition of the demand for intermediate good j:

$$y_{j,t} = \left(\frac{p_{j,t}}{p_t} \right)^{-\varepsilon} y_t$$

When a technology shock hits, those firms that can update their prices would like to lower them. But not all firms can update. Hence the aggregate price level falls, but some firms don't have their $p_{j,t}$ fall. This means that there is an undesired increase in the relative price of the goods of the non-adjusting price-setters, which works to lower their output relative to the case where all prices are flexible. Hence, output rises by less than it would in response to a technology shock.

Now consider an increase in the money supply. This makes those firms that can update raise their prices. But not all firms can update. These non-updating firms end up with relative prices that are too low; this over-stimulates demand for their goods, and makes overall output rise by more than it would if prices were flexible (which is to say prices wouldn't rise at all).

Hence, the New Keynesian model features distorted *relative* prices owing to price stickiness. This is what makes the dynamic responses of real variables to shocks different than if prices were flexible. The fact that some firms can't change their prices makes their relative price either too high or too low, and leads to different real effects of shocks. This conclusion forms the basic prescription for optimal monetary policy, which we will consider soon. With sticky prices optimal monetary policy should be concerned with price stability – if the aggregate price level never fluctuates, then firms don't have to change their prices, and the fact that prices are sticky becomes irrelevant.

6 A Taylor Rule Formulation

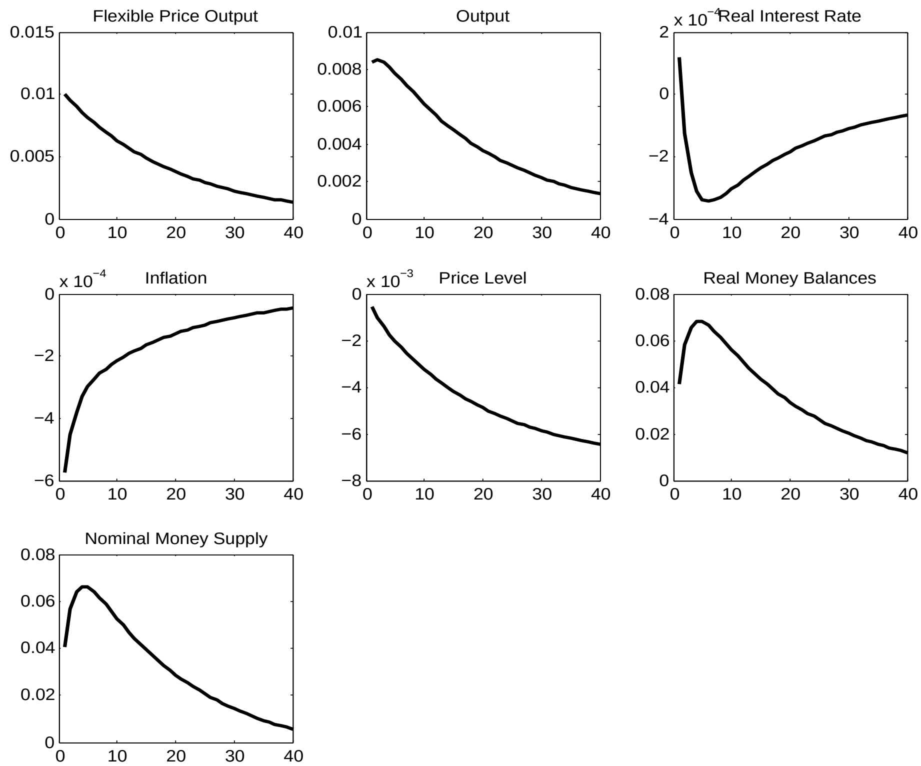
The money supply rule given above is considered somewhat archaic. In particular, most people (and indeed most central bankers) think of conducting monetary policy by setting nominal interest rates. We will use the term "Taylor rule" to generically refer to any nominal interest rate rule that features some feedback between economic variables and the nominal interest rate. In this sense the Taylor rule is nice because it (a) formulates policy in terms of interest rates, not money supply and (b) features an endogenous component of monetary policy.

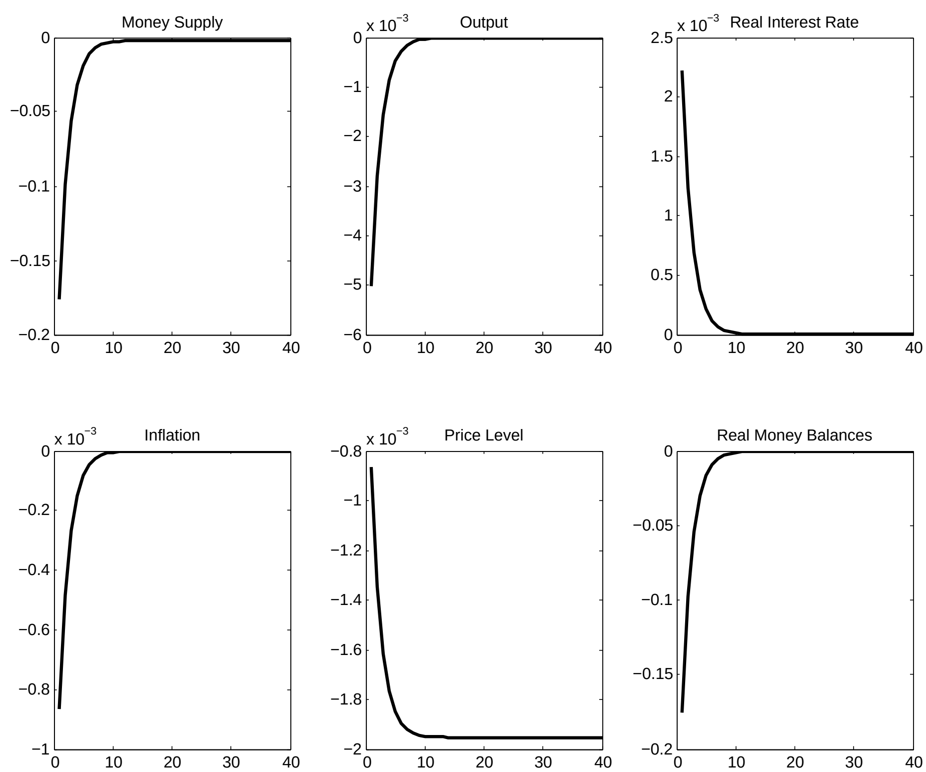
We will consider a partial adjustment version of John Taylor's (1992) celebrated rule as follows:

$$\widetilde{i}_t = \rho_m \widetilde{i}_{t-1} + (1-\rho_m)\phi_1 \widetilde{\pi}_t + (1-\rho_m)\phi_2 \left(\widetilde{y}_t - \widetilde{y}_t^f\right) + \varepsilon_{m,t} \tag{55}$$

Here I'm switching notation somewhat. Above $\widetilde{i}_{t+1}$ denotes the nominal interest from savings today that pays off tomorrow (and is hence known today). Here I'm calling this interest rate $\widetilde{i}_t$ instead (this is just more conventional . . . for the purposes of a Dynare code I need to re-write it this way anyway). ρ_m is the partial adjustment parameter, and ϕ_1 and ϕ_2 measure the response of nominal interest rates to inflation and output gaps. $\varepsilon_{m,t}$ is a monetary policy shock. We assume that the central bank raises nominal rates when inflation rises and also raises rates if the output gap turns positive. Loosely this means that the central bank wants to "fight" both inflation and output gaps (in either direction) by adjusting interest rates. The so-called "Taylor Principle" states that the Fed needs to raise nominal rates more than one for one with inflation so as to raise real rates (i.e. $\phi_1 > 1$), since it is real rates that matter for economic activity.

We can incorporate this Taylor rule into our quantitative model in place of the money supply rule. The money demand curve is still there; given the central bank's choice of the nominal interest rate, it must adjust the money supply so as to clear the money market. Let's consider the following parameterization of this rule: $\rho_m = 0.8$, $\phi_1 = 1.5$, and $\phi_2 = 0.5$. It is necessary for the coefficients of the Taylor rule to satisfy something akin to the Taylor Principle for determinacy of equilibrium; we will return to that issue later. Below are the impulse responses.


Flexible Price Output
Output
Real Interest Rate
Inflation
Price Level
Real Money Balances
Nominal Money Supply


Money Supply
Output
Real Interest Rate
Inflation
Price Level
Real Money Balances

For the impulse responses to the technology shock, I show the nominal money supply, since it actually does respond here. We observe that the Taylor rule involves the central bank increasing the nominal money supply in response to a technology shock. We also see that output rises by more on impact than it does under the money supply rule (this is in fact a result of the increase in the money supply). Output still does not rise by as much as $\widetilde{y}_t^f$ does, but it is much closer. Inflation and the aggregate price level fall by less here than they do in the money supply case; intuitively, with sticky prices, a smaller fall in the price level means that relative price distortions are smaller, and hence the real responses are closer to what would obtain without price stickiness.

The responses to the monetary shock look pretty reasonable. A "tightening" leads to a higher real interest rate, lower levels of economic activity, and falling prices.