The Simple New Keynesian Model

Graduate Macro II, Spring 2010
The University of Notre Dame
Professor Sims

1 Introduction

This document lays out the standard New Keynesian model based on Calvo (1983) staggered price-setting. The basic model is usually cast in a setting without physical capital, which means that there is no way in equilibrium to transfer resources across time (i.e. in equilibrium aggregate consumption is equal to output). Some argue that this isn't a problem, but I think it makes the model behave very differently. Aside from lacking physical capital, the model also differs from our benchmark in that it assumes imperfect competition (in particular monopolistic competition) on the firm side of the model. To think about price-stickiness you have to think about price-setting, and to think about price-setting you need some degree of pricing power. The household side of the model is basically identical to what we've seen before.

I'll begin with a model of imperfect competition with no price stickiness. Then we'll move to a model with price-stickiness.

2 The Model with No Price Stickiness

2.1 Households

The household side of the model is very standard and is similar to setups we have already seen. We assume that money enters the utility function in order to get households to hold money. The household's problem can be written as follows:

$$\max_{c_t,n_t,b_t,m_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}-1}{1-\sigma} + \theta \frac{(1-n_t)^{1-\xi}-1}{1-\xi} + \psi \frac{m_t^{1-v}-1}{1-v} \right)$$

s.t.

$$c_t + b_t + m_t \leq w_t n_t + \Pi_t + (1+i_{t-1}) \frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t}$$

The Lagrangian for the problem can be written:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{c_t^{1-\sigma}-1}{1-\sigma} + \theta \frac{(1-n_t)^{1-\xi}-1}{1-\xi} + \psi \frac{m_t^{1-v}-1}{1-v} + ... \\ ... + \lambda_t \left(w_t n_t + \Pi_t + (1+i_{t-1}) \frac{b_{t-1}}{1+\pi_t} + \frac{m_{t-1}}{1+\pi_t} - c_t - b_t - m_t \right) \end{array} \right\}$$

The first order conditions can be written:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= 0 \Leftrightarrow c_t^{-\sigma} = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial n_t} &= 0 \Leftrightarrow \theta(1-n_t)^{-\xi} = \lambda_t w_t \\
\frac{\partial \mathcal{L}}{\partial b_t} &= 0 \Leftrightarrow \lambda_t = \beta \lambda_{t+1} \frac{1+i_t}{1+\pi_{t+1}} \\
\frac{\partial \mathcal{L}}{\partial m_t} &= 0 \Leftrightarrow \beta^t \psi m_t^{-v} - \beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \frac{1}{1+\pi_{t+1}} = 0 \\
&\Rightarrow \psi m_t^{-v} = \lambda_t - \beta \lambda_{t+1} \frac{1}{1+\pi_{t+1}}
\end{aligned}
$$

We can simplify these using the Fisher relationship and simplifying:

$$
\begin{aligned}
(1-n_t)^{-\xi} &= c_t^{-\sigma} w_t \\
c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma}(1+r_t) \\
\psi m_t^{-v} &= c_t^{-\sigma} \left(\frac{i_t}{1+i_t} \right)
\end{aligned}
$$

2.2 Production

Production in these models is split into two stages – intermediate and final goods. The final goods production "technology" is simply a constant elasticity (CES) bundler of intermediate goods – there are no factors (i.e. labor) used to produce final goods. Profit maximization in the final goods sector (which is competitive) yields a downard sloping demand curve for intermediate goods producers, which gives them some pricing power. It is in the intermediate goods sector that we will assume some nominal rigidity (i.e. price-stickiness), which is in turn capable of generating meaningful non-neutralities.

What differentiates monopolistic competition from perfect competition is that a large number firms sell differentiated products and have some pricing power. Because of entry and exit, they earn no economic profits in the long run, however.

2.2.1 Final Goods

There is one final goods firm and a continuum (i.e. infinity) of intermediate goods firms. These firms are indexed along the unit interval. The "production function" for the final good is:

$$
y_t = \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}
$$

We require that $\varepsilon > 1$; this is the elasticity of substitution among the different intermediate goods. As long as $\varepsilon < \infty$, the intermediate goods are imperfect substitutes in consumption;

this is what gives them market power. Note that an integral is really just a sum – we're just taking a weighted sum of intermediate goods and then raising them all to a power. It is straightforward to verify that this "production function" has constant returns to scale – if you double all intermediate inputs, you double output.

The final goods firm wants to maximize profits (more generally they'd want to maximize the present discounted value of profits, but there is nothing that makes the problem interesting in a dynamic sense as they just buy the intermediate goods period by period, so maximizing value is equivalent to maximizing profits period by period). The objective function is written in nominal terms is:

$$\max_{y_t(j)} \quad p_t y_t - \int_0^1 p_t(j) y_t(j) dj$$

Total revenue is the final goods price times the amount of final good. Total cost is the sum over all intermediate goods of the price times quantity. Plug in the production function:

$$\max_{y_t(j)} \quad p_t \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 p_t(j) y_t(j) dj$$

The first order conditions come from taking the derivative with respect to each $y_t(j)$ and setting it equal to zero. Remember to treat the integral just like a sum in taking the derivatives.

$$p_t \frac{\varepsilon}{\varepsilon - 1} \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon - 1}{\varepsilon} y_t(j)^{\frac{\varepsilon-1}{\varepsilon}-1} = p_t(j) \qquad \forall \;\; j$$

Now play around with this and simplify and solve for the demand for each intermediate good:

$$
\begin{aligned}
p_t \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1} - \frac{\varepsilon-1}{\varepsilon-1}} y_t(j)^{\frac{\varepsilon-1}{\varepsilon} - \frac{\varepsilon}{\varepsilon}} &= p_t(j) \quad \forall \; j \\
p_t \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{1}{\varepsilon-1}} y_t(j)^{-\frac{1}{\varepsilon}} &= p_t(j) \quad \forall \; j \\
y_t(j)^{-\frac{1}{\varepsilon}} &= \left(\frac{p_t(j)}{p_t} \right) \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{-\frac{1}{\varepsilon-1}} \\
y_t(j) &= \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}} \\
y_t(j) &= \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t
\end{aligned}
$$

The last line follows from the definition of the final goods CES aggregator. This says that the demand for each intermediate good depends negatively on its relative price and positively on total production. We can interpret ε as the elasticity of demand – the requirement that $\varepsilon > 1$ is just say that monopolists produce on the elastic portion of the demand curve. As $\varepsilon \to \infty$ demand becomes perfectly elastic (equivalently, the intermediate goods are perfect substitutes), which will end up putting us back in the case of perfect competition.

Since the final good firm is competitive, profits are zero, which implies that:

$$
p_t y_t = \int_0^1 p_t(j) y_t(j) dj
$$

Plug in the demand functions for intermediate goods and solve for the price of the final good:

$$
p_t y_t = \int_0^1 p_t(j) \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t dj
$$

We can "take out of the integral" (i.e. sum) the variables not indexed by j on the right hand side, leaving:

$$
\begin{aligned}
p_t y_t &= p_t^{\varepsilon} y_t \int_0^1 p_t(j)^{1-\varepsilon} dj \\
p_t^{1-\varepsilon} &= \int_0^1 p_t(j)^{1-\varepsilon} dj \\
p_t &= \left[\int_0^1 p_t(j)^{1-\varepsilon} dj \right]^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

This can be thought of as the aggregate price index.

2.2.2 Intermediate Goods

Intermediate goods (remember, there an infinite number of them populated along the unit interval) produce output using a production function using labor and TFP. The level of TFP is common to all of them. Assume that this production function takes the form:

$$y_t(j) = a_t n_t(j)$$

Hence, production is linear in labor given TFP.

The typical intermediate goods firm optimizes along two dimensions – it must choose its employment and its price. We consider these problems sequentially.

Intermediate goods firms are price takers in factor markets (i.e. they take the wage as given). The market structure requires them to produce as much output as is demanded at a given price (they will be willing to do this since price, as we will show, will be above marginal cost). Nothing makes the value of the firm explicitly time dependent (i.e. firm's don't have factor attachment), so maximizing value is equivalent to maximizing profits period by period, which is in turn equivalent to minimizing costs period by period. It is easiest to think about the choice over labor as a cost minimization problem as follows:

$$\min_{n_t(j)} \quad W_t n_t(j)$$

s.t.

$$
\begin{aligned}
y_t(j) &\geq \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t \\
y_t(j) &= a_t n_t(j)
\end{aligned}
$$

Here, W_t is the nominal wage, which is common to all firms since they are competitive in factor markets. Profits are maximized when costs are minimized subject to two constraints – production is at least as much as demand and production is governed by the production "technology" given above. Minimzing a function is the same as maximizing the negative of the same function, so we can write the problem out as a standard Lagrangian:

$$\mathcal{L} = -W_t n_t(j) + \varphi_t \left(a_t n_t(j) - \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t \right)$$

The first order condition is:

$$W_t = \varphi_t a_t$$

The Lagrange multiplier, φ_t, has the interpretation of nominal marginal cost – how much nominal costs change (the objective function) if the constraint is relaxed (i.e. if the firm has to produce one more unit of its good). note that marginal cost is not indexed by j – constant returns to scale plus competitive factor markets insure that marginal cost is the same for all firms. Divide both sides of this expression by the aggregate price level (this puts this in terms of the consumption wage, which is what households care about). This leaves a relationship between the real wage, real marginal cost, and the marginal product of labor.

$$w_t = \frac{\varphi_t}{p_t} a_t$$

Here $w_t \equiv \frac{W_t}{p_t}$; i.e. the real wage. If markets were perfectly competitive, price would always be equal to marginal cost, and so real marginal cost would always be one, and the labor demand condition would be the familiar wage equals marginal product. More generally, real marginal cost will equal the real wage divided by the marginal product of labor.

Now consider the choice of the optimal price conditional on the optimal choice of labor. Again, since the firm can choose its price each and every period, we can write this as a static problem.

$$\max_{p_t(j)} \quad p_t(j) y_t(j) - W_t n_t$$

s.t.

$$\begin{aligned} y_t(j) &= \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t \\ W_t &= \varphi_t a_t \end{aligned}$$

In other words, the optimization is done subject to the demand function and the requirement that labor is chosen optimally. Techinically, we should be maximizing real profits, which would entail dividng by the aggregate price level, but given the static nature of the problem, doing so would not affect the optimal decision rule. We can plug these constraints in to write the problem as:

$$\max_{p_t(j)} \quad p_t(j) \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t - \varphi_t a_t n_t = p_t(j) \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t - \varphi_t \left(\frac{p_t(j)}{p_t} \right)^{-\varepsilon} y_t$$

Since the firm is small (i.e. there are an infinite number of them), it takes aggregate output, y_t, and the aggregate price level as given. Take the FOC:

$$(1-\varepsilon)p_t(j)^{-\varepsilon}p_t^{-\varepsilon}y_t + \varepsilon\varphi_t p_t(j)^{-\varepsilon-1}p_t^{-\varepsilon}y_t = 0$$

Simplifying:

$$\begin{aligned}(\varepsilon-1)p_t(j)^{-\varepsilon} &= \varepsilon\varphi_t p_t(j)^{-\varepsilon-1} \\ p_t(j) &= \frac{\varepsilon}{\varepsilon-1}\varphi_t\end{aligned}$$

Since $\varepsilon > 1$, $\frac{\varepsilon}{\varepsilon-1} > 1$. This means that the optimal price is a markup over marginal cost (i.e. price exceeds marginal cost). The extent of the markup depends on how "steep" the firm's demand curve is. As $\varepsilon \to \infty$, the firm faces a horizontal demand curve, $\frac{\varepsilon}{\varepsilon-1} \to 1$, and price is equal to marginal cost, and we're back in the perfectly competitive case.

2.3 Aggregation

We will restrict attention to a situation in which all firms behave identically (i.e. a "symmetric equilibrium"). This is not without loss of generality. Since firms operate in competitive factor markets, they all have the same marginal cost of production, φ_t. Since they all face the same demand elasticity, from above, we see that they will all choose the same price. But if they all choose the same price, they face the exact same demand. This in turn means that they will each produce an equal amount and will hire an equal amount of labor (since they all face the same aggregate TFP). Starting with the aggregate production function, we have:

$$y_t = \left[\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

Let $y_t(j)$ be the amount of output produced by the typical intermediate goods firm. Since it's the same across all j, we have can take it out of the integral and get:

$$y_t = y_t(j)\left[\int_0^1 dj\right]^{\frac{\varepsilon}{\varepsilon-1}} = y_t(j)$$

In other words, output of the final good is equal to output of the intermediate goods (or, more correctly, the production of the final good is equal to the sum of production of intermediate goods in the symetric equilibrium . . . since we are summing across the unit interval, the sum is equal to the amount produced by any one firm on the unit interval). Taking note of this fact, and using the intermediate goods production function, we have

$$y_t = y_t(j) = a_t n_t(j)$$

Note also that, since we're integrating over the unit interval and every firm produces the same amount, $y_t(j) = \int_0^1 y_t(j)dj$. Hence we can apply an integral above and get:

$$y_t = \int_0^1 a_t n_t(j)dj = a_t \int_0^1 n_t(j)dj = a_t n_t$$

This follows from the fact that employment supplied by the houshold is split amount the firms along the unit interval (i.e. $n_t = \int_0^1 n_t(j)dj$).

Since all intermediate goods firms are behaving the same, we get the same result that the aggregate price level is equal to the price level of the intermediate goods firm:

$$p_t = p_t(j)$$

From above, we know what each firm's price will be:

$$p_t(j) = \frac{\varepsilon}{\varepsilon - 1}\varphi_t$$

The labor demand condition for each intermediate goods firm is:

$$W_t = \varphi_t a_t$$

Divide both sides by the price level:

$$w_t = \frac{\varphi_t}{p_t} a_t$$

Now use the pricing condition:

$$w_t = \frac{\varepsilon - 1}{\varepsilon} a_t$$

$\frac{\varepsilon-1}{\varepsilon} < 1$, so the real wage is less than the marginal product.

We can summarize the entire model with the following equations:

$$c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma}(1 + r_t) \tag{1}$$

$$c_t = y_t \tag{2}$$

$$y_t = a_t n_t \tag{3}$$

$$\theta(1 - n_t)^{-\xi} = c_t^{-\sigma} w_t \tag{4}$$

$$w_t = \frac{\varepsilon - 1}{\varepsilon} a_t \tag{5}$$

$$\psi m_t^{-v} = c_t^{-\sigma}\left(\frac{i_t}{1+i_t}\right) \tag{6}$$

$$1 + r_t = \frac{1+i_t}{1+\pi_{t+1}} \tag{7}$$

$$dm_t + \pi_t = (1-\rho_m)\pi^* + \rho_m dm_{t-1} + \rho_m \pi_{t-1} + e_{m,t} \tag{8}$$

$$dm_t = \ln m_t - \ln m_{t-1} \tag{9}$$

$$\ln a_t = \rho \ln a_{t-1} + e_{a,t} \tag{10}$$

(1) is the Euler equation; (2) is the aggregate accounting identity; (3) is the production function; (4) is labor supply; (5) is labor demand; (6) is demand for real balances; (7) is the Fisher relationship; (8) is the exogenous process governing the growth rate of real balances; (9) defines the growth rate of real balances; and (10) is the familiar process for log technology.

3 The Model with Calvo Price Stickiness

Above firms could change their prices each period; each period, they would set prices as a constant markup over marginal cost, with the size of the markup related to the slope of the demand curve for their good. Now we assume that firms cannot change their prices freely each period. In particular, firms face a constant probability, $1-\phi$, of being able to adjust their price in any period. This hazard rate is constant across time.

The household side of the model is identical to above; the final goods production is also identical to above. The pricing decision is simlar but cannot be undertaken every period. Let's consider a firm who, at time t, is given the ability to adjust its price. It will do so to maximize the expected discounted value of profits, since it will, in expectation, be stuck with this price for more than just the current period. The firm discounts future profits by the gross real interest rate between today and future dates . . . i.e. $(1+r_{t,t+s})^{-1}$ for $s = 0, ...\infty$. From the households Euler equation, we can solve for this "long" real interest rate as:

$$(1+r_{t,t+s})^{-1} = \beta^s E_t \left(\frac{c_{t+s}}{c_t}\right)^{-\sigma}$$

This is often called the stochastic discount factor and is frequently used in the asset pricing literature. In addition, the firm will also discount future profit flows by the probability that it will be stuck with the price it chooses today. This probability is ϕ. If ϕ is small, then the firms get to update their prices frequently, and thus will heavily discount future profit flows when making current pricing decisions. On the other hand, if ϕ is large, it is very likely that a firm will be "stuck" with whatever price it chooses today for a long time, and will thus be relatively more concerned about the future when making its current pricing decisions.

Similarly to above but now taking account of the possibility of being stuck with a price, we can write the firm with the opportunity to change its price solves the following problem:

$$\max_{p_t(j)} \quad E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(\frac{p_t(j)}{p_{t+s}} \left(\frac{p_t(j)}{p_{t+s}} \right)^{-\varepsilon} y_{t+s} - \frac{\varphi_{t+s}}{p_{t+s}} \left(\frac{p_t(j)}{p_{t+s}} \right)^{-\varepsilon} y_{t+s} \right)$$

Here the problem is written as maximizing real profits discounted by the stochastic discount factor as well as the probability of being able to make price changes. For simplicity, I write $\Lambda_{t,t+s} = \left(\frac{c_{t+s}}{c_t}\right)^{-\sigma}$ – i.e. the ratio of marginal utility between period $t+s$ and period t. When $\phi = 0$, so that there is no price stickiness, it is straightforward to verify that the problem reduces to what we had above (because $(\phi\beta)^s = 0$ for every $s > 0$, so only current profits will factor into the pricing decision. Note that the firm's price isn't indexed by s, since it is choosing a price today that it won't be able to change in the future. The first order condition for this problem is:

$$E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left((1-\varepsilon) p_t(j)^{-\varepsilon} p_{t+s}^{-(1-\varepsilon)} y_{t+s} + \varepsilon p_t(j)^{-\varepsilon-1} \varphi_{t+s} p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right) = 0$$

Let's simplify:

$$E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left((\varepsilon-1) p_t(j)^{-\varepsilon} p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right) = E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(\varepsilon p_t(j)^{-\varepsilon-1} \varphi_{t+s} p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right)$$

Since the price they choose does not depend upon s, we can pull it out of the sums:

$$(\varepsilon-1) p_t(j)^{-\varepsilon} E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right) = \varepsilon p_t(j)^{-\varepsilon-1} E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(\varphi_{t+s} p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right)$$

Simplify:

$$p_t^{\#} = \frac{\varepsilon}{\varepsilon-1} \frac{E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(\varphi_{t+s} p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right)}{E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(p_{t+s}^{-(1-\varepsilon)} y_{t+s} \right)}$$

Above, I replace the $p_t(j)$ with $p_t^{\#}$, which is called the optimal reset price. Since firms face the same marginal cost and take aggregate variables as given, any firm that gets to update its price will choose the same price. Essentially, the current price that price-changing firms will choose is a present discount value of marginal costs. As noted, if there is no price-stickiness, so that $\phi = 0$, then the solution is the same as above, with $p_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\varphi_t$.

For ease of notation, let's write this expression as:

$$
\begin{aligned}
p_t^{\#} &= \frac{\varepsilon}{\varepsilon - 1}\frac{A_t}{B_t} \\
A_t &= E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(\varphi_{t+s} p_{t+s}^{-(1-\varepsilon)} y_{t+s}\right) \\
B_t &= E_t \sum_{s=0}^{\infty} (\phi\beta)^s \Lambda_{t,t+s} \left(p_{t+s}^{-(1-\varepsilon)} y_{t+s}\right)
\end{aligned}
$$

Now, when we go to the computer to solve this, the computer isn't going to like an infinite sum. Fortunately, we can write the expression for A_t and B_t as follows:

$$
\begin{aligned}
A_t &= \varphi_t p_t^{-(1-\varepsilon)} y_t + \phi\beta\Lambda_{t,t+1} E_t A_{t+1} \\
B_t &= p_t^{-(1-\varepsilon)} y_t + \phi\beta\Lambda_{t,t+1} E_t B_{t+1}
\end{aligned}
$$

Recall the definition of the aggregate price level:

$$
p_t = \left[\int_0^1 p_t(j)^{1-\varepsilon} dj\right]^{\frac{1}{1-\varepsilon}}
$$

We can split this intergral into a convex combination of two things – the optimal reset price and the previous price. This is because all firms that can reset will choose the same reset price, and the "average" price of the firms that cannot reset will equal the previous aggregate price level:

$$
\begin{aligned}
p_t &= \left[\int_0^1 \left((1-\phi) p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon}\right) dj\right]^{\frac{1}{1-\varepsilon}} \\
p_t &= \left[\int_0^{1-\phi} p_t^{\#1-\varepsilon} dj + \int_{1-\phi}^{1} p_{t-1}^{1-\varepsilon} dj\right]^{\frac{1}{1-\varepsilon}} \\
p_t &= \left[(1-\phi) p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

As a general matter we want to allow for the existence of steady state inflation (though in most linearizations it is assumed that there is zero steady state inflation), so we need to write this such that there is a well-defined steady state. To do this divide both sides by p_{t-1}:

$$
\begin{aligned}
\frac{p_t}{p_{t-1}} &= p_{t-1}^{-1}\left[(1-\phi)p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \\
\frac{p_t}{p_{t-1}} &= \left[p_{t-1}^{-(1-\varepsilon)}((1-\phi)p_t^{\#1-\varepsilon} + \phi p_{t-1}^{1-\varepsilon})\right]^{\frac{1}{1-\varepsilon}} \\
\frac{p_t}{p_{t-1}} &= \left[(1-\phi)\left(\frac{p_t^{\#}}{p_{t-1}}\right)^{1-\varepsilon} + \phi\left(\frac{p_{t-1}}{p_{t-1}}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

Defining $1+\pi_t = \frac{p_t}{p_{t-1}}$, we can write this:

$$
1+\pi_t = \left[(1-\phi)\left(\frac{p_t^{\#}}{p_{t-1}}\right)^{1-\varepsilon} + \phi\right]^{\frac{1}{1-\varepsilon}}
$$

Thus, to get an expression for current inflation, we need to find an expression for "reset price inflation", which I'll call $\frac{p_t^{\#}}{p_{t-1}}$. Go back to the expression for the rest price:

$$
p_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\frac{A_t}{B_t}
$$

Divide both sides by p_{t-1}:

$$
\frac{p_t^{\#}}{p_{t-1}} = \frac{\varepsilon}{\varepsilon-1}\frac{1}{p_{t-1}}\frac{A_t}{B_t}
$$

Let's deal with this part by part. Note that:

$$
\begin{aligned}
\frac{A_t}{p_{t-1}} &= \frac{1}{p_{t-1}}\left(\varphi_t p_t^{-(1-\varepsilon)} y_t + \phi\beta\Lambda_{t,t+1}E_t A_{t+1}\right) \\
\frac{A_t}{p_{t-1}} &= \frac{\varphi_t p_t^{-(1-\varepsilon)} y_t}{p_{t-1}} + \frac{\phi\beta\Lambda_{t,t+1}E_t A_{t+1}}{p_{t-1}}
\end{aligned}
$$

Defining $mc_t = \frac{\varphi_t}{p_t}$ as real marginal cost, we can write this as:

$$
\frac{A_t}{p_{t-1}} = mc_t\left(\frac{p_t}{p_{t-1}}\right)p_t^{-(1-\varepsilon)} y_t + \frac{\phi\beta\Lambda_{t,t+1}E_t A_{t+1}}{p_{t-1}}
$$

We need to play around further with the dates on the very end of the expression on the right hand side:

$$
\frac{A_t}{p_{t-1}} = mc_t\left(\frac{p_t}{p_{t-1}}\right)p_t^{-(1-\varepsilon)} y_t + \phi\beta\Lambda_{t,t+1}\left(\frac{p_t}{p_{t-1}}\right)E_t\frac{A_{t+1}}{p_t}
$$

To save on notation, let's go ahead and call $\frac{A_t}{p_{t-1}} = \widehat{A}_t$. Thus, we can write this as:

$$
\widehat{A}_t = (1+\pi_t)\left(mc_t p_t^{-(1-\varepsilon)} y_t + \phi\beta\Lambda_{t,t+1}\widehat{A}_{t+1}\right)
$$

Given this, we can write reset price inflation as:

$$\frac{p_t^{\#}}{p_{t-1}} = \frac{\varepsilon}{\varepsilon-1}\frac{\widehat{A}_t}{B_t}$$

Now, we're not yet done because both $\widehat{A}_t$ and B_t have a $p_t^{-(1-\varepsilon)}$ component in them. Fortunately, we can divide both numerator and denominator by $p_t^{-(1-\varepsilon)}$ without changing the equality. Define $\widehat{a}_t = \widehat{A}_t/p_t^{-(1-\varepsilon)}$ and $\widehat{b}_t = B_t/p_t^{-(1-\varepsilon)}$:

$$\frac{p_t^{\#}}{p_{t-1}} = \frac{\varepsilon}{\varepsilon-1}\frac{\widehat{a}_t}{\widehat{b}_t}$$

Now we need to find expression for $\widehat{a}_t$ and $\widehat{b}_t$:

$$\begin{aligned}
\widehat{a}_t &= \frac{\widehat{A}_t}{p_t^{-(1-\varepsilon)}} = (1+\pi_t)\left(mc_t y_t + E_t\phi\beta\Lambda_{t,t+1}\frac{\widehat{A}_{t+1}}{p_t^{-(1-\varepsilon)}}\right) \\
\widehat{a}_t &= \frac{\widehat{A}_t}{p_t^{-(1-\varepsilon)}} = (1+\pi_t)\left(mc_t y_t + E_t\phi\beta\Lambda_{t,t+1}\left(\frac{p_{t+1}}{p_t}\right)^{-(1-\varepsilon)}\frac{\widehat{A}_{t+1}}{p_{t+1}^{-(1-\varepsilon)}}\right) \\
\widehat{a}_t &= (1+\pi_t)\left(mc_t y_t + E_t\phi\beta\Lambda_{t,t+1}\left(1+\pi_{t+1}\right)^{-(1-\varepsilon)}\widehat{a}_{t+1}\right)
\end{aligned}$$

$$\begin{aligned}
\widehat{b}_t &= \frac{B_t}{p_t^{-(1-\varepsilon)}} = \frac{1}{p_t^{-(1-\varepsilon)}}\left(p_t^{-(1-\varepsilon)}y_t + E_t\phi\beta\Lambda_{t,t+1}B_{t+1}\right) \\
\widehat{b}_t &= y_t + E_t\phi\beta\Lambda_{t,t+1}\frac{B_{t+1}}{p_t^{-(1-\varepsilon)}} \\
\widehat{b}_t &= y_t + E_t\phi\beta\Lambda_{t,t+1}\left(\frac{p_{t+1}}{p_t}\right)^{-(1-\varepsilon)}\frac{B_{t+1}}{p_{t+1}^{-(1-\varepsilon)}} \\
\widehat{b}_t &= y_t + E_t\phi\beta\Lambda_{t,t+1}\left(1+\pi_{t+1}\right)^{-(1-\varepsilon)}\widehat{b}_{t+1}
\end{aligned}$$

A small technical point is that, for this "trick" to work (i.e. writting $\widehat{a}_t$ and $\widehat{b}_t$ not as infininite sums but rather as as current plus "continuation values") it must be the case that the effective discount factor be less than one in the steady state. Since $\Lambda^* = 1$, this means that $\phi\beta\left(1+\pi^*\right)^{-(1-\varepsilon)} < 1$. $\phi\beta < 1$, so if steady state inflation is zero, this is never an issue. But if steady state inflation is very high, or ε is very large, then this may not hold. Given the auxilliary variables $\widehat{a}_t$ and $\widehat{b}_t$, which, subject to the caveat above, have been rendered stationary, we can write actual price inflation as:

$$\begin{aligned}
1+\pi_t &= \left[(1-\phi)\left(\frac{p_t^{\#}}{p_{t-1}}\right)^{1-\varepsilon} + \phi\right]^{\frac{1}{1-\varepsilon}} \\
\frac{p_t^{\#}}{p_{t-1}} &= \frac{\varepsilon}{\varepsilon-1}\frac{\widehat{a}_t}{\widehat{b}_t}
\end{aligned}$$

Given this, we can write down the equations characterizing equilibrium of the model with price stickiness as follows:

$$c_t^{-\sigma} = \beta E_t c_{t+1}^{-\sigma}(1+r_t) \tag{11}$$

$$c_t = y_t \tag{12}$$

$$y_t = a_t n_t \tag{13}$$

$$\theta(1-n_t)^{-\xi} = c_t^{-\sigma} w_t \tag{14}$$

$$w_t = mc_t a_t \tag{15}$$

$$1+\pi_t = \left[(1-\phi)\left(1+\pi_t^{\#}\right)^{1-\varepsilon} + \phi\right]^{\frac{1}{1-\varepsilon}} \tag{16}$$

$$1+\pi_t^{\#} = \frac{\varepsilon}{\varepsilon-1}\frac{\widehat{a}_t}{\widehat{b}_t} \tag{17}$$

$$\widehat{a}_t = (1+\pi_t)\left(mc_t y_t + E_t \phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{a}_{t+1}\right) \tag{18}$$

$$\widehat{b}_t = y_t + E_t\phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{b}_{t+1} \tag{19}$$

$$\Lambda_{t,t+1} = \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \tag{20}$$

$$\psi m_t^{-\upsilon} = c_t^{-\sigma}\left(\frac{i_t}{1+i_t}\right) \tag{21}$$

$$1+r_t = \frac{1+i_t}{1+\pi_{t+1}} \tag{22}$$

$$dm_t + \pi_t = (1-\rho_m)\pi^* + \rho_m dm_{t-1} + \rho_m \pi_{t-1} + e_{m,t} \tag{23}$$

$$dm_t = \ln m_t - \ln m_{t-1} \tag{24}$$

$$\ln a_t = \rho \ln a_{t-1} + e_t \tag{25}$$

Note that I have fifteen equations and fifteen variables. Some of these (in fact many of these) variables can be eliminated from the solution.

I calibrate the parameters of the model as follows:

Parameter	Value
β	0.99
σ	1
ξ	1
υ	1
θ	3.5
ϕ	0.75
ρ	0.9
ρ_m	0.5
ψ	1
ε	11
π^*	0.01
σ_e	0.007
σ_{em}	0.002

How can these parameters be interpreted? The discount factor of 0.99 implies a steady state real interest rate of about one percent (or about four percent expressed at an annual frequency). Coupled with steady state inflation of 0.01, this means that the steady state nominal interest rate is about 0.02. The power coefficients in preferences being all equal to one means that the within period utility function is "log-log-log". $\theta = 3.5$ means that steady state hours per capita will be roughly 0.2. The shock standard deviations and autoregressive coefficients in the technology and money growth specifications are similar to what we've been using.

The two new parameters here that need some discussion are ϕ, which governs price-stickiness and is often called the "Calvo parameter", and ε, which controls market power. ε is easier to deal with, so we begin there. Recall from our derivation that the steady state (or average) markup of price over marginal cost is equal to $\frac{\varepsilon}{\varepsilon-1}$. In the data, average markups appear to be about 10% (Basu and Fernald (1997)). This means that $\frac{\varepsilon}{\varepsilon-1} = 1.1$, or $\varepsilon = 11$.

The Calvo parameter will govern the average duration between price changes. Conditional on changing a price in the current period, what is the expected duration until your next price change? Well, the probability of getting to change prices next period is $1 - \phi$. The probability of getting to change prices in two periods is $1 - \phi$ times the probability of not changing prices after one period, or $(1 - \phi)\phi$. The probability of getting to change prices in three periods is $1 - \phi$ times the probability of not getting to change prices for two consecutive periods, or $(1 - \phi)\phi^2$. More compactly:

Duration	Probability
1	$1 - \phi$
2	$(1 - \phi)\phi$
3	$(1 - \phi)\phi^2$
4	$(1 - \phi)\phi^3$
$\vdots$	$\vdots$
j	$(1 - \phi)\phi^{j-1}$

The expected duration between price changes is then just the sum of probabilities times duration:

$$\begin{aligned}\text{Expected Duration between Price Changes} &= \sum_{j=1}^{\infty}(1-\phi)\phi^{j-1}j \\ &= (1-\phi)\sum_{j=1}^{\infty}\phi^{j-1}j\end{aligned}$$

We can write the part inside the summation as:

$$S = 1 + 2\phi + 3\phi^2 + 4\phi^3 + 5\phi^4 + ... = \sum_{j=1}^{\infty}\phi^{j-1}j$$

Multiply everything by ϕ:

$$S\phi = \phi + 2\phi^2 + 3\phi^3 + 4\phi^4 + 5\phi^5 + ...$$

Subtract the former from the latter:

$$\begin{aligned}S - S\phi &= 1 + (2-1)\phi + (3-2)\phi^2 + (4-3)\phi^3 + (5-4)\phi^4 + ... \\ (1-\phi)S &= 1 + \phi + \phi^2 + \phi^3 + \phi^4 + ...\end{aligned}$$

Now multiply this expression by ϕ:

$$(1-\phi)\phi S = \phi + \phi^2 + \phi^3 + \phi^4 + ...$$

Now subtract this from the former:

$$(1-\phi)S - (1-\phi)\phi S = 1$$

This follows from the fact that, as $j \to \infty$, $\phi^{j+1} = \phi^j = 0$. Simplifying:

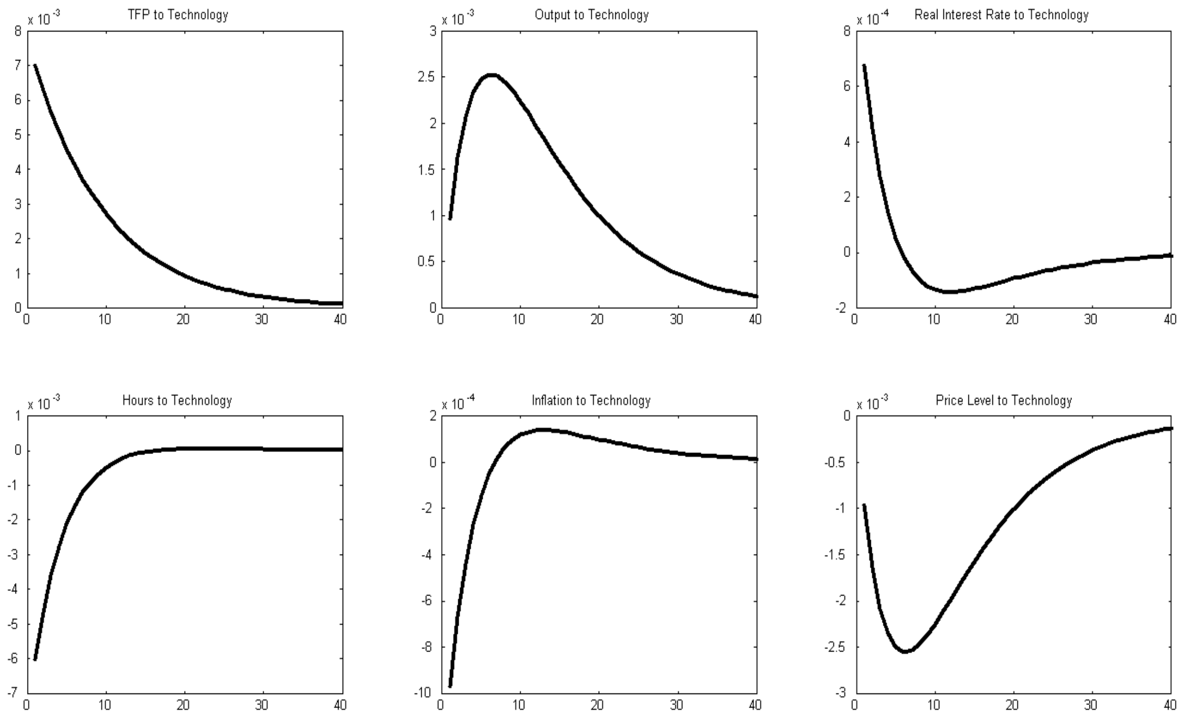
$$\begin{aligned}(1-\phi)^2 S &= 1 \\ S &= \frac{1}{(1-\phi)^2}\end{aligned}$$

Now plugging this back in to the original expression, we have:

$$\text{Expected Duration between Price Changes} = (1-\phi)\frac{1}{(1-\phi)^2} = \frac{1}{(1-\phi)}$$

Thus, we can calibrate ϕ by looking at data on the average duration between price changes. Bils and Klenow (2004) find that it's between 6 months and one year. We'll go with the long end of that range (four quarters), which suggests that $\phi = 0.75$.

Below are impulse responses to technology shock:



We see that a technology shock leads to an increase in output and the real interest rate on impact, with decreases in inflation, hours, and the price level. The fall in hours may seem non-intuitive at first. To see why hours fall, look at the money demand specification:

$$\psi m_t^{-v} = c_t^{-\sigma}\left(\frac{i_t}{1+i_t}\right)$$

Rewrite this in terms of the nominal money supply, the price level, and output (since consumption is equal to output in equilibrium):

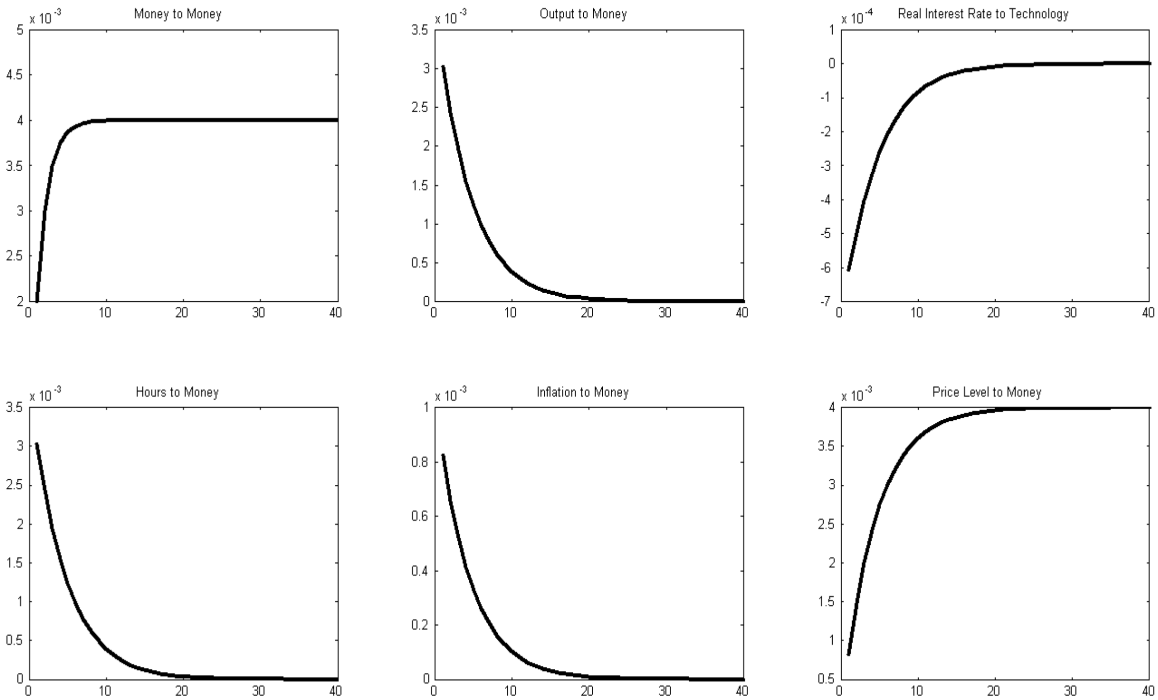
$$\left(\frac{M_t}{p_t}\right) = y_t^{\frac{\sigma}{v}}\psi^{\frac{1}{v}}\left(\frac{1+i_t}{i_t}\right)^{\frac{1}{v}}$$

To make this as simple to see as possible, suppose that both v and σ are very big, so that $\frac{\sigma}{v} \approx 1$ and $\frac{1}{v} \approx 0$. Then we recover exactly the simple quantity equation:

$$M_t = p_t y_t$$

If prices were fully flexible, when technology increases prices would fall by the amount of the increase in output. But because we have here assumed price stickiness, prices cannot fall by that much, so output cannot rise by as much as it would if prices were fully flexible. This means that hours cannot rises by as much as they do when prices are fully flexible; since in the way I wrote down preferences hours actually do not respond at all to a technology shock when prices are perfectly flexible, this necessitates a decrease in hours on impact.

Next, consider the responses to a money growth shock.



These responses look reasonably intuitive. An increase in money growth raises output, inflation, and the price level, while lowering nominal and real interest rates. The intuition for why this happens can be gained from the quantity theoretic equation above as well. The price level cannot adjust upward the same amount it would if prices were flexible when the money supply increases – therefore, output must rise to make the money market clear. Note that the Matlab filed used to produce these figures is titled nk_basic_notzero.mod and can be run from new_keynesian.m.

3.0.1 Log-Linearizing

Suppose that we want to log-linearize this expression about a steady state. The conventional linearization is about the zero inflation steady state, so that $\pi^* = 0$. AS short hand, let's call $\frac{p_t^{\#}}{p_{t-1}} = 1 + \pi_t^{\#}$. Log-linearize the inflation equation by first taking logs of both sides:

$$\ln(1+\pi_t) = \frac{1}{1-\varepsilon}\ln\left[(1-\phi)\left(1+\pi_t^{\#}\right)^{1-\varepsilon} + \phi\right]$$

Now do the Taylor series expansion about the point $\pi^* = 0$, which will mean $\pi^{*\#} = 0$ as well:

$$\ln(1+0) + \frac{d\pi_t}{1+0} = \frac{1}{1-\varepsilon}\ln(1) + \frac{1}{1-\varepsilon}\frac{1}{1}(1-\varepsilon)(1-\phi)(1+0)^{-\varepsilon}d\pi_t^{\#}$$

Some of this follows from the fact that $(1-\phi)(1+0)^{1-\varepsilon} + \phi = 1$. Simplify, we have:

$$d\pi_t = (1-\phi)d\pi_t^{\#}$$

Since inflation is already in a percentage rate, we want to leave it as an absolute rather than percentage deviation. Therefore, let $\widetilde{\pi}_t = d\pi_t$ and $\widetilde{\pi}_t^{\#} = d\pi_t^{\#}$:

$$\widetilde{\pi}_t = (1-\phi)\widetilde{\pi}_t^{\#}$$

Quite naturally, then, this says that deviation of inflation from 0 is equal to the fraction of firms changing prices times the amount by which they are changing prices. To close this out, we now need an expression for $\widetilde{\pi}_t^{\#}$. Log-linearize that expression by first taking logs:

$$\ln(1+\pi_t^{\#}) = \ln\varepsilon - \ln(\varepsilon-1) + \ln\widehat{a}_t - \ln\widehat{b}_t$$

Now do a Taylor series expansion about the zero inflation steady state:

$$\begin{aligned}
\ln(1+0) + d\pi_t^{\#} &= \ln\varepsilon - \ln(\varepsilon-1) + \ln\widehat{a}^* - \ln\widehat{b}^* + \frac{d\widehat{a}_t}{\widehat{a}^*} - \frac{d\widehat{b}_t}{\widehat{b}^*} \\
\widetilde{\pi}_t^{\#} &= \ln\varepsilon - \ln(\varepsilon-1) + \ln\left(\frac{\widehat{a}^*}{\widehat{b}^*}\right) + \widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t
\end{aligned}$$

Where $\widetilde{\pi}_t^{\#} = d\pi_t^{\#}$. Now, what is $\frac{\widehat{a}^*}{\widehat{b}^*}$? Note that $\Lambda_{t,t+1}^* = 1$. Solve for them individually using the definitions:

$$\begin{aligned}
\widehat{a}_t &= (1+\pi_t)\left(mc_t y_t + E_t\phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{a}_{t+1}\right) \\
\widehat{a}^* &= mc^* y^* + \phi\beta\widehat{a}^* \\
\widehat{a}^* &= \frac{mc^* y^*}{1-\phi\beta}
\end{aligned}$$

$$\begin{aligned}
\widehat{b}_t &= \left(y_t + E_t\phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{b}_{t+1}\right) \\
\widehat{b}^* &= y^* + \phi\beta\widehat{b}^* \\
\widehat{b}^* &= \frac{y^*}{1-\phi\beta}
\end{aligned}$$

To derive the above I'm using the assumption that inflation is zero in the steady state. Thus, I have:

$$\frac{\widehat{a}^*}{\widehat{b}^*} = mc^*$$

From above, we know that price is equal to a markup over nomina marginal cost. Thus real marginal cost is equal to the inverse of that markup, or, in the steady state:

$$mc^* = \frac{\varepsilon-1}{\varepsilon}$$

This means that:

$$\ln\left(\frac{\widehat{a}^*}{\widehat{b}^*}\right) = \ln(\varepsilon - 1) - \ln\varepsilon$$

Now plugging this in above, we see that the εs disappear, leaving:

$$\tilde{\pi}_t^{\#} = \widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t$$

So now we need $\widetilde{\widehat{a}}_t$ and $\widetilde{\widehat{b}}_t$. Begin with the first by first taking logs:

$$\begin{aligned}
\widehat{a}_t &= (1+\pi_t)\left(mc_t y_t + E_t\phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{a}_{t+1}\right) \\
\ln\widehat{a}_t &= \ln(1+\pi_t) + \ln\left(mc_t y_t + E_t\phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{a}_{t+1}\right)
\end{aligned}$$

Now do the Taylor series expansion evaluated at the steady state. Before proceeding, note that $mc^* y^* + E_t\phi\beta\widehat{a}^* = \widehat{a}^*$ since steady state inflation is 0:

$$\ln\widehat{a}^* + \frac{d\widehat{a}_t}{\widehat{a}^*} = \ln(1+0) + d\pi_t + \ln\widehat{a}^* + \frac{dmc_t y^*}{\widehat{a}^*} + \frac{dy_t mc^*}{\widehat{a}^*} + \ldots$$

$$\ldots + \frac{\phi\beta d\Lambda_{t,t+1}\widehat{a}^*}{\widehat{a}^*} - \frac{(1-\varepsilon)\phi\beta d\pi_{t+1}\widehat{a}^*}{\widehat{a}^*} + \frac{\phi\beta d\widehat{a}_{t+1}}{\widehat{a}^*}$$

Simplifying, we have:

$$\widetilde{\widehat{a}}_t = d\pi_t + \frac{dmc_t y^*}{\widehat{a}^*} + \frac{dy_t mc^*}{\widehat{a}^*} + \phi d\Lambda_{t,t+1} - (1-\varepsilon)\phi\beta d\pi_{t+1} + \phi\beta\widetilde{\widehat{a}}_{t+1}$$

Leave this alone for a minute. Now go to $\widetilde{\widehat{b}}_t$:

$$\ln\widehat{b}_t = \ln\left(y_t + E_t\phi\beta\Lambda_{t,t+1}(1+\pi_{t+1})^{-(1-\varepsilon)}\widehat{b}_{t+1}\right)$$

As above, note that $y^* + E_t\phi\beta\widehat{b}^* = \widehat{b}^*$. Proceed with the first order Taylor series expansion:

$$\ln\widehat{b}^* + \frac{d\widehat{b}_t}{\widehat{b}^*} = \ln\widehat{b}^* + \frac{dy_t}{\widehat{b}^*} + + \frac{\phi\beta d\Lambda_{t,t+1}\widehat{b}^*}{\widehat{b}^*} - \frac{(1-\varepsilon)\phi\beta d\pi_{t+1}\widehat{b}^*}{\widehat{b}^*} + \frac{\phi\beta d\widehat{b}_{t+1}}{\widehat{b}^*}$$

Now simplify some:

$$\widetilde{\widehat{b}}_t = \frac{dy_t}{\widehat{b}^*} + \phi d\Lambda_{t,t+1} - (1-\varepsilon)\phi\beta d\pi_{t+1} + \phi Q^*\widetilde{\widehat{b}}_{t+1}$$

We can now rewrite part of this as:

$$\begin{aligned}
\widetilde{\widehat{a}}_t &= d\pi_t + \frac{y^*}{\widehat{a}^*}dmc_t + \frac{mc^*}{a^*}dy_t + \phi d\Lambda_{t,t+1} - (1-\varepsilon)\phi\beta d\pi_{t+1} + \phi\beta\widetilde{\widehat{a}}_{t+1} \\
\widetilde{\widehat{b}}_t &= \frac{dy_t}{\widehat{b}^*} + \phi d\Lambda_{t,t+1} - (1-\varepsilon)\phi\beta d\pi_{t+1} + \phi\beta\widetilde{\widehat{b}}_{t+1}
\end{aligned}$$

Now subtract the latter from the former:

$$\widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t = d\pi_t + \frac{y^*}{\widehat{a}^*}dmc_t + \frac{mc^*}{\widehat{a}^*}dy_t - \frac{dy_t}{\widehat{b}^*} + \phi\beta\left(\widetilde{\widehat{a}}_{t+1} - \widetilde{\widehat{b}}_{t+1}\right)$$

Note that $\frac{y^*}{\widehat{a}^*} = \frac{(1-\phi\beta)}{mc^*}$ and $\frac{mc^*}{\widehat{a}^*} = \frac{1}{\widehat{b}^*}$. Using these facts, we can write:

$$\widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t = \widetilde{\pi}_t + (1-\phi\beta)\widetilde{mc}_t + \phi\beta\left(\widetilde{\widehat{a}}_{t+1} - \widetilde{\widehat{b}}_{t+1}\right)$$

Now note that $\widetilde{\pi}_t = (1-\phi)\left(\widetilde{\widehat{a}}_t - \widetilde{\widehat{b}}_t\right)$ and $\left(\widetilde{\widehat{a}}_{t+1} - \widetilde{\widehat{b}}_{t+1}\right) = \frac{E_t\widetilde{\pi}_{t+1}}{1-\phi}$:

$$\widetilde{\pi}_t = (1-\phi)\widetilde{\pi}_t + (1-\phi)(1-\phi\beta)\widetilde{mc}_t + \phi\beta E_t\widetilde{\pi}_{t+1}$$

Now solve for $\widetilde{\pi}_t$:

$$\begin{aligned}\phi\widetilde{\pi}_t &= (1-\phi)(1-\phi\beta)\widetilde{mc}_t + \phi\beta E_t\widetilde{\pi}_{t+1} \\ \widetilde{\pi}_t &= \frac{(1-\phi)(1-\phi\beta)}{\phi}\widetilde{mc}_t + \beta E_t\widetilde{\pi}_{t+1}\end{aligned}$$

The above relationship is what is often called the New Keynesian Phillips Curve.

It is actually quite common to see the Phillips Curve expressed not in terms of the log-deviation of real marginal cost, but rather in terms of an "output gap". To get to that specification, let's start with what defines real marginal cost and then go from there:

$$mc_t = \frac{w_t}{a_t}$$

We can substitute out for the wage using the household's first order condition for labor supply:

$$w_t = c_t^{\sigma}\theta(1-n_t)^{-\xi}$$

Now use the accounting identity fact that consumption equals income to get:

$$mc_t = \frac{y_t^{\sigma}\theta(1-n_t)^{-\xi}}{a_t}$$

Now let's log-linearize this expression. Begin by taking logs:

$$\ln mc_t = \sigma \ln y_t + \ln\theta - \xi\ln(1-n_t) - \ln a_t$$

Now do the first order Taylor series expansion about the steady state:

$$\begin{aligned}\ln mc^* + \frac{dmc_t}{mc^*} &= \ln mc^* + \sigma\frac{dy_t}{y^*} + \xi\frac{dn_t}{1-n^*} - \frac{da_t}{a^*} \\ \widetilde{mc}_t &= \sigma\widetilde{y}_t + \xi\frac{n^*}{1-n^*}\widetilde{n}_t - \widetilde{a}_t\end{aligned}$$

Now, note that, from the aggregate production function, $\widetilde{n}_t = \widetilde{y}_t - \widetilde{a}_t$:

$$\widetilde{mc}_t = \sigma \widetilde{y}_t + \xi \frac{n^*}{1-n^*} \left(\widetilde{y}_t - \widetilde{a}_t\right) - \widetilde{a}_t$$

Simplifying:

$$\widetilde{mc}_t = \left(\sigma + \xi \frac{n^*}{1-n^*}\right) \widetilde{y}_t - \left(1 + \xi \frac{n^*}{1-n^*}\right) \widetilde{a}_t$$

The output gap is defined as the deviation between the actual level of output and the "flexible price" level of output, $\widetilde{y}_t^f$ which is the level of output which would obtain in the absence of price stickiness. If prices are not sticky, price is a constant markup over nominal marginal cost, which implies that real marginal cost is constant, or equivalently that $\widetilde{mc}_t = 0$ (i.e. the log deviation of a constant is zero). We can then solve for the flexible price equilibrium level of output in terms of the exogenous driving variable using this fact and the above expression:

$$\begin{aligned} 0 &= \left(\sigma + \xi \frac{n^*}{1-n^*}\right) \widetilde{y}_t^f - \left(1 + \xi \frac{n^*}{1-n^*}\right) \widetilde{a}_t \\ \widetilde{y}_t^f &= \frac{1 + \xi \frac{n^*}{1-n^*}}{\sigma + \xi \frac{n^*}{1-n^*}} \widetilde{a}_t \end{aligned}$$

Note that, if we have log utility over consumption (i.e. $\sigma = 1$), then $\widetilde{y}_t^f = \widetilde{a}_t$ (i.e. employment is constant in the flexible price equilibrium. Using the above, we can eliminate $\widetilde{a}_t$ from the expression for the log deviation of real marginal cost:

$$\begin{aligned} \widetilde{mc}_t &= \left(\sigma + \xi \frac{n^*}{1-n^*}\right) \widetilde{y}_t - \left(\sigma + \xi \frac{n^*}{1-n^*}\right) \widetilde{y}_t^f \\ \widetilde{mc}_t &= \left(\sigma + \xi \frac{n^*}{1-n^*}\right) \left(\widetilde{y}_t - \widetilde{y}_t^f\right) \end{aligned}$$

Letting $\kappa = \left(\sigma + \xi \frac{n^*}{1-n^*}\right)$, we can re-write the Phillips Curve in terms of the output gap as:

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} \kappa \left(\widetilde{y}_t - \widetilde{y}_t^f\right) + \beta E_t \widetilde{\pi}_{t+1}$$

Holding expected inflation fixed, we see that positive output gaps put upward pressure on current inflation.

We can also log-linearize the rest of the model. Start with the Euler equation, after having already imposed the accounting identity:

$$y_t^{-\sigma} = \beta E_t (y_{t+1}^{-\sigma} (1 + r_t))$$

Take logs:

$$-\sigma \ln y_t = \ln \beta - \sigma \ln y_{t+1} + r_t$$

Above I have imposed the approximation that $\ln(1 + r_t) \approx r_t$. Now do the first order Taylor series expansion:

$$-\sigma \ln y^* - \sigma \frac{dy_t}{y^*} = \ln \beta - \sigma \ln y^* + r^* - \sigma \frac{dy_{t+1}}{y^*} + dr_t$$

Defining $\widetilde{y}_t = \frac{dy_t}{y^*}$ and $\widetilde{r}_t = dr_t$, we have:

$$\begin{aligned} -\sigma \widetilde{y}_t &= -\sigma \widetilde{y}_{t+1} + \widetilde{r}_t \\ \widetilde{y}_t &= \widetilde{y}_{t+1} - \frac{1}{\sigma} \widetilde{r}_t \end{aligned}$$

The log-linearized Euler equation is often referred to as the "New Keynesian IS" curve, as it shows a negative relationship between current spending and the current real interest rate, holding fixed expected future spending.

Now let's log-linearize the money supply curve (written in terms of real balances). It can be written out as follows:

$$\ln m_t - \ln m_{t-1} + \pi_t = (1 - \rho_m)\pi^* + \rho_m(\ln m_{t-1} - \ln m_{t-2}) + \rho_m \pi_{t-1} + e_m$$

Since this equation is already in logs and already linear, we can write it exactly the same way but interpreting the variables as log deviations $\widetilde{m}_t = \frac{dm_t}{m^*}$ and $\widetilde{\pi}_t = d\pi_t$:

$$\widetilde{m}_t = (1 - \rho_m)\pi^* + \widetilde{m}_{t-1} + \rho_m(\widetilde{m}_{t-1} - \widetilde{m}_{t-2}) - \widetilde{\pi}_t + \rho_m \widetilde{\pi}_{t-1} + e_m$$

Now let's log-linearize the money demand function. First take logs:

$$\ln \psi - v \ln m_t = -\sigma \ln y_t + \ln i_t - \ln(1 + i_t)$$

Do the first order Taylor series expansion:

$$\ln \psi - v \ln m^* - v \frac{dm_t}{m^*} = -\sigma \ln y^* + \ln i^* - \ln(1 + i^*) - \sigma \frac{dy_t}{y^*} + \frac{di_t}{i^*} - \frac{di_t}{1 + i^*}$$

Simplifying and use the tilde notation:

$$-v \widetilde{m}_t = -\sigma \widetilde{y}_t + \left(\frac{1}{i^*} - \frac{1}{1 + i^*} \right) \widetilde{i}_t$$

Simplifying further:

$$\widetilde{m}_t = \frac{\sigma}{v} \widetilde{y}_t - \left(\frac{1}{v i^* (1 + i^*)} \right) \widetilde{i}_t$$

Equilibrium requires that money demand be equal to money supply, so we can eliminate money altogether from the set of equations by equating demand with supply:

$$\frac{\sigma}{v}\widetilde{y}_t - \left(\frac{1}{vi^*(1+i^*)}\right)\widetilde{i}_t = (1-\rho_m)\pi^* + \widetilde{m}_{t-1} + \rho_m(\widetilde{m}_{t-1} - \widetilde{m}_{t-2}) - \widetilde{\pi}_t + \rho_m\widetilde{\pi}_{t-1} + e_m$$

Simplify by solving for the current log deviation of output:

$$\widetilde{y}_t = \left(\frac{1}{\sigma i^*(1+i^*)}\right)\widetilde{i}_t + \frac{v}{\sigma}(1-\rho_m)\pi^* + \frac{v}{\sigma}\widetilde{m}_{t-1} + \frac{v}{\sigma}\rho_m(\widetilde{m}_{t-1} - \widetilde{m}_{t-2}) - \frac{v}{\sigma}\widetilde{\pi}_t + \frac{v}{\sigma}\rho_m\widetilde{\pi}_{t-1} + \frac{v}{\sigma}e_m$$

We can write this in terms of the real interest rate by using the linearized Fisher relationship ($\widetilde{i}_t = \widetilde{r}_t + \widetilde{\pi}_{t+1}$):

$$\widetilde{y}_t = \left(\frac{1}{\sigma i^*(1+i^*)}\right)(\widetilde{r}_t + \widetilde{\pi}_{t+1}) + \frac{v}{\sigma}(1-\rho_m)\pi^* + \frac{v}{\sigma}\widetilde{m}_{t-1} + \frac{v}{\sigma}\rho_m(\widetilde{m}_{t-1} - \widetilde{m}_{t-2}) - \frac{v}{\sigma}\widetilde{\pi}_t + \frac{v}{\sigma}\rho_m\widetilde{\pi}_{t-1} + \frac{v}{\sigma}e_m$$

The expression above can be interpreted as an LM curve from intermediate macro – it is the set of points in $(\widetilde{r}_t, \widetilde{y}_t)$ space consistent with the money market clearing. The IS curve is the set $(\widetilde{r}_t, \widetilde{y}_t)$ pairs consistent with the "goods market" clearing, which means that consumption is equal to income and the Euler equation holds. The IS equation is downward sloping, while the LM curve is upward sloping.

Above we derived an expression for the flexible price equilibrium level of output as:

$$\widetilde{y}_t^f = \frac{1+\xi\frac{n^*}{1-n^*}}{\sigma+\xi\frac{n^*}{1-n^*}}\widetilde{a}_t$$

For notational ease, call $\gamma = \frac{1+\xi\frac{n^*}{1-n^*}}{\sigma+\xi\frac{n^*}{1-n^*}}$, so:

$$\widetilde{y}_t^f = \gamma\widetilde{a}_t$$

Now plug in this process for technology:

$$\widetilde{y}_t^f = \gamma\rho\widetilde{a}_{t-1} + \gamma e_t$$

Now we know that $\widetilde{a}_{t-1} = \frac{1}{\gamma}\widetilde{y}_{t-1}^f$, so we can write this as:

$$\widetilde{y}_t^f = \rho\widetilde{y}_{t-1}^f + \gamma e_t$$

The full set of log-linearized equations which allow us to solve the model are then:

$$\widetilde{y}_t = \widetilde{y}_{t+1} - \frac{1}{\sigma}\widetilde{r}_t \tag{26}$$

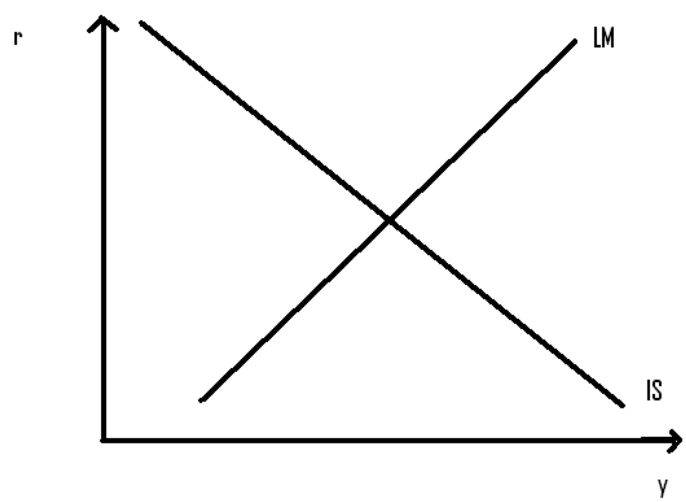
$$\widetilde{y}_t = \left(\frac{1}{\sigma i^*(1+i^*)}\right)(\widetilde{r}_t + \widetilde{\pi}_{t+1}) + \frac{v}{\sigma}(1-\rho_m)\pi^* + \frac{v}{\sigma}\widetilde{m}_{t-1} + \frac{v}{\sigma}\rho_m(\widetilde{m}_{t-1} - \widetilde{m}_{t-2}) - \frac{v}{\sigma}\widetilde{\pi}_t + \frac{v}{\sigma}\rho_m\widetilde{\pi}_{t-1} + \frac{v}{\sigma}e_m \tag{27}$$

$$\widetilde{y}_t^f = \rho \widetilde{y}_{t-1}^f + \gamma e_t \tag{28}$$

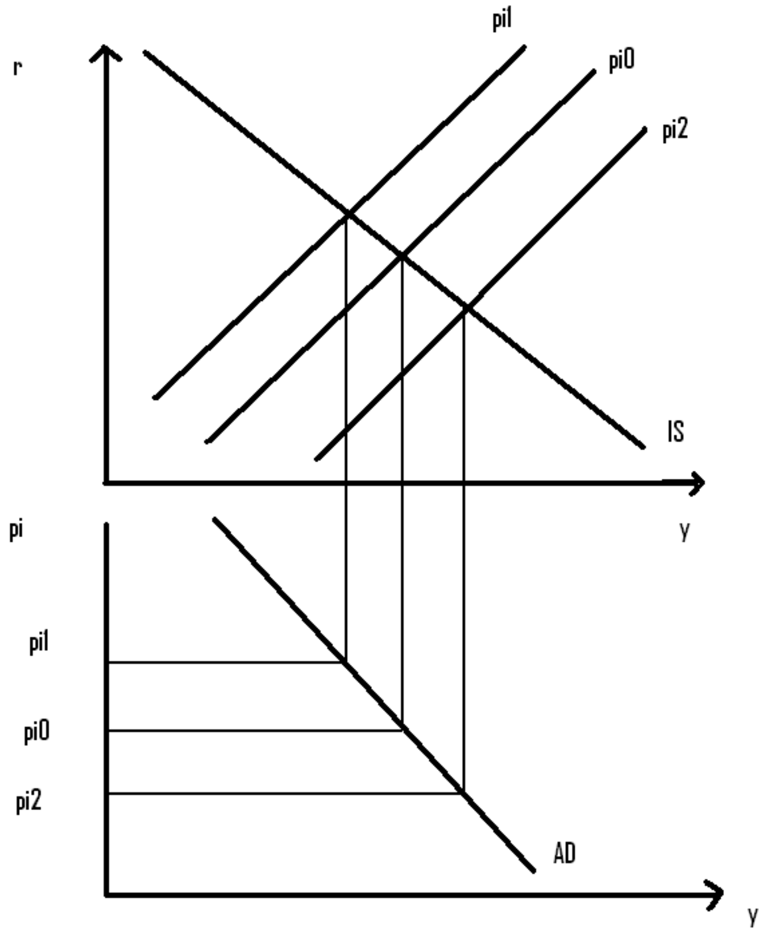
$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} \kappa \left(\widetilde{y}_t - \widetilde{y}_t^f \right) + \beta E_t \widetilde{\pi}_{t+1} \tag{29}$$

Equation (26) is the IS curve, (27) is the LM curve, (28) is the process for the "supply shock", and (29) is the Phillips Curve. There are four equations and four variables (output, real interest rate, the flexible price level of output, and inflation).

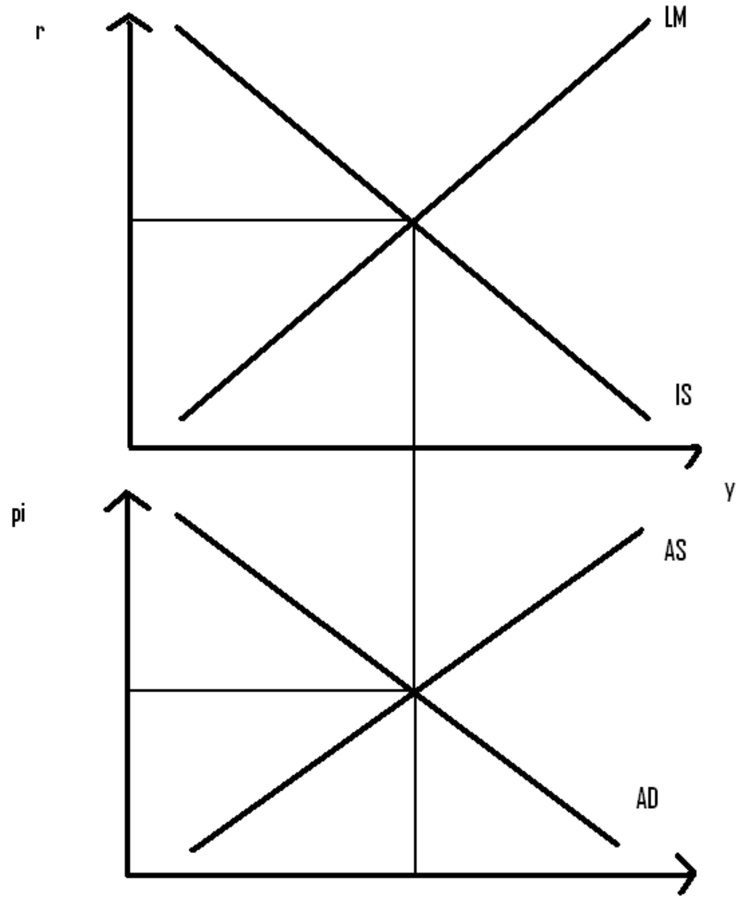
It turns out there is a graphical interpretation of this model that is is visually similar to what one sees in intermediate macro. Holding the values of all future and past variables fixed, as well as the value of current inflation, we can plot out the IS and LM curves as follows:



Recall that the LM curve is drawn holding current inflation fixed (the IS curve does not depend on current inflation). Effectively what this does is define an equilibrium level of output and the interest rate for each level of current inflation possible. If inflation goes up, the LM curve shifts horizontally to the left (i.e. holding the real interest rate fixed output must fall when inflation goes up). The opposite holds when inflation goes down. We can then trace out an aggregate demand curve in $(\widetilde{\pi}_t, \widetilde{y}_t)$ space as follows:

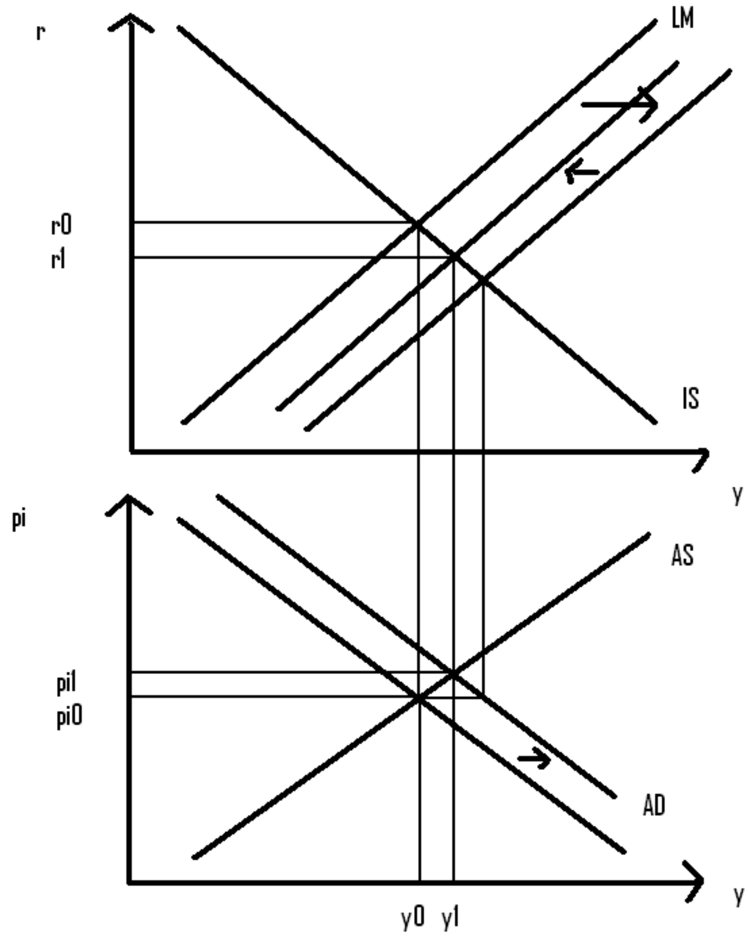


When inflation is relative high, the LM curve is relatively far in, and so output is relatively low, and vice versa. Tracing out the points, the AD curve is downward sloping. We can complete the model by adding in the Phillips curve, which is an upward sloping AS relationship, defined for a give value of the flexible price level of output and a given expected future inflation:



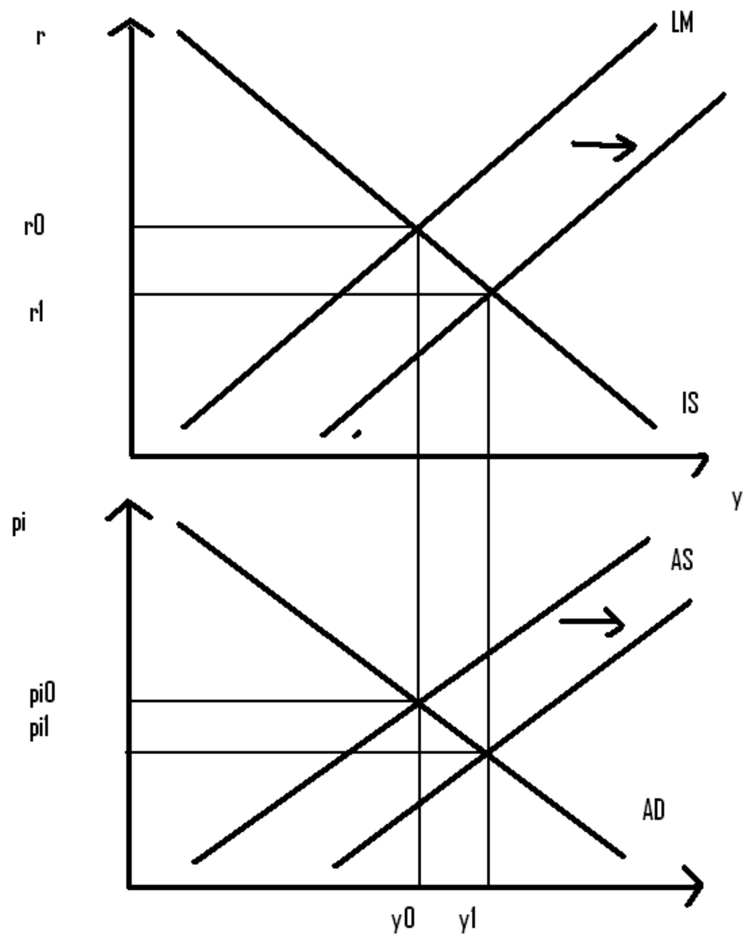
Given this framework, we can graphically conduct comparative statics exercises. I should be very upfront that this exercise is frought with hazards – there are lots of expected future endogenous variables in these equations, all of which will, in general, move when exogenous variables change. This means that shifting curves holding expectations of future endogenous variables constant really isn't correct. Nevertheless, if shocks are transitory enough, this will provide a very good approximation.

Let's first consider a monetary policy shock – this will cause the LM curve to shift right (i.e. a positive innovation to e_m raises output for a given interest rate).



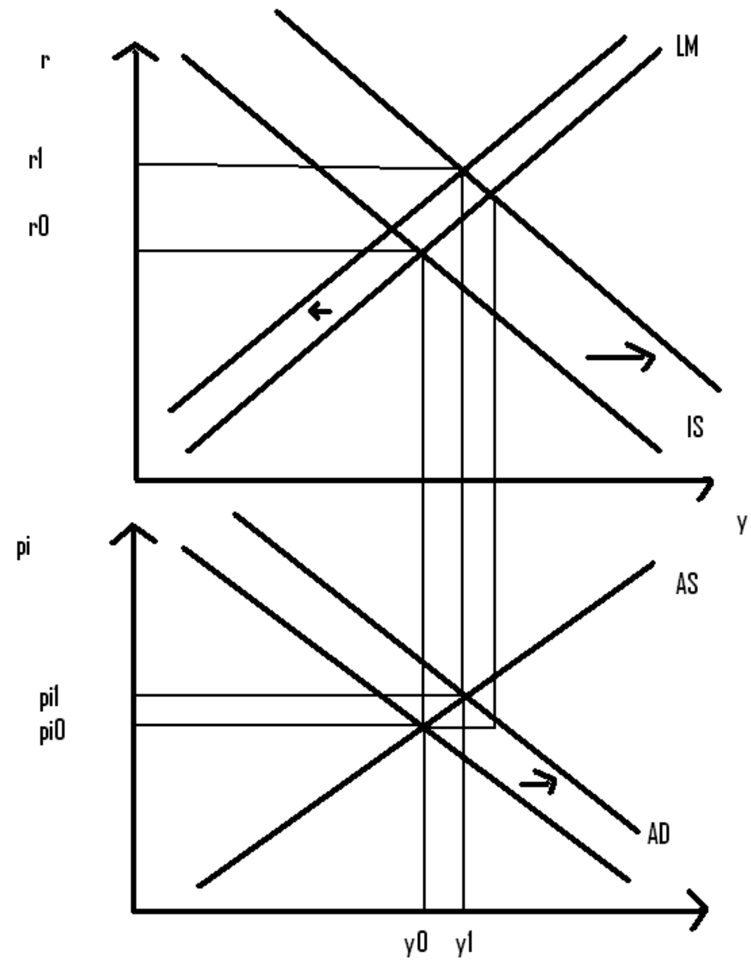
The increase in money supply shifts the LM curve out – this raises the equilibrium level of output for a given level of inflation, shifting the AD curve horizontally. In order to also be on the Phillips Curve/AS relationship, inflation rises. This means that output rises by less than the horizontal shift in the AD curve. The rise in inflation causes the LM curve to shift back in some, so as to intersect the IS curve at the same level of output. We see that, in equilibrium the real interest rate is lower, output is higher, and inflation is lower – in other words, more or less exactly what our undergraduate intuition is. Furthermore, we see that the increase in output due to monetary shocks is increasing in the flatness of the Phillips Curve. When is the Phillips Curve flat? When ϕ, the probability of *not* being able to adjust one's price, is big. In other words, money supply shocks have a bigger effect on output (and a smaller effect on inflation), the stickier are prices. If prices are flexible, so that $\phi = 0$, then the Phillips Curve is vertical at the flexible price level of output, which means that monetary shocks have no real effect and just lead to inflation.

Now let's consider a "supply shock" – i.e. a shock to the flexible price level of output. From inspection of the Phillips Curve, this leads to an outward shift of the AS relationship. Graphically:



The outward shift in the AS relationship raises output and lowers inflation. The lower inflation forces the LM curve outward. At the end of the day, the supply shock leads to higher output, lower inflation, and a lower real interest rate. Note that the increase in output is smaller than if the AS/Phillips Curve were perfectly vertical. This is what necessitates the reduction in hours on impact in response to a technology shock in the model.

Finally, consider an "IS Shock". We don't formally have that in the model as specified, but would could think of it as a shock to expected future output. We will ignore the fact that this would influence expected inflation in equilibrium, which would in turn shift the Phillips Curve:



Here the outward shift of the IS curve shifts the AD curve out, which raises both output and inflation. The increase in inflation leads to the LM curve shifting back in some so as to restore equilibrium. At the end of the day, output, inflation, and the real interest rate are all higher.

The above exercise shows that this dynamic, optimizing model can be thought of in terms very similar to what one learns in a typical intermdiate micro course. Of course, this is all approximate. Nevertheless, it restores a lot of the Keynesian intuition.