Optimal Monetary Policy

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Here I consider how a welfare-maximizing central bank can and should implement monetary policy in the standard New Keynesian model. I do not consider optimal monetary policy in models without nominal frictions; there we see that the Friedman rule (setting the nominal interest rate equal to zero) was optimal. The standard New Keynesian model can be reduced to a couple of log-linearized equations, the most important of which are the Phillips Curve and the IS relation. These can be written:

\[
\tilde{\pi}_t = \gamma \left( \tilde{y}_t - \tilde{y}_t^f \right) + \beta E_t \tilde{\pi}_{t+1} \\
\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma} \tilde{r}_t
\]

All variables are deviations (or percentage deviations) from steady state. \( \tilde{y}_t^f \) corresponds with the level of output that would obtain if prices were fully flexible. The parameter \( \gamma = \frac{(1-\phi)(1-\phi{\beta})}{\phi} (\sigma + \xi \frac{n^*}{1-n^*}) \). It is useful to define the natural rate of interest as the real interest rate that would obtain if prices were fully flexible. This can be obtained from the IS equation evaluated at the flexible price equilibrium:

\[
\tilde{y}_t^f = E_t \tilde{y}_{t+1}^f - \frac{1}{\sigma} \tilde{r}_t^f \\
E_t \tilde{y}_{t+1}^f - \tilde{y}_t^f = \frac{1}{\sigma} \tilde{r}_t^f
\]

Because this equality holds, we can add it to the original IS equation without changing the equality:

\[
\tilde{y}_t + E_t \tilde{y}_{t+1} - \tilde{y}_t^f = E_t \tilde{y}_{t+1} - \frac{1}{\sigma} \tilde{r}_t + \frac{1}{\sigma} \tilde{r}_t^f
\]

Simplifying:

\[
\tilde{y}_t - \tilde{y}_t^f = E_t \tilde{y}_{t+1} - E_t \tilde{y}_{t+1}^f - \frac{1}{\sigma} \left( \tilde{r}_t - \tilde{r}_t^f \right)
\]

In other words, we can write the IS equation in terms of the output gap and the interest rate gap. For notational ease, define \( \tilde{x}_t \equiv \tilde{y}_t - \tilde{y}_t^f \), and re-write the Phillips and IS curves as:

\[
\tilde{\pi}_t = \gamma \tilde{x}_t + \beta E_t \tilde{\pi}_{t+1} \\
\tilde{x}_t = E_t \tilde{x}_{t+1} - \frac{1}{\sigma} \left( \tilde{r}_t - \tilde{r}_t^f \right)
\]
The other equations of the model are a money demand relationship, the Fisher relation, and a process for the exogenous driving force(s) which move the flexible price levels of output around. We don’t impose a monetary policy rule because that’s what we’re going to try and derive. The money demand curve (expressed in terms of real balances) and the Fisher relationship can be written as:

\[-\nu \tilde{m}_t = -\sigma \tilde{y}_t + \left(\frac{1}{i^*} - \frac{1}{1+i^*}\right) \tilde{i}_t\]

\[\tilde{r}_t = \tilde{i}_t - E_t \tilde{\pi}_{t+1}\]

As shown in class and in previous course notes, assuming that the only real disturbance is a TFP shock with autoregressive parameter \(\rho\), we can write the process for the flexible price equilibrium level of output as:

\[\tilde{y}_t^f = \rho \tilde{y}_{t-1}^f + \gamma e_t\]

Taking the expectation forward one period, we can get an expression for the expected future flexible price level of output:

\[E_t \tilde{y}_{t+1}^f = \rho \tilde{y}_t^f\]

Since:

\[\tilde{r}_t^f = \sigma \left( E_t \tilde{y}_{t+1}^f - \tilde{y}_t^f \right) = \sigma (\rho - 1) \tilde{y}_t^f\]

Since \(\rho < 1\) and \(\sigma > 0\), this says that whenever the flexible price level of output is above its steady state the natural rate of interest is below its steady state (i.e. output must be expected to decline, so the real interest rate must below its steady state). Plugging in the expression for the evolution of the flexible price level of output:

\[\tilde{r}_t^f = \sigma (\rho - 1) \rho \tilde{y}_{t-1}^f + \sigma (\rho - 1) \gamma e_t\]

But since \(\tilde{y}_{t-1}^f = \frac{1}{\sigma (\rho - 1)} \tilde{r}_{t-1}^f\), we can write this as:

\[\tilde{r}_t^f = \rho \tilde{r}_{t-1}^f + \sigma (\rho - 1) \gamma e_t\]

Put differently, I can just as easily think about the exogenous driving force as being to the flexible price level of output as I can it being to the natural rate of interest.

We assume that the central bank does not like output gaps or inflation. As such, its objective function is to minimize the following quadratic loss function:

\[
\min \quad \frac{1}{2} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( \tilde{\pi}_t^2 + \omega \tilde{\pi}_t^2 \right) \right]
\]

We can think of \(\omega\) as representing the relative weight that the central bank places on output gap deviations relative to inflation deviation. It is implicitly assumed here that the target level of inflation is zero (which is consistent with the log-linearization of the model.
about the zero inflation steady state) and the target level of output is the flexible price level of output. One might ask the question why the central bank would target the flexible price equilibrium level of output. The answer is that the flexible price level of output differs from the first best level of output by a constant. Consider the first order condition for labor demand in our standard model with flexible prices:

\[ w_t = \frac{\epsilon - 1}{\epsilon} a_t \]

Since \( \epsilon > 1 \), we see that the real wage is less than the marginal product of labor. A welfare-maximizing social planner would set the real wage equal to the marginal product of labor. Once can show that this leads to output that is lower in the flexible price equilibrium relative to the first best by a constant which is monotonically decreasing in \( \epsilon \) (i.e. the more competitive is the economy the smaller is the distortion). Hence, minimizing deviations of \( \bar{x}_t \) is equivalent to minimizing deviations of actual output from the first best planner’s solution.

We can think of the central bank as picking time paths of \( \tilde{\pi}_t, \bar{x}_t, \) and \( \tilde{\mu}_t \) to minimize the objective function subject to the constraints, which are the IS and Phillips Curves plus the Fisher relationship. How does the central bank “pick” these variables, which are the outcome of an equilibrium model? Answer: by adjusting the money supply.

It is easiest to think about the problem in stages. In the first stage the central bank picks \( \tilde{\pi}_t \) and \( \tilde{\mu}_t \) to minimize the objective function subject to the constraints, which are the IS and Phillips Curves plus the Fisher relationship. Then it picks a time path of the nominal interest rate consistent with the IS and Fisher relationships to make this happen. Then it picks a time path of the nominal money supply consistent with the money demand curve to ensure this time path of the nominal interest rate.

We consider two cases: policy under discretion and policy under commitment. Discretion means that the central bank re-optimizes every period. Commitment means that the central bank picks a state contingent time path of interest rates at the beginning of time and is able to stick to it. We consider first the case of discretion. Because the model features no endogenous state variables, future values of the variables of the model will be unaffected by current choices. Hence, we can write the objective as a static problem with expected future values taken as given:

\[
\min_{\tilde{\pi}_t, \bar{x}_t} \frac{1}{2} \left( \tilde{\pi}_t^2 + \omega \bar{x}_t^2 \right)
\]

s.t.

\[
\tilde{\pi}_t = \gamma \bar{x}_t + \beta E_t \tilde{\pi}_{t+1}
\]

The Lagrangian is:

\[
\mathcal{L} = -\frac{1}{2} \tilde{\pi}_t^2 - \frac{1}{2} \omega \bar{x}_t^2 + \mu_t (\gamma \bar{x}_t + \beta E_t \tilde{\pi}_{t+1} - \tilde{\pi}_t)
\]

The first order conditions are:
\frac{\partial L}{\partial \tilde{\pi}_t} = 0 \Leftrightarrow -\tilde{\pi}_t = \mu_t \\
\frac{\partial L}{\partial \tilde{x}_t} = 0 \Leftrightarrow \mu_t \gamma = \omega \tilde{x}_t

Combining these two yields:

\tilde{x}_t = -\frac{\gamma}{\omega} \tilde{\pi}_t

Loosely speaking, we can interpret this first order condition as a “lean against the wind” strategy. If inflation is high, the central bank will allow output to contract (i.e. to go below the flexible price level of output, leaving a negative output gap) and vice-versa.

Now take this condition and plug it into the Phillips Curve, eliminating the output gap:

\begin{align*}
\tilde{\pi}_t &= -\frac{\gamma^2}{\omega} \tilde{\pi}_t + \beta E_t \tilde{\pi}_{t+1} \\
\left(1 + \frac{\gamma^2}{\omega}\right) \tilde{\pi}_t &= \beta E_t \tilde{\pi}_{t+1} \\
\tilde{\pi}_t &= \left(\frac{\beta \omega}{\omega + \gamma^2}\right) E_t \tilde{\pi}_{t+1}
\end{align*}

Note that, since \( \beta < 1 \), \( \frac{\beta \omega}{\omega + \gamma^2} < 1 \), so we could solve this forward for a non-explosive level of inflation. Now we go to the second stage. It is evident that one solution consistent with this first order condition is to have \( \tilde{x}_t = 0 \) and \( \tilde{\pi}_t = 0 \). From above, this would also imply that \( E_t \tilde{\pi}_{t+1} = 0 \) for the Phillips Curve to hold, which would then imply, since the central bank would optimize the same way in the next period, that \( E_t \tilde{x}_{t+1} = 0 \). Go to the IS equation

\tilde{x}_t = E_t \tilde{x}_{t+1} - \frac{1}{\sigma} \left(\tilde{r}_t - \tilde{r}_t^f\right)

Plug in zero for the current and future output gaps:

\begin{align*}
0 &= \frac{1}{\sigma} \left(\tilde{r}_t - \tilde{r}_t^f\right) \\
\Rightarrow \tilde{r}_t &= \tilde{r}_t^f
\end{align*}

Now take the Fisher relationship:

\tilde{i}_t = \tilde{r}_t + E_t \tilde{\pi}_{t+1}

Plug this in above:

\tilde{i}_t = \tilde{r}_t^f + E_t \tilde{\pi}_{t+1}
Now use the expression from the Phillips Curve to write this in terms of current inflation:

\[ \tilde{i}_t = \tilde{n}_t + \frac{\omega + \gamma^2}{\beta \omega} \tilde{\pi}_t \]

As argued above, \( \frac{\omega + \gamma^2}{\beta \omega} > 1 \). Thus, the central bank needs to adjust the nominal interest rate one for one with the natural rate of interest, and “threaten” to raise the nominal interest rate more than one for one with inflation. I put “threaten” in quotation marks because we know, from above, that inflation is actually always zero in equilibrium. This “threatening” to respond to inflation turns out to be important. If the central bank followed a rule with \( \tilde{i}_t = \tilde{n}_t \), there would be no determinate equilibrium (Sargent and Wallace (1975)).

Finally, we consider what must be true of the money supply in order to obtain this nominal interest rate given money demand. Recall the money demand relationship:

\[ v e_m t = e_y f t + i_t \]

Plug in what we know to be true from above (i.e. that \( \tilde{i}_t = \tilde{n}_t \) in equilibrium and \( \tilde{y}_t = \tilde{y}_t^f \)):

\[-v \tilde{m}_t = -\sigma \tilde{y}_t^f + \left( \frac{1}{\tilde{i}^*} - \frac{1}{1 + \tilde{i}^*} \right) \tilde{i}_t \]

To ease notation, define \( \zeta \equiv \frac{1}{\tilde{i}^*} - \frac{1}{1 + \tilde{i}^*} > 0 \). We can write the required money supply as:

\[ \tilde{m}_t = \frac{\sigma \tilde{y}_t^f - \zeta \tilde{i}_t^f}{v} \]

From above, we know that \( \tilde{y}_t^f = \frac{1}{\sigma (\rho - 1)} \tilde{i}_t^f \). Plugging this in:

\[ \tilde{m}_t = \frac{\sigma}{v} \frac{1}{\sigma (\rho - 1)} \tilde{i}_t^f - \frac{\zeta}{v} \tilde{i}_t^f = \left( \frac{1}{v \sigma (\rho - 1)} - \frac{\zeta}{v} \right) \tilde{i}_t^f \]

We also know that

\[ \tilde{i}_t^f = \rho \tilde{i}_{t-1}^f + \sigma (\rho - 1) \gamma e_t \]

Plug this in:

\[ \tilde{m}_t = \left( \frac{1}{v \sigma (\rho - 1)} - \frac{\zeta}{v} \right) \rho \tilde{i}_{t-1}^f + \sigma (\rho - 1) \gamma \left( \frac{1}{v \sigma (\rho - 1)} - \frac{\zeta}{v} \right) e_t \]

But since:

\[ \tilde{i}_{t-1}^f = \left( \frac{1}{v \sigma (\rho - 1)} - \frac{\zeta}{v} \right)^{-1} \tilde{m}_{t-1} \]

We can write this as:

\[ \tilde{m}_t = \rho \tilde{m}_{t-1} + \sigma (\rho - 1) \gamma \left( \frac{1}{v \sigma (\rho - 1)} - \frac{\zeta}{v} \right) e_t \]
Note that $\sigma (\rho - 1) \gamma \left( \frac{1}{\nu\sigma (\rho - 1)} - \frac{\xi}{v} \right) > 0$ (since $\frac{1}{\nu\sigma (\rho - 1)} - \frac{\xi}{v} < 0$ and $\sigma (\rho - 1) \gamma < 0$). Hence, we conclude that the optimal monetary policy should follow an AR(1) with autoregressive parameter equal to the persistence of the technology shock and a positive response coefficient on the technology shock. Note that, since inflation will always be zero, an AR(1) in real balances and an AR(1) in nominal money are identical.

The model is solved in Dynare with the following linear equations:

\begin{align*}
  \pi &= \gamma x + \beta \pi(+1); \\
  x(+1) &= x + \frac{1}{\sigma}(r-r_f); \\
  m &= (\sigma/v)y + \nu \text{int}; \\
  r_f &= \rho r_f(-1) + \sigma (\rho-1) \gamma e; \\
  r &= \text{int} - \pi(+1); \\
  \text{int} &= rf + (\omega + \gamma^2)/(\beta \omega) \pi; \\
  x &= y - y_f; \\
  y_f &= \rho y_f(-1) + e;
\end{align*}

Here the variable “nu” is equal to $\frac{1}{\nu}$. Note that I have to specify the nominal interest rate rule as above – I can’t include both the money demand and the derived implicit money supply relationship as above, because the process for the money supply I derived uses the money demand curve. Because the coefficient on inflation in the nominal interest rate rule is greater than unity, the Taylor principle is satisfied. The parameters are calibrated as follows: $\sigma = 1$, $\rho = 0.9$, $\omega = 1$, $\beta = 0.98$, $\phi = 0.75$, $\nu^* = 0.2$, $\xi = 1$, and $v = 1$. Here are the impulse responses to a technology shock:

We can see that the output gap and inflation are completely stabilized. The natural interest rate falls; so too does the actual real interest rate and the nominal rate, both by
the same amount. The money supply rises. It is easy to verify that the time path of the money supply is exactly the same as that given by the implicit money supply relationship above. Note also that the solution is independent of the parameterization of $\omega$, the relative weight that the central bank places on output gap fluctuations. The intuition here is that the Fed should “accommodate” supply shocks by printing more/less money.

Studying this same problem under commitment is not interesting, as the solution is identical to above. This is because the bank is already setting its loss function to zero each and every period under this set up; there is no gain from the bank credibly committing to a rule and solving the dynamic problem instead of solving the problem period by period.

There is a sense in which monetary policy is too “simple” here. Basically, policy can completely neutralize the effects of nominal rigidities – the flexible price equilibrium obtains every period here. Further, there is no trade-off between stabilizing inflation and stabilizing the output gap. The central bank can stabilize both. There is obviously something that is lacking in reality, as central bankers evidently implicitly feel that there is an inflation/output trade-off, and there is a feeling that there are welfare gains to be had from commitment as opposed to discretion.

To address these issues, the basic model needs to be modified. In particular, there needs to be some feature which prevents the central bank from completely stabilizing both the output gap and inflation each period. In the literature this is typically done by introducing a “cost-push shock” into the Phillips Curve. Let the new Phillips Curve be written:

$$\pi_t = \gamma \tilde{x}_t + \beta E_t \pi_{t+1} + \tilde{u}_t$$

Assume that $\tilde{u}_t$ follows an AR(1) process:

$$\tilde{u}_t = \rho \tilde{u}_{t-1} + e_{u,t}$$

The structural interpretation of the “cost-push” shock is not always very clear; most authors interpret it as reflecting forces which drive a wedge between real marginal cost and the output gap, or as forces which induce time variation in the gap between the first best (planner’s) equilibrium and the flexible price equilibrium. It is a convenient short cut to make the central bank’s problem a little more interesting.

The optimization problem under discretion is the same as above, subject to this new Phillips Curve. But because the central bank takes $\tilde{u}_t$ as given, the solution is the same, yielding the same first order condition:

$$\tilde{x}_t = -\frac{\gamma}{\omega} \tilde{\pi}_t$$

There is a gain a “lean against the wind” interpretation of this condition. What is different is that $\tilde{x}_t = \pi_t = 0$ will not be a viable solution in general, since there is no reason to expect $\tilde{u}_t = 0$ (i.e. if $\tilde{u}_t \neq 0$ then $\tilde{x}_t = \pi_t = 0$ will not be consistent with the Phillips Curve holding). As before, plug this in to the Phillips Curve:

$$\pi_t = -\frac{\gamma^2}{\omega} \pi_t + \beta E_t \pi_{t+1} + \tilde{u}_t$$

Simplify:
As argued above, \( \frac{\beta \omega}{\omega + \gamma^2} < 1 \). Hence, we can “solve forward” for a stable solution by successively plugging in for \( E_t \bar{\pi}_{t+1} \):

\[
\bar{\pi}_t = \left( \frac{\beta \omega}{\omega + \gamma^2} \right) \left( \frac{\beta \omega}{\omega + \gamma^2} \right) E_t \bar{\pi}_{t+2} + \left( \frac{\omega}{\omega + \gamma^2} \right) \bar{u}_{t+1} + \left( \frac{\omega}{\omega + \gamma^2} \right) \bar{u}_t
\]

Going forward and using the terminal condition that \( \lim_{T \to \infty} \left( \frac{\beta \omega}{\omega + \gamma^2} \right)^T E_t \bar{\pi}_{t+T} = 0 \), we have:

\[
\bar{\pi}_t = \left( \frac{\omega}{\omega + \gamma^2} \right) E_t \sum_{j=0}^{\infty} \left( \frac{\beta \omega}{\omega + \gamma^2} \right)^j \bar{u}_{t+j}
\]

We can simplify this even further by imposing the autoregressive feature that \( E_t \bar{u}_{t+j} = \rho_u^j \bar{u}_t \):

\[
\bar{\pi}_t = \left( \frac{\omega}{\omega + \gamma^2} \right) \bar{u}_t \sum_{j=0}^{\infty} \left( \frac{\rho_u \beta \omega}{\omega + \gamma^2} \right)^j
\]

Simplifying this, we get:

\[
\bar{\pi}_t = \left( \frac{\omega}{\omega + \gamma^2} \right) \bar{u}_t \frac{1}{1 - \rho_u \beta \omega} = \left( \frac{\omega}{\omega + \gamma^2 - \rho_u \beta \omega} \right) \bar{u}_t
\]

Now plugging this back into the first order condition, we get an expression for the output gap:

\[
\bar{x}_t = -\frac{\gamma}{\omega(1 - \rho_u \beta) + \gamma^2} \bar{u}_t
\]

In other words, the central bank should allow inflation to rise and the output gap to decline in response to a “cost-push” shock, while the central bank should stabilize both
inflation and the output gap in response to a real supply shock (i.e. a shock in which \( \bar{u}_t = 0 \)).

Now we need to figure out the required time path of the nominal interest rate in order to implement this strategy. Start with the IS equation:

\[
\bar{x}_t = E_t \bar{x}_{t+1} - \frac{1}{\sigma} \left( \bar{r}_t - \bar{r}_t^f \right)
\]

\[
\bar{r}_t = \bar{r}_t^f + \sigma (E_t \bar{x}_{t+1} - \bar{x}_t)
\]

\[
\bar{r}_t = \bar{r}_t^f + \sigma \left( \frac{\gamma}{\omega(1 - \rho_u \beta)} + \frac{\gamma^2}{\omega(1 - \rho_u \beta)} \right) (1 - \rho_u) \bar{u}_t
\]

Using the autoregressive structure of the cost push shock, we can simplify:

\[
\bar{r}_t = \bar{r}_t^f + \sigma \left( \frac{\gamma}{\omega(1 - \rho_u \beta)} + \frac{\gamma^2}{\omega(1 - \rho_u \beta)} \right) (1 - \rho_u) \bar{u}_t
\]

Now plug in the Fisher relationship to write this in terms of the nominal interest rate:

\[
\bar{i}_t - E_t \bar{\pi}_{t+1} = \bar{r}_t^f + \sigma \left( \frac{\gamma}{\omega(1 - \rho_u \beta)} + \frac{\gamma^2}{\omega(1 - \rho_u \beta)} \right) (1 - \rho_u) \bar{u}_t
\]

\[
\bar{i}_t = \bar{r}_t^f + E_t \bar{\pi}_{t+1} + \sigma \left( \frac{\gamma}{\omega(1 - \rho_u \beta)} + \frac{\gamma^2}{\omega(1 - \rho_u \beta)} \right) (1 - \rho_u) \bar{u}_t
\]

From above, we know that:

\[
\bar{u}_t = \frac{\omega(1 - \rho_u \beta) + \gamma^2}{\omega} \bar{\pi}_t
\]

\[
E_t \bar{\pi}_{t+1} = \left( \frac{\omega + \gamma^2}{\beta \omega} \right) \bar{\pi}_t - \frac{1}{\beta} \bar{u}_t
\]

\[
E_t \bar{\pi}_{t+1} = \frac{1}{\beta} \left( \frac{\omega + \gamma^2}{\omega - \omega(1 - \rho_u \beta) + \gamma^2} \right) \bar{\pi}_t = \rho_u \bar{\pi}_t
\]

Make these substitutions yields the optimal nominal interest rate rule:

\[
\bar{i}_t = \bar{r}_t^f + \rho_u \bar{\pi}_t + \frac{\sigma \gamma}{\omega} (1 - \rho_u) \bar{\pi}_t
\]

\[
\bar{i}_t = \left( \bar{r}_t^f + \left( \rho_u + \frac{\sigma \gamma}{\omega (1 - \rho_u)} \right) \bar{\pi}_t \right)
\]

This says that the central bank should adjust the nominal interest rate one for one with changes in the natural rate of interest and should raise nominal interest rates when inflation
goes up. A potential problem arises in that \( \rho_u + \frac{\sigma\gamma}{\omega}(1 - \rho_u) \) is not guaranteed to be greater than one, and being greater than one is necessary for there to exist a determinate rational expectations equilibrium. It is straightforward to verify that \( \sigma\gamma > \omega \) for this to be greater than unity. Since \( \gamma \), the “slope coefficient” in the Phillips Curve, is typically quite small, this is unlikely to be satisfied if \( \omega \) is very big (i.e. if the central bank puts much weight on output fluctuations.

Suppose that I try to solve the model with the same parameterization as above, with \( \rho_u = 0.75 \) and the standard deviation of the cost-push shock equal to 0.002. When I do that, I get the following error message:

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Error using ==> print_info at 42
Blanchard Kahn conditions are not satisfied: indeterminacy
```

With the above parameterization, \( \rho_u + \frac{\sigma\gamma}{\omega}(1 - \rho_u) = 0.77 \), so the model does not have a determinate solution. In order for determinacy to obtain, the central bank needs to lower \( \omega \) (i.e. it needs to pay less attention to output gap fluctuations). Suppose I set \( \omega = 0.1 \), which essentially implies that the central bank cares about inflation ten times as much as the output gap. Then the coefficient on inflation becomes 1.02, and the solution is determinate. Below are the impulse responses to both the technology shock and the cost-push shock:
We can observe that the responses to the technology are identical to case without cost push shocks. Whether there are cost push shocks are not, it is optimal for the central bank to completely stabilize both inflation and the gap in response to supply side shocks. What is different here is that the gap goes negative (i.e. output falls) and inflation rises in response to the cost push shock. The extent to which this happens depends on the relative weight on output, again noting that the relative weight on output cannot be too big for determinacy to obtain.

One way out of the indeterminacy issue is for the central bank to adopt a forward-looking nominal interest rate rule. In particular, we saw above that:

\[ E_t \tilde{\pi}_{t+1} = \rho_u \tilde{\pi}_t \]

Plug this in to the rule above, substituting out current inflation:

\[ \tilde{i}_t = \tilde{\pi}_t^f + \left( \rho_u + \frac{\sigma \gamma}{\omega} (1 - \rho_u) \right) \frac{1}{\rho_u} E_t \tilde{\pi}_{t+1} \]

\[ \tilde{i}_t = \tilde{\pi}_t^f + \left( 1 + \frac{\sigma \gamma}{\rho_u \omega} (1 - \rho_u) \right) E_t \tilde{\pi}_{t+1} \]

It is apparent that \( 1 + \frac{\sigma \gamma}{\rho_u \omega} (1 - \rho_u) > 1 \) for any permissible value of the parameters. As discussed in Woodford (2003) Chapter 4 (in particular p. 257), this is a necessary but not sufficient condition for determinacy. The coefficient on inflation must also be less than \( 1 + \frac{2\sigma(1+\beta)}{\gamma} \). Since \( \gamma \) is small, this is very likely to hold. The solutions under either rule are identical; the only thing that argues in favor of the forward-looking rule is that it is
almost guaranteed to yield a determinant equilibrium, whereas it is very unlikely to have a

determinant equilibrium under the current inflation rule.

Now let’s switch gears and consider the problem under commitment. That is, instead

of solving the problem each period, the central bank solves the problem at the beginning

of time and commits to its rule. Furthermore, it takes into accounts its current choices on

future variables.

We solve the problem similarly to before, but now the objective function is a discounted

sum, not just the current period:

$$
\min \quad E_0 \frac{1}{2} \sum_{t=0}^{\infty} (\bar{\pi}_t^2 + \omega \bar{x}_t^2) 
$$

s.t.

$$
\bar{\pi}_t = \gamma \bar{x}_t + \beta E_t \bar{\pi}_{t+1} + \bar{u}_t 
$$

Write out a current value Lagrangian:

$$
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{2} \left( \bar{\pi}_t^2 + \omega \bar{x}_t^2 \right) + \mu_t \left( \gamma \bar{x}_t + \beta \bar{\pi}_{t+1} + \bar{u}_t - \bar{\pi}_t \right) \right\} 
$$

We have a slight complication in that there is a boundary issue – the central bank cannot

choose $E_{-1} \bar{\pi}_0$. Hence the first order condition for inflation in period 0 is just:

$$
\frac{\partial \mathcal{L}}{\partial \bar{\pi}_0} = 0 \iff -\bar{\pi}_0 = \mu_0 
$$

For periods after period 0, the first order condition is:

$$
\frac{\partial \mathcal{L}}{\partial \bar{\pi}_t} = 0 \iff \beta^{t-1} \beta \mu_{t-1} - \beta^t \bar{\pi}_t - \beta^t \mu_t = 0 \quad \forall \quad t > 0
$$

$$
\bar{\pi}_t = \mu_{t-1} - \mu_t 
$$

The first order condition for the output gap is identical to what obtained under the case

of no commitment:

$$
\frac{\partial \mathcal{L}}{\partial \bar{x}_t} = 0 \iff \bar{x}_t = \frac{\gamma}{\omega} \mu_t 
$$

Eliminating the Lagrange multipliers, we get:

$$
\bar{x}_0 = -\frac{\gamma}{\omega} \bar{\pi}_0 
$$

$$
\bar{x}_t = \frac{\gamma}{\omega} \mu_{t-1} - \frac{\gamma}{\omega} \bar{\pi}_t 
$$

Note that $\mu_{t-1} = \frac{\omega}{\gamma} \bar{x}_{t-1}$, so the second first order condition becomes:
\[ \tilde{x}_t = \tilde{x}_{t-1} - \frac{\gamma}{\omega} \tilde{\pi}_t \]

We can this combine the period 0 constraint with the remainder of the constraints to get:

\[ \tilde{x}_1 = \tilde{x}_0 - \frac{\gamma}{\omega} \tilde{\pi}_1 = -\frac{\gamma}{\omega} \tilde{\pi}_0 - \frac{\gamma}{\omega} \tilde{\pi}_1 = -\frac{\gamma}{\omega} (\tilde{\pi}_0 + \tilde{\pi}_1) \]
\[ \tilde{x}_2 = \tilde{x}_1 - \frac{\gamma}{\omega} \tilde{\pi}_2 = -\frac{\gamma}{\omega} (\tilde{\pi}_0 + \tilde{\pi}_1) - \frac{\gamma}{\omega} \tilde{\pi}_2 = -\frac{\gamma}{\omega} (\tilde{\pi}_0 + \tilde{\pi}_1 + \tilde{\pi}_2) \]
\[ \tilde{x}_t = -\frac{\gamma}{\omega} \sum_{j=0}^{t} \tilde{\pi}_{t-j} \]

Now what is the sum of inflation rates? It’s just the price level relative to its initial position before time began! \[ \tilde{\pi}_t = \tilde{p}_t - \tilde{p}_{t-1} \]. Hence \[ \sum_{j=0}^{t} \tilde{\pi}_{t-j} = \tilde{p}_t - \tilde{p}_{t-1} \]. Normalizing \( \tilde{p}_{t-1} = 0 \), we get the optimality condition:

\[ \tilde{x}_t = -\frac{\gamma}{\omega} \tilde{p}_t \]

In other words, the central bank allows the output gap to contract if the price level is above an implicit target, where the implicit target is the price level that existed before the solution to the problem was carried out. Hence, under commitment there is an inverse relationship between the output gap and the price level, whereas under discretion the optimal policy leads to an inverse relationship between the output gap and inflation.

As before, we are now interested in characterizing the kind of rule that will implement this policy. Go to the Euler/IS equation:

\[ \tilde{x}_t = E_t \tilde{x}_{t+1} - \frac{1}{\sigma} \left( \tilde{r}_t - \tilde{r}_t^f \right) \]

Plug in the first order condition for the output gap:

\[ -\frac{\gamma}{\omega} \tilde{p}_t = -\frac{\gamma}{\omega} E_t \tilde{p}_{t+1} - \frac{1}{\sigma} \left( \tilde{r}_t - \tilde{r}_t^f \right) \]

Simplify:

\[ \tilde{r}_t = \tilde{r}_t^f - \frac{\sigma \gamma}{\omega} E_t \tilde{p}_{t+1} + \frac{\sigma \gamma}{\omega} \tilde{p}_t \]
\[ \tilde{r}_t = \tilde{r}_t^f - \frac{\sigma \gamma}{\omega} E_t \tilde{\pi}_{t+1} \]

Now use the Fisher relationship to write this in terms of the nominal interest rate:

\[ \tilde{r}_t = \tilde{r}_t^f + \left( 1 - \frac{\sigma \gamma}{\omega} \right) E_t \tilde{\pi}_{t+1} \]

Note that we are not done yet . . . this cannot be the rule because we have not ensured that the solution is consistent with the Phillips Curve. So go to the Phillips Curve now:
\[ \tilde{\pi}_t = \gamma \tilde{x}_t + \beta E_t \tilde{\pi}_{t+1} + \tilde{u}_t \]

Plug in the first order condition for the output gap:

\[ \tilde{\pi}_t = -\frac{\sigma \gamma^2}{\omega} \tilde{p}_t + \beta E_t \tilde{\pi}_{t+1} + \tilde{u}_t \]

Now solve for expected inflation:

\[ E_t \tilde{\pi}_{t+1} = \frac{1}{\beta} \tilde{\pi}_t + \frac{\sigma \gamma^2}{\beta \omega} \tilde{p}_t - \frac{1}{\beta} \tilde{u}_t \]

Now plug this back into the expression for the nominal interest rate:

\[ \tilde{i}_t = \tilde{r}_t^f + \left( 1 - \frac{\sigma \gamma}{\omega} \right) \left( \frac{1}{\beta} \tilde{\pi}_t + \frac{\sigma \gamma^2}{\beta \omega} \tilde{p}_t - \frac{1}{\beta} \tilde{u}_t \right) \]

Simplify:

\[ \tilde{i}_t = \tilde{r}_t^f + \left( 1 - \frac{\sigma \gamma}{\omega} \right) \frac{1}{\beta} \tilde{\pi}_t + \left( 1 - \frac{\sigma \gamma}{\omega} \right) \frac{\sigma \gamma^2}{\beta \omega} \tilde{p}_t - \left( 1 - \frac{\sigma \gamma}{\omega} \right) \frac{1}{\beta} \tilde{u}_t \]

This rule says that the central bank should move the nominal interest rate one for one with fluctuations in the natural rate of interest, and should move the nominal rate with movements in current inflation, the current price level, and the current cost-push shock. Whether that movement is up or down depends upon the sign of \( 1 - \frac{\sigma \gamma}{\omega} \). For this to be positive, it must be the case that \( \omega > \sigma \gamma \). It’s also the case that a necessary and sufficient condition for determinacy is that the coefficient on the price level be positive (just positive . . . doesn’t need to be greater than unity . . . see Woodford Chapter 4), so we need this condition to be satisfied. Note that this determinacy condition is exactly the opposite of what we needed above in the case of discretion.

As a benchmark I calibrate \( \omega = 1 \), so that \( 1 - \frac{\sigma \gamma}{\omega} = 0.89 \), ensuring determinacy. The impulse responses to both shocks under this rule are shown below:
First of all, we see that the responses to the “real shock” are identical as they have been before. Whether there is commitment or discretion, it is always optimal and possible for the central bank to completely stabilize both inflation and the output gap in response to a
shock which moves the natural rate of interest. As before, the central bank allows inflation to rise and the output gap to go negative in response to a cost push shock.

It is very difficult to tell how these responses differ from the case without commitment, however. To make that point somewhat clearer, I show the impulse responses using the same parameters under different rules (the forward-looking rule in the case of discretion, this rule in the case of commitment). See below. The Matlab file is called comparison_commitvdiscretion.m.

![Inflation to Cost Push Shock](image1)
![Output Gap to Cost Push Shock](image2)
![Price Level to Cost Push Shock](image3)

I only show the responses to the cost push shock because the responses to the natural rate shock are identical. We see that the impulse responses of inflation in the case of commitment lies everywhere below the response under discretion. While the output gap decline on impact is smaller with commitment than with discretion, it is more persistent in the case of commitment. The biggest difference is in the response of the price level. Under commitment, where the central bank obeys an implicit price level target, the price level eventually returns to where it started. Under the case of discretion, however, the price level responds permanently. This is actually fairly intuitive. Under commitment, the central bank has to “correct” for past “mistakes” and therefore contracts the economy long after the cost-push shock is gone in order to get the price level back to target. Under the case of discretion, which is effectively an inflation target, inflation being above target in the past
has no bearing on what the central bank does today, so there is no “correction” off in the future.

It is not immediately obvious from looking at these pictures that welfare is higher under the case of commitment. It turns out that it is, however. The expected value of the objective function in response to a cost push shock (the only thing which allows the objective function to differ from zero). In the case of discretion, the minimized value of the objective function (times 1000) is 0.1174. With commitment, the minimized value is 0.0714. In other words, there is almost a 40 percent welfare gain from commitment.

We can make the following conclusions from our analysis so far:

- Without “cost-push” shocks distorting the Phillips Curve relationship, the central bank faces no inflation-output trade-off. It can perfectly stabilize both by following a rule in which it adjusts the nominal interest rate one for one with fluctuations in the natural rate of interest and promising to raise interest rates more than one for one in response to changes in inflation (which never actually happens in equilibrium). Since the objective function can be set to zero, there is no gain from commitment vs. discretion

- Introduction of “cost-push” shocks introduces a trade-off between inflation and the output gap. Under discretion – which amounts to the central bank re-optimizing period by period and taking future values of endogenous variables as given – the optimal rule is an implicit inflation target. If inflation goes up due to cost push shocks, the central bank “leans against the wind” by contracting the output gap. An interest rate rule can be derived that is similar to what we had in the case of no-cost push shocks, though parameter restrictions need to be made in order to ensure determinacy of equilibrium. Alternatively, we could write the rule in terms of the central bank moving the nominal rate one for one with the natural rate and reacting more than one for one to expected inflation.

- There are gains to be had from commitment vs. discretion upon the introduction of cost push shocks. The optimal rule is an implicit price level target as opposed to inflation target. In order for a determinate rational expectations equilibrium, however, we needed a sufficiently strong weight on the output gap in the objective function.

All of these rules require that the central bank move the nominal interest rate one for one with the natural rate of interest; the rule from commitment requires the central bank to also respond directly to the cost-push shock itself. The natural rate of interest is a theoretical construct; it is not directly observable. This is somewhat problematic for the implementation of these rules.

A simpler alternative is a “Taylor Rule” named after John Taylor (1993). It takes the following form:

$$\tilde{i}_t = i^* + \theta_\pi \tilde{\pi}_t + \theta_x \tilde{x}_t$$

\(i^*\) is the steady state nominal interest rate consistent with the long run inflation target, which in this case is zero (the constant here disappears in the linearization). Taylor suggested values of \(\theta_\pi = 1.5\) and \(\theta_x = 0.5\). While cosmetically similar to the rules discussed above,
the main difference is the constancy of the intercept here as opposed to the time-varying intercept reflecting fluctuations in the natural rate of interest above. Here are the impulse responses from the model with this rule:

Here, the optimized value of the objective function (times 1000) is 0.1265, compared with 0.1174 to the optimal policy with discretion and 0.0714 with commitment. Put differently,
the welfare loss from following the “simple” Taylor rule is relatively small, especially compared to the rule with discretion. The extent to which these differ will depend upon the extent to which the natural rate of interest moves around. If real shocks are very persistent, then fluctuations in the natural rate of interest are very small, and a constant intercept rule like this is pretty close to the optimal rule in which the intercept moves with the natural rate.