Instructions You may work on this problem set in groups of up to four people. Should you choose to do so, please make sure to legibly write each group member’s name on the first page of your solutions. This problem set is due in class on Thursday, September 12.

(1) Production Function Math: This problem asks you to work through some properties of production functions. In what follows, $A$ is an exogenous productivity term, $K_t$ is physical capital, $N_t$ is labor input, and $Y_t$ is output. Greek letters are parameters.

(a) $Y_t = A (\alpha K_t + (1 - \alpha)N_t), \ 0 < \alpha < 1$

(i) Does this production function have constant returns to scale?

(ii) Calculate the first partial derivatives of $Y_t$ with respect to $K_t$ and $N_t$, e.g. $\partial Y_t / \partial K_t$ and $\partial Y_t / \partial N_t$.

(iii) Calculate the own second derivatives, e.g. $\partial^2 Y_t / \partial K_t^2$ and $\partial^2 Y_t / \partial N_t^2$. Is this production function concave?

(iv) Calculate the first cross-partial derivative, e.g. $\partial^2 Y_t / \partial K_t \partial N_t$ (take the derivative with respect to $N_t$ of the first derivative with respect to $K_t$. . . the “order” in which you calculate partial derivatives does not matter).

(b) $Y_t = AK_t^{\alpha} N_t^{1-\alpha}, \ 0 < \alpha < 1$

(i) Repeat steps (i)-(ii) from above for this production function.

(ii) Show that you can write the first partial derivatives as functions of the output to factor ratios; e.g. the partial with respect to $K_t$ can be written as a function of $\frac{Y_t}{K_t}$, and similarly for $N_t$.

(c) $Y_t = A \left( \alpha K_t^{-\gamma} + (1 - \alpha)N_t^{-\gamma} \right)^{\frac{1}{\gamma}}, \ 0 < \alpha < 1, \ \gamma \geq 0$

(i) Repeat steps (i)-(ii) from part (a) for this production function.

(ii) As $\gamma \to \infty$, what does this production function become?

(iii) Look at your first derivatives of this production function. When $\gamma \to 1$, what do these derivatives look like? Hint: try to write the derivatives in terms of output/factor ratios, as in (ii) of (b). What can you intuit about the form of this production function as $\gamma \to 1$?

(2) The Golden Rule: Suppose that we have a standard Solow model. There is no population or technology growth.

(a) The firm problem is to maximize profits, where profits are given by:

$$\Pi_t = AK_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t K_t$$
The firm gets to choose \( N_t \) and \( K_t \), and takes the factor prices as given. Find the first order conditions characterizing optimal firm behavior.

(b) Use the first order conditions from (a) to show that \( \frac{w_t N_t}{Y_t} = 1 - \alpha \), where \( Y_t = AK_t^\alpha N_t^{1-\alpha} \).

(c) The household is endowed with labor and owns the capital stock and leases it to firms on a period-by-period basis. Firms remit any profits back to households. The household budget constraint is given by:

\[
C_t + I_t = w_t N_t + R_t K_t + \Pi_t
\]

Use your answers from above to show that the right hand side reduces to \( Y_t = AK_t^\alpha N_t^{1-\alpha} \).

(d) The capital accumulation equation is standard:

\[
K_{t+1} = I_t + (1 - \delta)K_t
\]

Where \( K_t \) is given in period \( t \), as it is inherited from past decisions. Assume that the household consumes a constant fraction of its income each period, \( 1 - s \), and supplies labor inelastically, with \( N_t = 1 \). Re-write the capital accumulation equation as a difference equation relating \( K_{t+1} \) to \( K_t \) and exogenous variables and parameters only.

(e) Create a graph plotting \( K_{t+1} \) against \( K_t \). Argue that there exists a steady state, \( K^* \), at which \( K_{t+1} = K_t \).

(f) Algebraically solve for the steady state capital stock, \( K^* \), as well as steady state output, \( Y^* \), and consumption, \( C^* \).

(g) What value of \( s \) would maximize \( Y^* \)? Do you think the household would like this saving rate? Why or why not?

(h) What value of \( s \) would maximize current \( C_t \)? Do you think it would be a good thing to have this saving rate?

(i) What value of \( s \) would maximize steady state consumption, \( C^* \)? Please derive an analytical expression for the \( s \) that makes \( C^* \) as big as possible.

(j) A reasonable value for \( \alpha \) is 0.33. With a saving rate between 10-15 percent, would the US be near the “Golden Rule” saving rate you found in (i)? If not, would you necessarily recommend that we increase our saving rate?