Instructions You may work on this problem set in groups of up to four people. Should you choose to do so, please make sure to legibly write each group member’s name on the first page of your solutions. This problem set is due in class on Thursday, September 13.

(1) The Golden Rule: Suppose that we have a standard Solow model. There is no population or technology growth.

(a) The firm problem is to maximize profits, where profits are given by:
\[ \Pi_t = AK_t^\alpha N_t^{1-\alpha} - w_t N_t - R_t K_t \]
The firm gets to choose \( N_t \) and \( K_t \), and takes the factor prices as given. Find the first order conditions characterizing optimal firm behavior.

(b) Use the first order conditions from (a) to show that \( \frac{w_t N_t}{Y_t} = 1 - \alpha \), where \( Y_t = AK_t^\alpha N_t^{1-\alpha} \).

(c) The household is endowed with labor and owns the capital stock and leases it to firms on a period-by-period basis. Firms remit any profits back to households. The household budget constraint is given by:
\[ C_t + I_t = w_t N_t + R_t K_t + \Pi_t \]
Use your answers from above to show that the right hand side reduces to \( Y_t = AK_t^\alpha N_t^{1-\alpha} \).

(d) The capital accumulation equation is standard:
\[ K_{t+1} = I_t + (1 - \delta)K_t \]
Where \( K_t \) is given in period \( t \), as it is inherited from past decisions. Assume that the household consumes a constant fraction of its income each period, \( 1 - s \), and supplies labor inelastically, with \( N_t = 1 \). Re-write the capital accumulation equation as a difference equation relating \( K_{t+1} \) to \( K_t \) and exogenous variables and parameters only.

(e) Create a graph plotting \( K_{t+1} \) against \( K_t \). Argue that there exists a steady state, \( K^* \), at which \( K_{t+1} = K_t \).

(f) Algebraically solve for the steady state capital stock, \( K^* \), as well as steady state output, \( Y^* \), and consumption, \( C^* \).

(g) What value of \( s \) would maximize \( Y^* \)? Do you think the household would like this saving rate? Why or why not?
(h) What value of \( s \) would maximize current \( C_t \)? Do you think it would be a good thing to have this saving rate?

(i) What value of \( s \) would maximize steady state consumption, \( C^* \)? Please derive an analytical expression for the \( s \) that makes \( C^* \) as big as possible.

(j) A reasonable value for \( \alpha \) is 0.33. With a saving rate between 10-15 percent, would the US be near the “Golden Rule” saving rate you found in (i)? If not, would you necessarily recommend that we increase our saving rate?

(2) A Quantitative Solow Model Exercise: Consider a Solow model with both population and technological growth. Assume that the production function is Cobb-Douglas, as in (1), and let lowercase variables with a “hat” denote per efficiency units of labor variables, e.g. \( \hat{k}_t = K_t/\bar{N}_t \). The central Solow model equation can be written:

\[
(1 + g_z)(1 + g_n)\hat{k}_{t+1} = sA\hat{k}_t^\alpha + (1-\delta)\hat{k}_t
\]

(a) Analytically solve for expression for the steady state capital stock per effective workers, \( \hat{k}^* \), output per effective worker, \( \hat{y}^* \), and consumption per effective worker, \( \hat{c}^* \).

(b) In the steady state, what will be the growth rate of per capita capital, output, and consumption?

(c) Suppose that the values of the parameters in the model are as follows: \( g_n = 0.01, g_z = 0.02, \delta = 0.1, \alpha = 0.33, s = 0.2, A = 1 \). Create an Excel spreadsheet, with rows corresponding to time periods (e.g. 1, 2, etc.) and columns corresponding to variables. Have \( \hat{k}, \hat{y}, \) and \( \hat{c} \) sit in their steady states for periods 1-9. In period 10, suppose that the saving rate increases to \( s = 0.25 \). Use your Excel spreadsheet to compute the time paths of \( \hat{k}, \hat{y}, \) and \( \hat{c} \) following this change (go up to period 100). Plot these time paths out in separate plots (for periods 1-100).

(d) Approximately how many periods does it take \( \hat{k} \) to get one-half of the way to its new steady state?

(e) Calculate the growth rate of \( \hat{y}_t \) for each period of the simulation (you don’t have an observed growth rate in the first period) using the log first difference approximation. Plot the simulated time path of this growth rate.

(f) Re-do exercises (c)-(e), but this time assume that \( \alpha = 0.67 \). Describe any differences, paying particular attention to the exercises in (d) and (e).

(g) Given your answers on (f), speculate intelligently on how the desirability of trying to encourage a higher saving rate depends on the value of \( \alpha \).