(1) Basic Consumer Theory: Suppose that a consumer lives for two periods and has lifetime utility given by:

\[ U = \ln C_t + \beta \ln C_{t+1} \]

The household faces a sequence of budget constraints, where \( Y_t \) and \( Y_{t+1} \) are exogenously given and the household takes the real interest rate, \( r_t \), as given:

\[
\begin{align*}
C_t + S_t &= Y_t \\
C_{t+1} &= Y_{t+1} + (1 + r_t)S_t
\end{align*}
\]

(a) Explain in words why \( S_{t+1} = 0 \), and therefore does not appear in the second period budget constraint.

(b) What are the units of the real interest rate?

(c) Combine the two period budget constraints into one unified budget constraint.

(d) Derive an optimality condition that characterizes optimal behavior (i.e., the Euler equation).

(e) Combine this Euler equation with the budget constraint to solve for an analytic expression for the consumption function – i.e. \( C_t \) as a function of \( Y_t, Y_{t+1}, \) and \( r_t \).

(f) Take the partial derivatives of your consumption function with respect to \( Y_t, Y_{t+1}, \) and \( r_t \). Show them here, and discuss if there is any ambiguity in the sign.

(g) Suppose that \( Y_t = 1, Y_{t+1} = 1, r_t = 0.05, \) and \( \beta = 0.95 \). Use your derived consumption function and Euler equation to numerically solve for the optimal levels of current and future consumption, \( C_t \) and \( C_{t+1} \), as well as the total level of utility associated with this consumption bundle. Do this in Excel, which you will need for the next part.

(h) Numerically create indifference curves in Excel. You can do this by following these steps:

1. Create a “grid” of values for \( C_t \). Start at 0.5, and go up to 1.5, with a space between each point in the grid of 0.01.

2. Start with the level of lifetime utility you found in part (g).
3. Solve for the level of $C_{t+1}$ that achieves that fixed level of utility for the different levels of $C_t$ you created in 1. For example, suppose that you want to find the $C_{t+1}$ that generates a level of utility of 1, with a value of first period consumption of 1.5. You would solve the following equation for $C_{t+1}$:

$$1 = \ln(1.5) + \beta \ln C_{t+1}$$

Doing so, you get:

$$\ln C_{t+1} = \frac{1}{\beta} (1 - \ln(1.5))$$

Take the exponential of both sides to get this in terms of the level of $C_{t+1}$:

$$\exp(\ln C_{t+1}) = C_{t+1} = \exp\left(\frac{1}{\beta} (1 - \ln(1.5))\right)$$

Do this for every point in the grid for $C_t$ that you created. You should then be able to plot $C_{t+1}$ against $C_t$. This is the indifference curve associated with the optimized level of utility you found in part (g).

4. Repeat this for levels of utility $+/- 0.5$ from the optimized level of utility from part (g). Plot all three indifference curves together. Verify that they have the right-looking shape and that they do not cross, with the indifference curves associated with higher levels of utility to the “northeast” of indifference curves associated with lower levels of utility.

(i) Numerically create a budget line. To do this, simply solve for the $C_{t+1}$ that exhausts resources in the intertemporal budget constraint for the different values of $C_t$ in the grid that you created in the previous problem. Plot the budget line along with the three indifference curves, and visually verify that the indifference curve associated with the optimal level of utility is tangent to the budget line at the optimal $(C_t, C_{t+1})$ bundle.

(j) Suppose that current income, $Y_t$, increases from 1 to 1.1. Use Excel to calculate a new numeric level of current consumption with this new level of income. Compute the numeric change in consumption divided by the change in income, and compare it to the analytic expression from the partial derivative you found in (f).

(k) Revert to assuming that $Y_t = 1$. Suppose that $r_t$ increases from 0.05 to 0.07. Plot a new budget line, and comment on how it looks different.

(l) Numerically compute the new optimal level of $C_t$ with this new higher interest rate. Compute the change in consumption divided by the change in the real interest rate. Compare the answer to the analytic expression you found in (f).

(2) Income Persistence: Assume the same setup of the problem from above. In particular, a household lives for two periods, takes current and future income as given, and takes the real interest rate as given. Your starting point for this problem is the consumption function you derived in part (e) above. Suppose that future income goes up by a fraction, $0 \leq \rho \leq 1$, of any change in current income. That is, assume that $dY_{t+1} = \rho dY_t$. Use the consumption function you derived to get an expression for $\frac{dC_{t+1}}{dY_t}$, the change in consumption for a change in current income. How does the magnitude of $\rho$ affect the magnitude of the consumption response to the income change? What is the intuition for this?