Problem Set 4
Graduate Macro II, Spring 2010
The University of Notre Dame
Professor Sims

Instructions: You may consult with other members of the class, but please make sure to turn in your own work. Where applicable, please print out figures and codes. This problem set is due by 12:00 noon on Friday, February 26.

(1) Comparing Solution Techniques: Deterministic Growth Model: In this problem you will compare policy functions from a deterministic neoclassical growth model using (a) value function iteration and (b) log-linearization.

The equilibrium of the economy can be described as the solution to the following social planner’s problem:

\[
\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma}
\]

s.t.

\[
k_{t+1} = k_t^\alpha - c_t + (1 - \delta)k_t \quad k_0 \text{ given}
\]

(a) Use L’Hopital’s rule to prove that, as \( \sigma \to 1 \), the within period utility function goes to \( \ln c_t \).

(b) Set this problem up as a dynamic programming problem. What is the state variable? What is the control variable? Write down the Bellman equation. Find the first order condition necessary for an optimal solution.

(c) Find an expression for the steady state capital stock and the steady state value of consumption.

(d) Suppose that the parameter values are as follows: \( \beta = 0.99, \alpha = 0.36, \sigma = 2, \) and \( \delta = 0.025 \). What are the numerical values of the steady state capital stock and consumption for these parameters? Write your own code to numerically solve for the value and policy functions. To do so, create a grid of the capital stock, with the minimum value 0.25 of the steady state capital stock and the maximum value 1.75 times the steady state capital stock, with 300 grid points between. Show a graph of both the final value function and the policy function.
(e) Now set up the problem using a Lagrangian. Write out the first order conditions, including the transversality condition. Provide a verbal explanation for the intuition behind the transversality condition.

(f) Log-linearize the first order conditions about the steady state. Form a VAR(1) of the form:

\[ X_{t+1} = MX_t \]

Where \( X_t \) contains the variables expressed as percentage deviations about the steady state. Write out an expression for \( M \).

(g) Solve for the linear policy function mapping the state variable into the jump variable. Write out the numerical policy function here.

(h) Show a plot of the linearized policy function and the policy function from the value function iteration procedure obtained above together. Be sure to transform the linearized policy function, which is expressed as a percentage deviation about the steady state, into actual levels so as to make the comparison appropriate. Comment on the quality of the linear approximation.

(i) Repeat the exercise in (h) for the following different values of \( \sigma \): 5 and 10. How does the quality of the linear approximation vary with \( \sigma \)? Why does this make sense?

(2) Linearizing the Stochastic Growth Model: Consider a similar problem as above, but now assume that there are stochastic shocks to the production function. The social planner’s problem can be written:

\[
\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}
\]

s.t.

\[
k_{t+1} = a_t k_t^\alpha - c_t + (1-\delta)k_t
\]

\( k_0 \) given

Assume that the technology shifter, \( a_t \), follows an AR(1) process in the log:

\[ \ln a_t = \rho \ln a_{t-1} + e_t \]

The shock is \( e_t \) is drawn from a white noise process. Assume that \( 0 < \rho < 1 \).

(a) What is the unconditional mean of log technology? What, therefore, is the mean of the level of log technology? What is the unconditional variance of log technology if the variance of the white noise process \( e_t \) is \( \sigma_e^2 \)?

(b) Write down the first order conditions necessary for the solution to this problem.
(c) Log-linearize the first order conditions about the steady state. Form a VAR(1) representation of the form:

\[ E_t X_{t+1} = MX_t \]

Write out the elements of \( M \) as a function of the deep parameters of the model.

(d) Solve for the linearized policy function mapping the states into jump variables. Assume the same parameter values as in (1), with \( \rho = 0.9 \). Assume the benchmark value \( \sigma = 2 \). Find log-linear policy functions for the "static" variables output and investment as well. Use the fact that output is given by \( y_t = a_t k_t^\alpha \) and investment by \( i_t = y_t - c_t \).

(e) Compute the impulse responses to a one unit shock to \( \ln a_t \) at time 0. Show graphs of the responses of consumption, the capital stock, technology, output, and investment to the shock for 20 periods.

(f) Simulate a data set with \( T = 1000 \) observations. Begin your simulation at the steady state, and draw technology shocks from a standard normal distribution with standard deviation of 0.1. HP filter the simulated data with smoothing parameter \( \lambda = 1600 \). Calculate the standard deviations (volatilities) of HP detrended consumption, output, and investment.

(g) Repeat the exercise in (f) with two different values of \( \rho \): \( \rho = 0.1 \) and \( \rho = 0.99 \). Compare the ratio of consumption and investment volatility to output volatility for the three different values of \( \rho \). Can you notice a pattern? Can you provide any economic intuition for why you see that pattern?

(3) A Decentralized Real Business Cycle Economy: In this problem you will solve for the decentralized equilibrium for a neoclassical growth economy with variable labor.

Households solve the following maximization problem period by period:

\[
\max_{c_t, l_t, b_{t+1}} \quad \sum_{j=0}^{\infty} \beta^j \left( \ln c_{t+j} + \psi \ln l_{t+j} \right)
\]

s.t.

\[
l_{t+j} = 1 - n_{t+j}
\]
\[
c_{t+j} + b_{t+j+1} = w_{t+j} n_{t+j} + (1 + r_{t+j}) b_{t+j} + \Pi_{t+j}
\]

The household derives utility from consumption and leisure, \( l_t \). The time endowment is normalized to unity, and households must split their time between leisure and labor, \( n_t \). \( b_t \) denotes the stock of (real) household saving (it can be positive or negative), with \( (1 + r_t) \) the (gross) real interest rate on saving. The stock of saving in period \( t \) is predetermined – households can only influence the future stock of saving with actions today. \( w_t \) is the real wage rate, and \( \Pi_t \) is dividends or other exogenous sources of household income.
(a) Find the first order conditions characterizing the solution to the household problem, including the transversality condition (hint: use the first constraint to eliminate leisure as a choice variable, instead writing the problem as one of choosing labor). Provide verbal intuition for the transversality condition.

Now consider the firm’s problem. Its objective is to maximize its value, which is equal to the present discounted value of profits. The firm hires labor in spot markets period by period taking the real wage as given. It owns its capital stock, and chooses investment each period. The relative price of investment to output is unity – an extra unit of investment reduces profits by one. Formally, the problem is:

\[
\max_{n_t, i_t, k_{t+1}} E_t \sum_{j=0}^{\infty} \prod_{k=0}^{j} (1 + r_{t+k})^{-1} \left( a_{t+j} k_{t+j}^\alpha n_{t+j}^{1-\alpha} - w_{t+j} n_{t+j} - i_{t+j} \right)
\]

s.t.

\[k_{t+j+1} = i_{t+j} + (1 - \delta)k_{t+j}\]

The \( \prod_{k=0}^{j} \) is the “product” operator. In long hand,

\[
\prod_{k=0}^{j} (1 + r_{t+k})^{-1} = \left( \frac{1}{1 + r_t} \right) \cdot \left( \frac{1}{1 + r_{t+1}} \right) \cdot \left( \frac{1}{1 + r_{t+2}} \right) \cdot \ldots \cdot \left( \frac{1}{1 + r_{t+j}} \right)
\]

It is straightforward then to see that:

\[
\sum_{j=0}^{\infty} \prod_{k=0}^{j} (1 + r_{t+k})^{-1} = \left( \frac{1}{1 + r_t} \right) + \left( \frac{1}{1 + r_t} \right) \cdot \left( \frac{1}{1 + r_{t+1}} \right) + \left( \frac{1}{1 + r_t} \right) \cdot \left( \frac{1}{1 + r_{t+1}} \right) \cdot \left( \frac{1}{1 + r_{t+2}} \right) + \ldots
\]

If the interest rate is not varying over time, it is straightforward then to see that this reduces to the “usual” discounting rule.

(b) Find the first order conditions necessary for an interior solution to the firm’s problem.

(c) Combine the first order conditions for the firm’s problem with the first order conditions for the household’s problem to eliminate \( w_t \) and \( r_{t+1} \).

(d) In US data people seem to spend about 20 percent of their time endowment working (there are 168 hours in a week, and the average workweek is about 36 hours). This implies that \( n^* = 0.2 \). Assuming no trend growth in \( a_t \), use this fact to find expressions for the steady state capital stock, consumption, and output as functions of the deep parameters of the model.

(e) The Frisch labor supply elasticity is defined as the elasticity of employment with respect to the real wage holding the level of consumption fixed. In particular, define:
What is the Frisch elasticity evaluated at the steady state, assuming that $n^* = 0.2$?

(f) Microeconomists often argue that the labor supply elasticity is roughly zero. They are usually talking about an uncompensated elasticity – i.e. the elasticity of employment with respect to the real wage *not* holding consumption fixed. Derive an expression for the uncompensated elasticity of labor supply (evaluated at the steady state) as a function of the elasticity of consumption with respect to the real wage. What must be true about the elasticity of consumption with respect to the real wage for the uncompensated elasticity of labor supply to be zero? Use your intuition from the permanent income hypothesis to discuss whether temporary or permanent changes in real wages are likely to have bigger effects on employment.