Problem Set 4

Graduate Macro II, Spring 2013
The University of Notre Dame
Professor Sims

Instructions: You may consult with other members of the class, but please make sure to turn in your own work. Where applicable, please print out figures and codes. This problem set is due in class on Thursday, March 7.

(1) ABCs (and Ds) of DSGE Models and VARs: Consider a standard stochastic growth model. A planner solves the following dynamic problem:

$$\max_{C_t, K_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0$$

s.t.

$$K_{t+1} = A_t K_t^{\alpha} - C_t + (1-\delta)K_t$$

K_0 is given and it is assumed that A_t follows a stationary, mean zero AR(1) in the log:

$$\ln A_t = \rho \ln A_{t-1} + e_t, \;\; 0 < \rho < 1, \;\; e_t \sim N(0, \sigma_e^2)$$

(a) Find the first order conditions necessary for an optimal solution to the planner's problem.

(b) Find analytic expressions for the non-stochastic steady state values of C^* and K^*.

(c) Log-linearize the first order conditions of the system about the non-stochastic steady state. Form a first order systme of difference equations, $E_t \mathbf{X}_{t+1} = \mathbf{M}\mathbf{X}_t$, where $\mathbf{M}$ is a function of the parameters of the model. Assume parameter values of: $\alpha = 1/3$, $\sigma = 1$, $\delta = 0.025$, $\beta = 0.99$, $\rho = 0.95$, and $\sigma_e^2 = 0.01^2$. Use the method outlined in class to find the linearized policy function mapping the states into the jump variable in Matlab. The policy function should take the form:

$$\widetilde{C}_t = \mathbf{\Phi} \begin{bmatrix} \widetilde{K}_t \\ \widetilde{A}_t \end{bmatrix}$$

There are multiple different ways to write the state space of the model. One way uses coefficient matrixes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, and writes the system as follows:[1]

$$\begin{bmatrix} \widetilde{K}_t \\ \widetilde{A}_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} \widetilde{K}_{t-1} \\ \widetilde{A}_{t-1} \end{bmatrix} + \mathbf{B} e_t$$

$$\widetilde{C}_t = \mathbf{C} \begin{bmatrix} \widetilde{K}_{t-1} \\ \widetilde{A}_{t-1} \end{bmatrix} + \mathbf{D} e_t$$

[1] In what follows, note that e_t is 1×1, so the dimension of $\mathbf{B}$ is 2×1. One could alternative define a shock vector to be 2×1, with a 0 in the (1,1) element since the capital stock is not affected by the productivity shock. In this case $\mathbf{B}$ would be 2×2.

It may be more convenient to define $\mathbf{X}_{2,t} = \begin{bmatrix} \widetilde{K}_t \\ \widetilde{A}_t \end{bmatrix}$ (the state vector). Under this notation, the state space model can be written:

$$\mathbf{X}_{2,t} = \mathbf{A}\mathbf{X}_{2,t-1} + \mathbf{B}e_t \quad (1)$$
$$\widetilde{C}_t = \mathbf{C}\mathbf{X}_{2,t-1} + \mathbf{D}e_t \quad (2)$$

(d) Find expression for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ in terms of $\mathbf{M}$ and your numeric policy function, $\mathbf{\Phi}$.

(e) Solve for e_t in terms of $\widetilde{C}_t$ and $\mathbf{X}_{2,t-1}$ from (2). Plug this into (1), eliminating e_t.

(f) Using this modification from (e), solve the modified equation (1) "backwards." You should be able to express $\mathbf{X}_{2,t}$ as a linear function of $X_{2,0}$ (i.e. the state at the beginning of time) and the observed values of $\widetilde{C}_{t-j}$, for $j = 0, ..., t$ (i.e. the current and lagged values of consumption going back to the "beginning of time," which is period 0).

(g) For the expression derived in (f), provide a condition for the coefficient matrix on the initial state vector to go to zero as the sample size gets big (i.e. as $t \to \infty$). Numerically verify that this condition is satisfied for this calibration of the model.

(h) Use your result from (g) to show that you can write $\widetilde{C}_{1,t}$ as a AR(∞) of the form:

$$\widetilde{C}_t = \sum_{j=1}^{\infty} \rho_j \widetilde{C}_{t-j} + \mathbf{D}e_t$$

Provide analytic expressions for the coefficients, ρ_j. These analytic expressions will be functions of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, and j. You can then evaluate these numerically using the calibrated values of the deep parameters, which will yield quantitative measures of the ρ_j. HINT: you won't have an infinite number of different coefficients to compute, there should be a pattern that emerges.

(i) Simulate data from the model, drawing the e_t shocks from a standard normal distribution with the variance as calibrated above. Start the simulation in the steady state (initial value of $\widetilde{K}_0 = 0$), and simulate for $T = 1000$ periods. Estimate an AR(p) on the simulated consumption data for different values of p: $p = 1$, $p = 2$, $p = 4$, $p = 8$, and $p = 12$. Note that the *true* AR representation has an infinite number of coefficients, but you are estimating a finite number of lags. Comment on how closely your estimated AR coefficients correspond to the true AR coefficients computed above in (h), and how the closeness of these estimates depends on the lag order, p.

(j) Briefly outline (you do not have to actually do this) an algorithm for recovering estimates of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ after the estimation of the AR(p) on consumption. How many lags p would you have to include to be able to get unique estimates of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$?

(2) A Simple Real Business Cycle Model: Consider a simple RBC model, the solution of which can be characterized as the solution to a planner's problem:

$$\max_{C_t, K_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \psi \frac{N_t^{1+\phi}}{1+\phi} \right), \quad \sigma > 0, \psi > 0, \phi \geq 0$$

s.t.

$$K_{t+1} = A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta)K_t$$

There is no trend population or productivity growth. K_0 is given and it is assumed that A_t follows a stationary, mean zero AR(1) in the log:

$$\ln A_t = \rho \ln A_{t-1} + e_t, \;\; 0 < \rho < 1, \;\; e_t \sim N(0, \sigma^2)$$

(a) Set up a Lagrangian and find the first order conditions of the model.

(b) Solve for the non-stochastic steady state of the variables of the model.

(c) Log-linearize the conditions characterizing the equilibrium of the model about the steady state.

(d) Use one of the linearized first order conditions to eliminate $\widetilde{N}_t$ from the equilibrium conditions.

(e) Use numeric values of the coefficients of $\beta = 0.99$, $\delta = 0.025$, $\psi = 2$, $\phi = 1$, $\alpha = 0.33$, $\rho = 0.95$, and $\sigma_e^2 = 0.01^2$. Write the modified equilibrium conditions in matrix form as $E_t \mathbf{X}_{t+1} = \mathbf{M}\mathbf{X}_t$, where $\mathbf{X}_t = [\widetilde{C}_t \; \widetilde{K}_t \; \widetilde{N}_t]'$. Use Matlab to find the numeric policy function mapping the state vector into the control, e.g. $\widetilde{C}_t = \mathbf{\Phi}[\widetilde{K}_t \; \widetilde{A}_t]'$.

(f) Use your policy function from (e) and your condition from (d) to derive a policy function for $\widetilde{N}_t$, e.g. $\widetilde{N}_t = \mathbf{\Phi}_N[\widetilde{K}_t \; \widetilde{A}_t]'$.

(g) Compute impulse responses functions of all the variables of the model to a one standard deviation productivity shock. Show these impulse response functions over a 40 period horizon.

(h) Re-do the impulse responses for different values of ϕ: 0.25 and 3. Comment on how the impulse responses are different relative to the baseline case in (g).

(i) Re-do the impulse responses for different values of ρ: 0.5 and 0.99 (hold ϕ at its original value of 1). comment on how the impulse responses are different relative to the baseline case in (g).