Problem Set 5

Instructions: You may consult with other members of the class, but please make sure to turn in your own work. Where applicable, please print out figures and codes. This problem set is due in class on Tuesday, March 26.

(1) A Two Sector Real Business Cycle Model: Consider a two-sector version of an otherwise standard real business cycle model. The two sectors are consumption goods and investment goods, both of which are produced using capital and labor. Capital is the existing stock of previously produced (and non-depreciated) investment goods. The total capital stock is predetermined, but capital can flow freely between the consumption and investment goods sectors within period.

The production of new consumption and investment goods are given by the following Cobb-Douglas production technologies, with common share parameter, $0 < \alpha < 1$:

$$C_t = A_{c,t} K_{c,t}^{\alpha} N_{c,t}^{1-\alpha}$$
$$I_t = A_{i,t} K_{i,t}^{\alpha} N_{i,t}^{1-\alpha}$$

$A_{c,t}$ and $A_{i,t}$ are productivity shifters in each sector; $K_{j,t}$, $j = c, i$ is the capital used in sector j in period t; and $N_{j,t}$, $j = c, i$ is the labor devoted to sector j in period t. Total capital and labor must be split between the two sectors each period: $K_t = K_{c,t} + K_{i,t}$ and $N_t = N_{c,t} + N_{i,t}$.

The overall capital stock accumulates according to the standard law of motion:

$$K_{t+1} = I_t + (1 - \delta) K_t$$

The household has flow utility from consuming and supplying labor:

$$U(C_t, N_t) = \ln C_t - \psi \frac{N_t^{1+\phi}}{1 + \phi}, \quad \phi \geq 0$$

The household discounts future utility flows with discount factor β, with $0 < \beta < 1$.

Given the current capital stock and the values of the productivity shifters, the planner's problem is to choose consumption, labor supply, the distribution of capital within sectors, the distribution of labor within sectors, and future capital:

$$\max_{C_t, N_t, N_{c,t}, N_{i,t}, K_{c,t}, K_{i,t}, K_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \psi \frac{N_t^{1+\phi}}{1 + \phi} \right)$$

s.t.

$$C_t = A_{c,t} K_{c,t}^{\alpha} N_{c,t}^{1-\alpha}$$
$$K_{t+1} = A_{i,t} K_{i,t}^{\alpha} N_{i,t}^{1-\alpha} + (1 - \delta) K_t$$
$$K_t = K_{c,t} + K_{i,t}$$
$$N_t = N_{c,t} + N_{i,t}$$

(a) Use a Lagrangian to find the first order conditions of the planner's problem.

(b) Use these first order conditions to argue that the capital labor ratios in each sector must be equal to one another, e.g. $\frac{K_{c,t}}{N_{c,t}} = \frac{K_{i,t}}{N_{i,t}}$, which must in turn be equal to the aggregate capital labor ratio, $\frac{K_t}{N_t}$.

(c) Suppose, for this part only, that $A_{c,t} = A_{i,t} = A_t$. Under this restriction, show that the first order conditions reduce to the standard first order condition of a one sector real business cycle, and that there exists an aggregate production function of the form $Y_t = C_t + I_t = A_t K_t^\alpha N_t^{1-\alpha}$.

Drop the assumption that $A_{i,t} = A_{c,t} = A_t$. Suppose that $A_{i,t}$ and $A_{c,t}$ obey the following stochastic processes:

$$A_{c,t} = A_{c,t}^T A_{c,t}^S$$
$$A_{i,t} = A_{i,t}^T A_{i,t}^S$$

$A_{c,t}^T$ and $A_{i,t}^T$ are the trend components, while $A_{c,t}^S$ and $A_{i,t}^S$ are stationary components. The trend components are deterministic and obey:

$$A_{c,t+1}^T = (1 + g_c) A_{c,t}^T$$
$$A_{i,t+1}^T = (1 + g_i) A_{i,t}^T$$

The stationary terms, $A_{c,t}^S$ and $A_{i,t}^S$, share a common component, a_t, and have idiosyncratic components as well, $a_{c,t}$ and $a_{i,t}$. The stochastic processes here are:

$$A_{c,t}^S = a_t a_{c,t}$$
$$A_{i,t}^S = a_t a_{i,t}$$

These processes follow stationary, mean zero AR(1) processes in natural logs (mean one in the level):

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_{a,t}$$
$$\ln a_{c,t} = \rho_c \ln a_{c,t-1} + \varepsilon_{c,t}$$
$$\ln a_{i,t} = \rho_i \ln a_{i,t-1} + \varepsilon_{i,t}$$

The shocks are drawn from mean zero normal distributions with known variances, σ_a^2, σ_c^2, and σ_i^2.

(d) Focus on the long run and balanced growth. This means we can effectively think of "shutting down" the stationary stochastic terms. Along a balanced growth path, all variables grow at a constant (though not necessarily the same) rate. Find the balanced growth path growth rates of C_t, I_t, K_t, N_t, $K_{c,t}$, $K_{i,t}$, $N_{c,t}$, and $N_{i,t}$.

(e) Find an expression for the growth rate of the ratio of investment to consumption along the balanced growth path?

Even though we've written the model down as a planner's problem, it is useful to consider, for a moment, a decentralized version of the model so that we can say something about the relative price of investment to consumption (which will be useful in a moment). You can think about a perfectly competitive investment goods firm and a perfectly competitive consumption goods firm, both of whom rent capital and labor in

competitive markets at common factor prices, R_t and w_t. The firms are price-takers, taking their own prices, $P_{i,t}$ and $P_{c,t}$, as given. The profit maximization problems for each firm are:

$$\max_{K_{i,t},N_{i,t}} \quad P_{i,t}A_{i,t}K_{i,t}^{\alpha}N_{i,t}^{1-\alpha} - R_tK_{i,t} - w_tN_{i,t}$$

$$\max_{K_{c,t},N_{c,t}} \quad P_{c,t}A_{c,t}K_{c,t}^{\alpha}N_{c,t}^{1-\alpha} - R_tK_{c,t} - w_tN_{c,t}$$

(f) Find the first order conditions characterizing the firms' optimization problems and use this to derive an expression for the relative price of investment to consumption.

(g) The nominal ratio of investment to consumption is given by: $\frac{P_{i,t}I_t}{P_{c,t}C_t}$. Use your expression from (f) to derive an expression for the growth rate of the nominal investment-consumption ratio along the balanced growth path.

(h) Download data from the BEA website on investment and consumption. Website is here: `http://www.bea.gov/iTable/index_nipa.cfm`. The BEA measures of consumption and investment do not perfectly correspond to the economic definitions of these terms. Define "consumption" as the sum of non-durable and services consumption, and "investment" as gross private fixed investment plus durable goods consumption. By their definition durable goods provide utility flows into the future, and so purchasing new durable goods is more like purchasing more capital than it is pure consumption (durable goods provide a *service flow* of consumption, but the BEA measure of purchases measures changes in the *stock* of durable goods, and should therefore be counted as investment). To construct the *nominal* levels of these one need only add up the "current dollar" values of these different components, which can be downloaded from Table 1.1.5 (billions of dollars, seasonally adjusted at annual rates). Download the data from 1947q1 - 2012q4. Show a plot of the ratio of nominal investment to consumption, and comment on whether this ratio appears to have a trend or not.

(i) Now download data on price deflators from the BEA (Table 1.1.4). You can download individual deflators for durable goods consumption, non-durable goods consumption, services consumption, and fixed investment. Construct real measures of each of these individual components by simply dividing by the individual deflators.

(j) While constructing real measures of these individual components is straightforward, it is not straightforward to construct real measures of the composite series – you cannot simply add up the real series if the nominal price ratios are not constant. To proceed, take the following steps:

- Construct the growth rates of the real series for each component, e.g. $\Delta C_{s,t} = \frac{C_{s,t}-C_{s,t-1}}{C_{s,t-1}}$ for services consumption.

- Define the growth rate of the composite consumption and investment series as the (lagged) *nominal share-weighted* real growth rates of the individual components. The real growth rate of non-durable and services consumption is (where the "s" subscript denotes services and the "nd" subscript denotes non-durables):

$$\Delta C_{nds,t} = \left(\frac{P_{s,t-1}C_{s,t-1}}{P_{s,t-1}C_{s,t-1} + P_{nd,t-1}C_{nd,t-1}}\right)\Delta C_{s,t} + \left(\frac{P_{nd,t-1}C_{nd,t-1}}{P_{s,t-1}C_{s,t-1} + P_{nd,t-1}C_{nd,t-1}}\right)\Delta C_{nd,t}$$

For investment, we have (where the "f" subscript denotes fixed investment while the "d" subscript denotes durables):

$$\Delta I_t = \left(\frac{P_{f,t-1}I_{f,t-1}}{P_{f,t-1}I_{f,t-1} + P_{d,t-1}C_{d,t-1}}\right)\Delta I_{f,t} + \left(\frac{P_{d,t-1}C_{d,t-1}}{P_{f,t-1}I_{f,t-1} + P_{d,t-1}C_{d,t-1}}\right)\Delta C_{d,t}$$

- Normalize the initial value of composite investment and consumption to be 1. Use the growth rates from the previous part to compute a level series for consumption and investment.

- Re-scale your level series so that it is equal to the nominal component in 2005q3 (this is consistent with the normalization currently used by the BEA where the price indexes are normalized to 100 in 2005q2/2005q3).

- You can compute implicit price indexes for consumption and investment as the ratio of the nominal values to the real values.

Now create a graph plotting the ratio of real investment to real consumption and a graph plotting the relative price of investment to consumption. Comment on the patterns evident in the plots.

(k) Compute the average growth rate of the real investment-consumption ratio in the data for the period 1947-2012. Use this number, along with your answer to (e), to determine the average relative growth rate of $A_{i,t}$ to $A_{c,t}$ (e.g. $g_i - g_c$).

(l) The above part (k) identifies the average *relative* growth rates of investment-specific and consumption-specific technological change. It does not identify the levels of these growth rates. We can do that by looking at the average growth rates of real consumption and real investment (not just the growth rate of their ratio) and comparing that to the data. Compute the average growth rates of real investment and real consumption in your data sample. Technically, we need to correct for population growth as well. In post-war US data, average population growth is about 0.003 per quarter. The average per-capita growth rates of investment and consumption will then be their average growth rates minus 0.003. Use these in conjunction with your answer to the previous question to pin down numeric values for g_i and g_c, assuming a value of $\alpha = 0.33$.

(m) To solve for the approximate solution of this model we need to re-write the variables in the model in terms of stationary variables. Use your answers from part (d) to rewrite the trending variables of the model as:

$$c_t = \frac{C_t}{X_t} \qquad k_t = \frac{K_t}{Z_t}$$

Determine what X_t and Z_t need to be.

(n) Re-write the first order conditions of the model in terms of the stationary variables.

(o) Solve for analytic expressions for the non-stochastic steady state of the stationary transformed variables.

(p) Create a Dynare .mod file to solve for the policy functions of the model written in stationary form. Use the following parameter values: $\psi = 2$, $\phi = 1$, $\beta = 0.99$, $\delta = 0.025$, and $\alpha = 0.33$. Use values you calculated in part (l) to come up with parameter values for g_i and g_c. Use values of the AR parameters in the exogenous processes of $\rho = \rho_i = \rho_c = 0.95$, and assume that the standard deviation of each of the three technology shocks is 0.01. Compute impulse responses to each of the three technology shocks (a "neutral" shock to a_t, a "consumption-specific" shock to $a_{c,t}$, and an "investment-specific" shock to $a_{i,t}$). Create a plot of the impulse responses to turn in with your assignment.

Unlike part (c), if $A_{c,t} \neq A_{i,t}$ we cannot define real output as the sum of real consumption and real investment because the relative price of consumption and investment will not be equal to one. Let nominal output (already measured in terms of stationary variables) be defined as:

$$P_t y_t = P_{c,t} c_t + P_{i,t} i_t$$

One approach is to set output as the numeraire, and to normalize $P_t = 1$ (since there is no money in the model that would determine this price anyway). We would then have $P_{c,t} = \frac{1}{a_{c,t}}$ and $P_{i,t} = \frac{1}{a_{i,t}}$, so that the relative price of consumption goods to investment goods corresponds to what you should have found in part (f).

(q) Use this definition of output to derive an aggregate production function (hint: use the fact that the capital to labor ratios must be equal in each sector). Define total factor productivity as (log) aggregate output minus (log) share-weighted aggregate capital and labor; e.g. $TFP_t = \ln y_t - \alpha \ln k_t - (1-\alpha) \ln N_t$. What exogenous driving force in the model does TFP_t measure here?

(r) Include this measure of y_t in your Dynare code and compute impulse responses of y_t to the three exogenous productivity shocks.

An alternative measurement would set either consumption or investment goods as the numeraire. Under a consumption goods measurement, we would have:

$$\frac{P_t y_t}{P_{c,t}} = y_t^c = c_t + \frac{P_{i,t}}{P_{c,t}} i_t$$

(s) Use your answer from (f) about the relative price of investment to consumption goods and this definition of aggregate output to derive an aggregate production function. What exogenous driving force(s) from the model would TFP_t measure?

(t) Finally, one could also use investment goods as the numeraire. This means that output would be measured as:

$$\frac{P_t y_t}{P_{i,t}} = y_t^i = \frac{P_{c,t}}{P_{i,t}} c_t + i_t$$

What exogenous driving force(s) of the model would TFP measure under this scenario?