Instructions: You may consult with other members of the class, but please make sure to turn in your own work. Where applicable, please print out figures and codes. This problem set is due in class on Tuesday, April 6.

(1) The Cagan Model and Seigniorage: This problem follows Cagan (1956) and will have you derive the seigniorage maximizing rate of inflation. Suppose that we have an ad-hoc money demand specification as follows:

\[ \ln M^d_t - \ln p_t = -\eta E_t \pi_{t+1} \]

In other words, the demand for real balances is equal to the negative of a coefficient, \( \eta \), times the rate of expected inflation. The central bank sets the money supply, \( \ln M_t \), exogenously. Market-clearing requires that \( \ln M^d_t = \ln M_t \).

(a) Provide some verbal intuition for why the demand for real balances ought to be inversely related to expected inflation.

(b) Note that \( \pi_{t+1} = \ln p_{t+1} - \ln p_t \). Derive an expression for the current log price level as a discounted value of the current and future money supply, assuming that \( \lim_{T \to \infty} \ln p_{t+T} < \infty \).

(c) Use your expression from (c) to show that if \( \ln M_t = \mu + \ln M_{t-1} + e_t \), \( E(e_t) = 0 \), then \( E_t - (\ln p_t - \ln p_{t-1}) = \mu \).

(d) Seigniorage is defined as the revenue that the government receives from printing money. It is given by:

\[ s_t = \frac{M_t - M_{t-1}}{P_t} \]

Note that these are levels of the variables, not logs. Assume that there is a constant growth rate of money with no uncertainty, so that \( \ln M_t - \ln M_{t-1} = \mu \ \forall \ t \). Use this, plus the definition of seigniorage, plus Cagan’s assumed money demand function, to derive an expression for seigniorage in terms only \( \mu \) and \( \eta \). In your derivation, use the approximation that \( \exp(\mu) = 1 + \mu \).

(e) Find an expression for the rate of money growth, \( \mu \), which maximizes seigniorage. Provide some verbal intuition for your answer.
(2) A RBC Model with Some Twists: Consider a decentralized version of the basic real business cycle model studied in class. The household solves the following problem:

\[
\max_{} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln c_t + \theta \ln(1 - n_t) + \psi \ln m_t \right\}
\]

s.t.

\[
c_t + b_t + m_t \leq w_t n_t + \Pi_t + (1 + i_{t-1}) \frac{b_{t-1}}{1 + \pi_t} + \frac{m_{t-1}}{1 + \pi_t}
\]

\[
n_t = 0 \quad \text{or} \quad n_t = \pi, \quad 0 < \pi < 1
\]

The variable \( m_t = \frac{M_t}{p_t} \), i.e. real money balances. The final constraint says that the choice over labor supply is restricted to two values – 0 or \( \pi \). So as to make the problem “well-behaved”, we assume that there are perfect insurance markets whereby households get paid labor income whether they work or not; there is a time-varying probability, call it \( \kappa_t \), of the household being called to work in any period.

(a) Show that the expected within period utility function (i.e. “felicity”) under these assumptions can be re-written as:

\[
\ln c_t - Bn_t + \psi \ln m_t
\]

Where \( n_t = \kappa_t \pi \). Derive the analytical expression for \( B \).

(b) Derive the first order conditions characterizing the solution of this reformulated version of the household’s problem.

(c) Now consider the problem of a firm that owns its capital stock and hires labor in competitive spot markets every period. The firm’s problem is different from normal in that it can utilize its capital stock to varying degrees – i.e. it can let its machines sit idle or it can work them overtime. Let \( u_t \) be the utilization rate, and \( u_t k_{t-1} \) be the firm’s “effective” capital stock in any period. The production function is:

\[
y_t = a_t(u_t k_{t-1})^\alpha n_t^{1-\alpha}
\]

The cost for increased utilization is faster depreciation (i.e. if you run the machines harder they wear out quicker). As such, the capital accumulation equation can be written:

\[
k_t = I_t + (1 - \delta(u_t))k_{t-1}
\]

Assume that the cost of utilization is convex and takes the form: \( \delta(u_t) = \delta_0 u_t^\phi \), \( \phi > 1 \). This is due to Burnside and Eichenbaum (1995). This specification implies that, when \( u_t > 1 \), \( \delta > \delta_0 \), and when \( u_t < 1 \), \( \delta < \delta_0 \).

The value of the firm is given by the present discounted value of profits, or:
\[ V_t = a_t(u_t k_{t-1})^{\alpha} n_t^{1-\alpha} - w_t n_t - I_t + \sum_{j=1}^{\infty} \prod_{i=1}^{j} (1 + r_{t-1+i})^{-1} (a_{t+j}(u_{t+j} k_{t+j-1})^{\alpha} n_{t+j}^{1-\alpha} - w_{t+j} n_{t+j} - I_{t+j}) \]

Derive the first order conditions necessary for the solution to the firm's problem.

(d) Assume that the nominal money supply follows an AR(1) in its growth rate:

\[ \ln M_t - \ln M_{t-1} = (1 - \rho_m) \pi^* + \rho_m (\ln M_{t-1} - \ln M_{t-2}) + u_t \]

\( \pi^* \) is an exogenous parameter set by the central bank. Re-write this process in terms of real balances, and verify that there exists a steady state in which real balances do not grow.

(e) Assume that TFP follows a mean zero, stationary AR(1) in its log, so that its unconditional mean in the level is unity:

\[ \ln a_t = \rho \ln a_{t-1} + \epsilon_t \]

Given the complete model described above, discuss how you would calibrate values of \( \beta \) and \( \alpha \). If we normalize \( u^* = 1 \) and want the average (i.e. steady state) rate of depreciation to be 0.03, what must \( \phi \) and \( \delta_0 \) be? If \( \pi = 1/2 \), so that people work one half their time allotment when they work, what must the parameter \( \theta \) be for \( n^* = 0.2 \)?

(f) Download data on M2 (you can get this from the St. Louis Fed website) and estimate an AR(1) on the growth rate of the money supply. The raw data will be given to you at a monthly frequency. Convert to a quarterly frequency by taking the within-quarter average. Make sure you download the seasonally adjusted data. Use the regression output to calibrate \( \pi^*, \rho_m, \) and \( \sigma_u \).

(g) Assume that \( \rho = 0.97 \) and \( \sigma_u = 0.01 \) (the standard deviation of the technology shock). Compute impulse responses to both technology and money growth shocks and show the graphs in your solution. Also show the impulse responses of the nominal money supply and the price level, which must be computed outside of the Dynare .mod file. Conduct your own simulation of the model. Simulate 500 data sets with 200 observations each. HP filter (smoothing parameter 1600) the simulated data and show a table of standard deviations and correlations with output. Does the correlation of nominal money with output match the data? What about the correlation of the price level and output?

(h) Do money growth shocks have effects on the real variables of the model in this setup? Discuss why your results are different than for the cash in advance version of the same economy.

(i) Repeat the exercise (just part (g)) on the exact same model but in which utilization is constant at 1 (i.e. \( \delta = \delta_0 \) and \( u_t = 1 \)). Compare and contrast your results in (g) and (i). How does utilization change the results?

(j) Compare your results for (g) and (i) with the standard RBC model with divisible labor and no variable utilization. Comment on how indivisible labor and utilization change the behavior of the model.