Graduate Macro Theory II:
The Real Business Cycle Model

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1 Introduction

This note describes the canonical real business cycle model. A couple of classic references here are Kydland and Prescott (1982), King, Plosser, and Rebelo (1988), and King and Rebelo (2000).

2 The Decentralized Model

I will set the problem up as a decentralized model, studying first the behavior of households and then the behavior of firms.

There are two primary ways of setting the model up, which both yield identical solutions. In both households own the firms, but management and ownership are distinct, and so households behave as though firm profits are given. In one formulation firms own the capital stock, and issue both debt and equity to households. In another formulation, households own the capital stock and rent it to firms; firms still issue debt and equity to households. We will go through both formulations. Intuitively these set-ups have to be the same because, either way, households own the capital stock (either directly or indirectly).

In both setups I abstract from trend growth, which, as we have seen, does not really make much of a difference anyway.

2.1 Firms Own the Capital Stock

Here we assume that firms own the capital stock. We begin with the household problem.

2.1.1 Household Problem

Households discount the future by $\beta < 1$. They supply labor (measured in hours), n_t and consume, c_t. They get utility from consumption and leisure; with the time endowment normalized to unity, leisure is $1 - n_t$. They earn a wage rate, w_t, which they take as given. They hold bonds, b_t, which pay interest rate r_t. $b_t > 0$ means that the household has a positive stock of savings; $b_t < 0$

means the household has a stock of debt. Note that "savings" is a stock; "saving" is a flow. The households take the interest rate as given. Their budget constraint says that each period, total expenditure must equal total income. Household earn wage income, $w_t n_t$, have profit distributions in the form of dividends, Π_t, and interest income on existing bond holds, $r_t b_t$ (note this can be negative, so that there is an interest cost of servicing debt). Household expenditure is composed of consumption, c_t and saving, $b_{t+1} - b_t$ (i.e. the accumulation of new savings). Hence we can write the constraint:

$$c_t + (b_{t+1} - b_t) = w_t n_t + \Pi_t + r_t b_t \tag{1}$$

Note a timing convention – r_t is the interest you have to pay today on existing debt. r_{t+1} is what you will have to pay tomorrow, but you choose how much debt to take into tomorrow today. Hence, we assume that households observe r_{t+1} in time t. Hence we can treat r_{t+1} as known from the perspective of time t. Households choose consumption, work effort, and the new stock of savings each period to maximize the present discounted value of welfare:

$$\max_{c_t, n_t, b_{t+1}} \quad E_t \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + v(1 - n_t) \right)$$

s.t.

$$c_t + b_{t+1} = w_t n_t + \Pi_t + (1 + r_t) b_t$$

We can form a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + v(1 - n_t) + \lambda_t (w_t n_t + \Pi_t + (1 + r_t) b_t - c_t - b_{t+1}) \right)$$

The first order conditions characterizing an interior solution are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow u'(c_t) = \lambda_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Leftrightarrow v'(1 - n_t) = \lambda_t w_t \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} (1 + r_{t+1}) \tag{4}$$

These can be combined together to yield:

$$u'(c_t) = \beta E_t \left(u'(c_{t+1})(1 + r_{t+1}) \right) \tag{5}$$

$$v'(1 - n_t) = u'(c_t) w_t \tag{6}$$

(5) and (6) have very intuitive, intermediate micro type interpretations. (5) says to equate the marginal rate of substitution between consumption today and tomorrow (i.e. $\frac{u'(c_t)}{\beta u'(c_{t+1})}$) to the relative price of consumption today (i.e. $1 + r_{t+1}$). (6) says to equate the marginal rate of substitution between leisure and consumption (i.e. $\frac{v'(1-n_t)}{u'(c_t)}$) to the relative price of leisure (i.e. w_t).

In addition, there is the transversality condition:

$$\lim_{t \to \infty} \quad \beta^t b_{t+1} u'(c_t) = 0 \tag{7}$$

2.1.2 The Firm Problem

The firm wants to maximize the present discounted value of (real) net revenues (i.e. cash flows). It discounts future cash flows by the stochastic discount factor. The stochastic discount factor puts cash flows (measured in goods) in terms of *current* utils (we take the current period to be $t = 0$). Define the stochastic discount factor as:

$$M_t = \beta^t E_0 u'(c_t)$$

The firm discounts by this because this is how consumers value future dividend flows. One unit of dividends returned to the household at time t generates $u'(c_t)$ additional units of utility, which must be discounted back to the present period (which we take to be 0), by β^t. The firm produces output, y_t, according to a constant returns to scale production function, $y_t = a_t f(k_t, n_t)$, with the usual properties. It hires labor, purchases new capital goods, and issues debt. I denote its debt as d_t, and it pays interest on its debt, r_t. Its revenue each period is equal to output. Its costs each period are the wage bill, investment in new physical capital, and services costs on its debt. It can raise its cash flow by issuing new debt (i.e. $d_{t+1} - d_t$ raises cash flow). It discounts future cash flows by the expected real interest rate. Its problem can be written as:

$$\max_{n_t, I_t, d_{t+1}} \quad V_0 = E_0 \sum_{t=0}^{\infty} M_t \left(a_t f(k_t, n_t) - w_t n_t - I_t + d_{t+1} - (1 + r_t) d_t \right)$$

s.t.

$$k_{t+1} = I_t + (1 - \delta) k_t$$

We can re-write the problem by imposing that the constraint hold each period:

$$\max_{n_t, I_t, d_{t+1}} \quad V_0 = E_0 \sum_{t=0}^{\infty} M_t \left(a_t f(k_t, n_t) - w_t n_t - k_{t+1} + (1-\delta)k_t + d_{t+1} - (1+r_t)d_t \right)$$

The first order conditions are as follows:

$$\frac{\partial V_0}{\partial n_t} = 0 \Leftrightarrow a_t f_n(k_t, n_t) = w_t \tag{8}$$

$$\frac{\partial V_0}{\partial k_{t+1}} = 0 \Leftrightarrow u'(c)_t = \beta E_t(u'(c_{t+1})((a_{t+1} f_k(k_{t+1}, n_{t+1}) + (1-\delta)) \tag{9}$$

$$\frac{\partial V_0}{\partial d_{t+1}} = 0 \Leftrightarrow u'(c_t) = \beta E_t(u'(c_{t+1})(1 + r_{t+1})) \tag{10}$$

(9) and (10) follow from the fact that $M_t = \beta^t u'(c_t)$ and $E_t M_{t+1} = \beta^{t+1} E_t u'(c_{t+1})$. Note that (10) is the same as (5), and therefore must hold in equilibrium as long as the household is optimizing. This means that the amount of debt the firm issues is indeterminate, since the condition will hold for any choice of d_{t+1}. This is essentially the Modigliani-Miller theorem – it doesn't matter how the firm finances its purchases of new capital – debt or equity – and hence the debt/equity mix is indeterminate.

2.1.3 Closing the Model

To close the model we need to specify a stochastic process for the exogenous variable(s). The only exogenous variable in the model is a_t. We assume that it is well-characterized as following a mean zero AR(1) in the log (we have abstracted from trend growth):

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_t \tag{11}$$

2.1.4 Equilibrium

A competitive equilibrium is a set of prices (r_{t+1}, w_t) and allocations $(c_t, n_t, k_{t+1}, d_{t+1}, b_{t+1})$ taking k_t, d_t, b_t, a_t and the stochastic process for a_t as given; the optimality conditions (5) - (7), (8)-(10), and the transversality condition holding; the labor and bonds market clearing ($n_t^d = n_t^s$ and $b_{t+1} = d_{t+1}$); and both budget constraints holding with equality.

Let's consolidate the household and firm budget constraints:

$$c_t + (b_{t+1} - b_t) = w_t n_t + r_t b_t + a_t f(k_t, n_t) - w_t n_t - I_t + d_{t+1} - (1+r_t)d_t \tag{12}$$

$$\Rightarrow \tag{13}$$

$$a_t f(k_t, n_t) = c_t + I_t \tag{14}$$

In other words, bond market-clearing plus both budget constraints holding just gives the standard accounting identity that output must be consumed or invested.

2.2 Households Own the Capital Stock

Now we consider a version of the decentralized problem in which the households own the capital stock and rent it to firms. Otherwise the structure of the problem is the same.

2.2.1 Household Problem

As before, households consume and supply labor. Now they also own the capital stock. They earn a rental rate for renting out the capital stock to firms each period, R_t. The household budget constraint is:

$$c_t + k_{t+1} - (1-\delta)k_t + b_{t+1} - b_t = w_t n_t + R_t k_t + r_t b_t + \Pi_t \tag{15}$$

The household has income comprised of labor income, capital income, interest income, and profits (again it takes profits as given). It can consume this, accumulate more capital (this is the $k_{t+1} - (1-\delta)k_t$ term), or accumulate more saving. Its problem is:

$$\max_{c_t, n_t, k_{t+1}, b_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + v(1-n_t)\right)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t + b_{t+1} - b_t = w_t n_t + R_t k_t + r_t b_t + \Pi_t$$

Form a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + v(1-n_t) + \lambda_t(w_t n_t + R_t k_t + (1+r_t)b_t + \Pi_t - c_t - k_{t+1} + (1-\delta)k_t - b_{t+1}\right))$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow u'(c_t) = \lambda_t \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Leftrightarrow v'(1-n_t) = \lambda_t w_t \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(R_{t+1} + (1-\delta)) \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1 + r_{t+1}) \tag{19}$$

These first order conditions can be combined to yield:

$$v'(1-n_t) = u'(c_t)w_t \tag{20}$$

$$u'(c_t) = \beta E_t u'(c_{t+1})(R_{t+1} + (1-\delta)) \tag{21}$$

$$u'(c_t) = \beta E_t u'(c_{t+1})(1 + r_{t+1}) \tag{22}$$

Note that (20) and (21) are the same as (5) and (9). (22) is the same as (6). One is tempted to claim that $r_{t+1} + \delta = R_{t+1}$ given that (19) and (20) must both hold. This is not quite right. r_{t+1} is known at time t; R_{t+1} is not. Hence one can take $1 + r_{t+1}$ outside of the expectations operator in (23) to get:

$$1 + r_{t+1} = \frac{u'(c_t)}{\beta E_t u'(c_{t+1})}$$

But once cannot do the same for (22). Intuitively, $E_t(u'(c_{t+1})(R_{t+1}+(1-\delta)) = E_t(u'(c_{t+1})E_t(R_{t+1}+(1-\delta)) + cov(u'(c_{t+1}), R_{t+1})$. In general, that covariance term is not going to be zero. It is likely to be negative –as we will see, $E_t R_{t+1}$ is the expected marginal product of capital. When the marginal product of capital is high (so R_{t+1} is high), then consumption is likely to be high (because MPK being high probably means that TFP is high), which means that marginal utility of consumption is low. To be compensated for holding an asset whose return covaries negatively with consumption, the household would demand a premium over the safe, riskless return r_{t+1}.

In a linearization of the model, that covariance term would drop out, and we could say that $r_{t+1} = E_t R_{t+1} - \delta$, but in general there is another term that is essentially the equity premium.

2.2.2 The Firm Problem

The firm problem is similar to before, but now it doesn't choose investment. Rather, it chooses capital *today* given the rental rate, R_t. Note that the firm can vary capital today even though the household cannot given that capital is predetermined. The labor choice and debt choice are similar. In fact, because the amount of the debt is going to end up being indeterminate, it is common to just assume that firms don't issue/hold debt and just solve a static problem. Again, the firm wants to maximize the present discounted value of cash flows.

$$\max_{n_t, k_t, d_{t+1}} \quad V_0 = E_0 \sum_{t=0}^{\infty} M_t \left(a_t f(k_t, n_t) - w_t n_t - R_t k_t + d_{t+1} - (1 + r_t)d_t\right)$$

The first order conditions are:

$$\frac{\partial V_0}{\partial n_t} = 0 \Leftrightarrow a_t f_n(k_t, n_t) = w_t \tag{23}$$

$$\frac{\partial V_0}{\partial k_t} = 0 \Leftrightarrow a_t f_k(k_t, n_t) = R_t \tag{24}$$

$$\frac{\partial V_0}{\partial d_{t+1}} = 0 \Leftrightarrow u'(c_t) = \beta E_t u'(c_{t+1})(1 + r_{t+1}) \tag{25}$$

(25) follows from the definition of the stochastic discount factor, and again is automatically satisfied; so again the amount of debt is indeterminate.

2.2.3 Equivalence to the Other Setup

Plug (24) into (21) and you get:

$$u'(c_t) = \beta E_t(u'(c_{t+1})(a_{t+1} f_k(k_{t+1}, n_{t+1}) + (1 - \delta)) \tag{26}$$

This is identical to (9). Also, (20) is equivalent to (6); (5) is equivalent to (22); and (23) is the same as (8). Hence, all the first order conditions are the same. The definition of equilibrium is the same. Both the firm and household budget constraints holding again give rise to the accounting identity (14). Hence, these setups give rise to identical solutions. It simply does not matter whether households own the capital stock and lease it to firms or whether firms own the capital stock. Since households own firms, these are equivalent ownership structures.

3 Equilibrium Analysis of the Decentralized Model

We can combined first order conditions from the firm and household problems (in either setup) to yield the equilibrium conditions:

$$u'(c_t) = \beta E_t \left(u'(c_{t+1})(a_{t+1} f_k(k_{t+1}, n_{t+1}) + (1 - \delta)\right) \tag{27}$$

$$v'(1 - n_t) = u'(c_t) a_t f_n(k_t, n_t) \tag{28}$$

$$k_{t+1} = a_t f(k_t, n_t) - c_t + (1 - \delta) k_t \tag{29}$$

$$\ln a_t = \rho \ln a_{t-1} + \varepsilon_t \tag{30}$$

$$y_t = a_t f(k_t, n_t) \tag{31}$$

$$y_t = c_t + I_t \tag{32}$$

$$u'(c_t) = \beta E_t u'(c_{t+1})(1 + r_{t+1}) \tag{33}$$

$$w_t = a_t f_n(k_t, n_t) \tag{34}$$

$$R_t = a_t f_k(k_t, n_t) \tag{35}$$

(27) can essentially be interpreted as an investment-saving equilibrium. (28) characterizes

equilibrium in the labor market, since the wage is equal to the marginal product of capital. (29) is just the capital accumulation equation, and (31) is the exogenous process for technology. (31) defines output and (32) defines investment. (33)-(35) just give us back the equilibrium factor prices. We have 1 truly forward-looking variable (consumption); two state/exogenous variables (capital and TFP); and six static variables (hours, output, investment, the real interest rate, the real wage, and the real rental rate). That's a total of mine variables and we have nine equations.

We need to specify functional forms. For simplicity, assume that $u(c_t) = \ln c_t$ and $v(1 - n_t) = \theta \ln(1 - n_t)$. Assume that the production function is Cobb-Douglas: $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$.

Given these parameter values we can analyze the steady state. The steady state is a situation in which $a^* = 1$ (its unconditional mean), $k_{t+1} = k_t = k^*$, and $c_{t+1} = c_t = c^*$. Given the steady state values of these variables, the steady state values of the static variables can be backed out. We can most easily solve for the steady state by beginning with the dynamic Euler equation, (27).

$$1 = \beta(\alpha k^{*\alpha-1} n^{*1-\alpha} + (1 - \delta))$$

Let's use this to solve for the steady state capital to labor ratio (life is much easier if you do it this way):

$$\frac{1}{\beta} - (1 - \delta) = \alpha \left(\frac{k^*}{n^*}\right)^{\alpha-1}$$
$$\Rightarrow$$
$$\frac{k^*}{n^*} = \left(\frac{\alpha}{\frac{1}{\beta} - (1 - \delta)}\right)^{\frac{1}{1-\alpha}} \tag{36}$$

Given the steady state capital-labor ratio, we now have the steady state factor prices:

$$w^* = (1 - \alpha)\left(\frac{k^*}{n^*}\right)^{\alpha} \tag{37}$$
$$R^* = \alpha \left(\frac{k^*}{n^*}\right)^{\alpha-1} \tag{38}$$
$$r^* = R^* - \delta \tag{39}$$

From the capital accumulation equation evaluated in steady state it is clear that $I^* = \delta k^*$. Use this and look at the production function combined with the accounting identity:

$$\left(\frac{k^*}{n^*}\right)^{\alpha} n^* = c^* + \delta k^*$$

Divide everything by n^*, and use this to express the consumption-hours ratio as a function of now-known things (the capital to hours ratio):

$$\frac{c^*}{n^*} = \left(\frac{k^*}{n^*}\right)^\alpha - \delta\left(\frac{k^*}{n^*}\right) \tag{40}$$

Hold on to this. Now go to (28), the intratemporal consumption-leisure tradeoff condition, evaluated at steady state:

$$\frac{\theta}{1-n^*} = \frac{1}{c^*}(1-\alpha)\left(\frac{k^*}{n^*}\right)^\alpha$$

Multiply and divide both sides by n^* and re-arrange to expression the consumption-hours ratio on the right hand side:

$$\frac{c^*}{n^*} = \left(\frac{1-n^*}{n^*}\right)\frac{1-\alpha}{\theta}\left(\frac{k^*}{n^*}\right)^\alpha \tag{41}$$

(40) and (41) essentially constitute two equations in two unknowns – the consumption-hours ratio and hours. Set them equal:

$$\left(\frac{1-n^*}{n^*}\right)\frac{1-\alpha}{\theta}\left(\frac{k^*}{n^*}\right)^\alpha = \left(\frac{k^*}{n^*}\right)^\alpha - \delta\left(\frac{k^*}{n^*}\right)$$

Now solve for n^*:

$$n^* = \frac{\frac{1-\alpha}{\theta}\left(\frac{k^*}{n^*}\right)^\alpha}{\frac{\theta+1-\alpha}{\theta}\left(\frac{k^*}{n^*}\right)^\alpha - \delta\left(\frac{k^*}{n^*}\right)} \tag{42}$$

Now that we have n^*, the rest of this is pretty easy to compute:

$$I^* = \delta\left(\frac{k^*}{n^*}\right)n^* \tag{43}$$

Steady state output is:

$$y^* = \left(\frac{k^*}{n^*}\right)^\alpha n^* \tag{44}$$

Steady state consumption then comes from the accounting identity:

$$c^* = n^*\left(\left(\frac{k^*}{n^*}\right)^\alpha - \delta\left(\frac{k^*}{n^*}\right)\right) \tag{45}$$

To do a quantitative analysis one would need to specify parameter values. Before doing that, let's think about analyzing the model qualitatively first. I'm going to log-linearize the first order conditions about the non-stochastic steady state. Start by taking logs of the static labor supply condition (28):

$$\ln\theta - \ln(1-n_t) = -\ln c_t + \ln(1-\alpha) + \ln a_t + \alpha\ln k_t - \alpha\ln n_t$$

The linearization (here I'm going to ignore the evaluation at steady state, which cancels out) is:

$$\frac{n_t - n^*}{1-n^*} = -\frac{c_t - c^*}{c^*} + \frac{a_t - a^*}{a^*} + \alpha\frac{k_t - k^*}{k^*} - \alpha\frac{n_t - n^*}{n^*}$$

Simplify into our "tilde" notation:

$$\left(\frac{n^*}{1-n^*}\right)\widetilde{n}_t = -\widetilde{c}_t + \widetilde{a}_t + \alpha\widetilde{k}_t - \alpha\widetilde{n}_t$$

So as to economize on notation, let's denote $\gamma = \frac{n^*}{1-n^*} > 0$. Then we get:

$$\widetilde{n}_t = -\left(\frac{1}{\gamma+\alpha}\right)\widetilde{c}_t + \left(\frac{1}{\gamma+\alpha}\right)\widetilde{a}_t + \left(\frac{\alpha}{\gamma+\alpha}\right)\widetilde{k}_t \tag{46}$$

Now let's linearize the accumulation equation. Begin by taking logs:

$$\ln k_{t+1} = \ln(a_t k_t^\alpha n_t^{1-\alpha} - c_t + (1-\delta)k_t)$$

Now linearize, again ignoring the evaluation at steady state part, which cancels out anyway:

$$\begin{aligned}\frac{k_{t+1} - k^*}{k^*} = \frac{1}{k^*}\big(&k^{*\alpha}n^{*1-\alpha}(a_t - a^*) + \alpha k^{*\alpha-1}n^{*1-\alpha}(k_t - k^*) + \dots \\ &+(1-\alpha)k^{*\alpha}n^{*-\alpha}(n_t - n^*) - (c_t - c^*) + (1-\delta)(k_t - k^*)\big)\end{aligned}$$

Now simplify and use our "tilde" notation to denote the percentage deviation of a variable from its steady state:

$$\begin{aligned}\widetilde{k}_{t+1} &= \left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\widetilde{a}_t + \alpha\widetilde{k}_t + (1-\alpha)\widetilde{n}_t\right) - \frac{c^*}{k^*}\widetilde{c}_t + (1-\delta)\widetilde{k}_t \\ \widetilde{k}_{t+1} &= \frac{1}{\beta}\widetilde{k}_t + \left(\frac{k^*}{n^*}\right)^{\alpha-1}\widetilde{a}_t + (1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\widetilde{n}_t - \frac{c^*}{k^*}\widetilde{c}_t\end{aligned}$$

The last simplification follows from the fact that $\alpha k^{*\alpha-1}n^{*1-\alpha} + (1-\delta) = \frac{1}{\beta}$. Now substitute the log-linearized expression for employment into this expression:

$$\widetilde{k}_{t+1} = \left(\frac{1}{\beta} + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\widetilde{k}_t + \left(\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\frac{1+\gamma}{\gamma+\alpha}\right)\right)\widetilde{a}_t - \left(\frac{c^*}{k^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\widetilde{c}_t \tag{47}$$

Now we need to log-linearize the consumption Euler equation. Begin by taking logs:

$$-\ln c_t = \ln \beta - \ln c_{t+1} + \ln\left(\alpha a_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta)\right)$$

Now linearize, ignoring the evaluation at steady state and making use of the fact that $\alpha k^{*\alpha-1} n^{*1-\alpha} + (1-\delta) = \frac{1}{\beta}$:

$$-\frac{c_t - c^*}{c^*} = -\frac{c_{t+1} - c^*}{c^*} + \beta\big(\alpha k^{*\alpha-1} n^{*1-\alpha}(a_{t+1} - a^*) + (\alpha-1)\alpha k^{*\alpha-2} n^{*1-\alpha}(k_{t+1} - k^*) + \dots$$
$$+(1-\alpha)\alpha k^{*\alpha-1} n^{*-\alpha}(n_{t+1} - n^*)\big)$$

This can be simplified using our tilde notation:

$$-\widetilde{c}_t = -\widetilde{c}_{t+1} + \beta\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\widetilde{a}_{t+1} + (\alpha-1)\widetilde{k}_{t+1} + (1-\alpha)\widetilde{n}_{t+1}\right) \tag{48}$$

Now eliminate $\widetilde{n}_{t+1}$ using (41):

$$-\widetilde{c}_t = -\widetilde{c}_{t+1} + \beta\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\widetilde{a}_{t+1} + \beta\alpha(\alpha-1)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\widetilde{k}_{t+1} + \dots$$
$$\dots + \beta\alpha(1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(-\left(\frac{1}{\gamma+\alpha}\right)\widetilde{c}_{t+1} + \left(\frac{1}{\gamma+\alpha}\right)\widetilde{a}_{t+1} + \left(\frac{\alpha}{\gamma+\alpha}\right)\widetilde{k}_{t+1}\right)$$

Simplifying:

$$-\widetilde{c}_t = -\left(1 + \beta\alpha(1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\frac{1}{\gamma+\alpha}\right)\right)\widetilde{c}_{t+1} + \left(\beta\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\left(\frac{1+\gamma}{\gamma+\alpha}\right)\widetilde{a}_{t+1} + \dots$$
$$\dots - \left(\beta\alpha(1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\left(\frac{\gamma}{\gamma+\alpha}\right)\widetilde{k}_{t+1} \tag{49}$$

Equations (47) and (49) (plus the exogenous process for TFP) define a system of linearized difference equations. Let's try to think about this in the context of a phase diagram. The $\widetilde{c}_{t+1} = \widetilde{c}_t$ isocline can be solved for from (44):

$$\beta\alpha(1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\frac{1}{\gamma+\alpha}\right)\widetilde{c}_{t+1} = \left(\beta\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\left(\frac{1+\gamma}{\gamma+\alpha}\right)\widetilde{a}_{t+1} - \left(\beta\alpha(1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\left(\frac{\gamma}{\gamma+\alpha}\right)\widetilde{k}_{t+1}$$

This simplifies greatly to yield:

$$\widetilde{c}_{t+1} = \left(\frac{1+\gamma}{1-\alpha}\right)\widetilde{a}_{t+1} - \gamma\widetilde{k}_{t+1}$$

To make things easier, evaluate this at t (we gain engage in the abuse of terminology in treating the two time periods as approximately the same):

$$\widetilde{c}_t = \left(\frac{1+\gamma}{1-\alpha}\right)\widetilde{a}_t - \gamma\widetilde{k}_t \tag{50}$$

This is the $\widetilde{c}_{t+1} = \widetilde{c}_t = 0$ isocline – i.e. the set of $(\widetilde{c}_t, \widetilde{k}_t)$ pairs where consumption is constant. In $(\widetilde{c}_t, \widetilde{k}_t)$ space it is downward sloping, and it will shift up if $\widetilde{a}_t$ were to change.

Now go to (47) to find the $\widetilde{k}_{t+1} = \widetilde{k}_t = 0$ isocline:

$$\left(\frac{c^*}{k^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\widetilde{c}_t = \left(\frac{1}{\beta} - 1 + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\widetilde{k}_t + \left(\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\frac{1+\gamma}{\gamma+\alpha}\right)\right)\widetilde{a}_t$$

Simplify a bit:

$$\widetilde{c}_t = \left(\frac{c^*}{k^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)^{-1}\left(\left(\frac{1}{\beta} - 1 + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\widetilde{k}_t + \left(\left(\frac{k^*}{n^*}\right)^{\alpha-1}\left(\frac{1+\gamma}{\gamma+\alpha}\right)\right)\widetilde{a}_t\right) \tag{51}$$

This is upward-sloping in $\widetilde{k}_t$. Furthermore, we actually know that the coefficient on the capital stock must be less than one. The coefficient can be written as follows:

$$\left(\frac{c^*}{k^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)^{-1}\left(\left(\frac{1}{\beta} - 1 + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right)\right)$$

We know that $\frac{1-\alpha}{\gamma+\alpha} > \frac{\alpha(1-\alpha)}{\gamma+\alpha}$ since $0 < \alpha < 1$. Hence, we can prove that this coefficient is less than unity if we can show that $\frac{1}{\beta} - 1$ is less than $\frac{c^*}{k^*}$ (because this would be sufficient to show that the numerator is less than the denominator, and hence the total coefficient is less than unity). We can find an expression for the consumption-capital ratio in the steady state by looking at the accounting identity and simplifying:

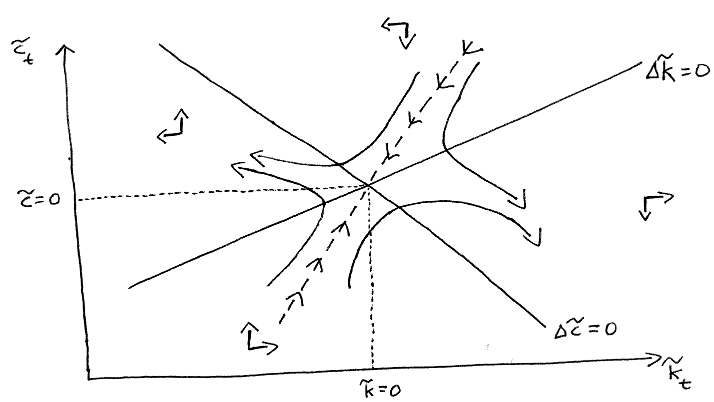
$$\begin{aligned}
c^* &= k^{*\alpha}n^{*1-\alpha} - \delta k^* \\
\frac{c^*}{k^*} &= \left(\frac{k^*}{n^*}\right)^{\alpha-1} - \delta \\
\frac{c^*}{k^*} &= \frac{\frac{1}{\beta} - (1-\delta)}{\alpha} - \delta \\
\frac{c^*}{k^*} &= \frac{\frac{1}{\beta} - 1}{\alpha} + \delta\left(\frac{1-\alpha}{\alpha}\right)
\end{aligned}$$

This has to be great than $\frac{1}{\beta} - 1$, since $0 < \alpha < 1$ and $\delta > 0$.

We thus have two isoclines. We want to plot these in a plane with $\widetilde{c}_t$ on the vertical axis and $\widetilde{k}_t$ on the horizontal axis. When $\widetilde{a}_t = 0$ (i.e. we were at the unconditional mean of technology), it is clear that the two isoclines cross at the point $\widetilde{c}_t = \widetilde{k}_t = 0$ – i.e. they cross at the non-stochastic steady state. We have that the $\widetilde{c}_{t+1} - \widetilde{c}_t = 0$ isocline is downward sloping and that the $\widetilde{k}_{t+1} - \widetilde{k}_t = 0$ isocline is downward sloping.

We want to examine the dynamics off of the isoclines so as to locate the saddle path. Quite intuitively, if $\widetilde{k}_t$ is "too big" relative to what it would be when $\widetilde{c}_{t+1} - \widetilde{c}_t = 0$ – i.e. we are to the right of the $\widetilde{c}_{t+1} - \widetilde{c}_t = 0$ iscoline – then consumption will be expected to decline overtime. The intuition for this is straight from the Euler equation – if the capital stock is "too big", then the marginal product of capital (and hence the real interest rate) is "too small" given concavity of the production function. The low interest rate means that consumption is high but expected to fall. Hence, anywhere to the right of $\widetilde{c}_{t+1} - \widetilde{c}_t = 0$ isocline the arrows point down. If $\widetilde{c}_t$ is "too big" relative to where it would be along the $\widetilde{k}_{t+1} - \widetilde{k}_t = 0$ isocline, then capital will be expected to decline – intuitively, people aren't saving enough. Hence, anywhere above the $\widetilde{k}_{t+1} - \widetilde{k}_t = 0$ iscoline, arrows point left; anywhere below, arrows point right.

We can see that the saddle path must be upward-sloping, moving from southwest to northeast in the picture. Note also that the slope of the saddle path must be steeper than the slope of the $\widetilde{k}_{t+1} - \widetilde{k}_t$ isocline. One can show (albeit tediously) that the $\widetilde{k}_{t+1} - \widetilde{k}_t$ isocline has slope less than 1.



This phase diagram tells us what $\widetilde{c}_0$ needs to be given $\widetilde{k}_0$ and $\widetilde{a}_0$ so as to be consistent with the first order conditions and the transversality condition holding. Given the two states, once we know $\widetilde{c}_0$, we can back out the values of the static variables (employment, output, and consumption) and the factor prices (the real wage and the real interest rate).

Begin by going to the labor market. Labor supply is implicitly determined by (6). Using our functional forms, this is:

$$\frac{\theta}{1-n_t} = \frac{1}{c_t} w_t$$

In other words, the amount of labor households want to supply depends negatively on consumption and positively on the real wage. Loosely speaking, we can think about consumption being in there as picking up wealth effects – when you're wealthier you want more leisure, which means less work. Let's log-linearize this equation:

$$\ln \theta - \ln(1-n_t) = -\ln c_t + \ln w_t$$
$$-\frac{n_t - n^*}{1-n^*} = -\frac{c_t - c^*}{c^*} + \frac{w_t - w^*}{w^*}$$

Simplify using the "tilde" notation to get:

$$\left(\frac{n^*}{1-n^*}\right) \widetilde{n}_t = -\widetilde{c}_t + \widetilde{w}_t \tag{52}$$

The term $\frac{1-n^*}{n^*}$ has a special name in economics – it is called the Frisch labor supply elasticity. It gives the percentage change in employment for a percentage change in the real wage, *holding consumption fixed*. If $n^* = \frac{1}{3}$, for example, then the Frisch elasticity would be 2. In terms of a graphical representation of the linearized labor supply function, it would be upward sloping in $\widetilde{w}_t$, with slope equal to the Frisch elasticity, and would shift in whenever consumption goes up (or out whenever consumption goes down).

Now go to the firm's first order condition which implicitly define a labor demand curve (i.e. equation (8)). Using our functional form assumptions, this is:

$$a_t(1-\alpha)k_t^{\alpha} n_t^{-\alpha} = w_t$$

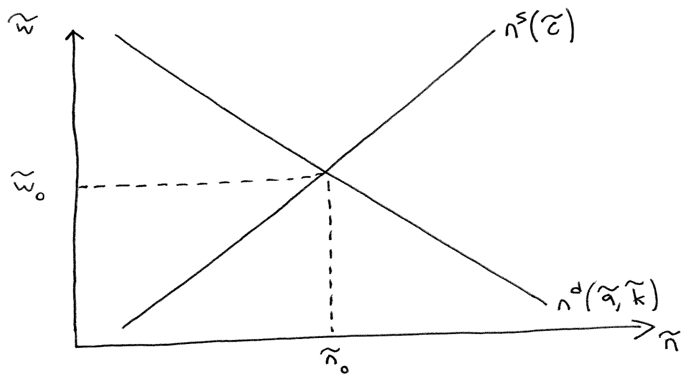
Let's log-linearize this:

$$\ln a_t + \ln(1-\alpha) + \alpha \ln k_t - \alpha \ln n_t = \ln w_t$$
$$\frac{w_t - w^*}{w^*} = \frac{a_t - a^*}{a^*} + \alpha \frac{k_t - k^*}{k^*} - \alpha \frac{n_t - n^*}{n^*}$$

Using the "tilde" notation, this simplifies to:

$$\widetilde{w}_t = \widetilde{a}_t + \alpha \widetilde{k}_t - \alpha \widetilde{n}_t \tag{53}$$

In terms of a graph, this is downward sloping in $\widetilde{w}_t$, and will shift out whenever $\widetilde{a}_t$ or $\widetilde{k}_t$ increase.



$\widetilde{a}_0$ and $\widetilde{k}_0$ are given. Once we know $\widetilde{c}_0$ from the phase diagram, we can determine the position of the labor supply curve. The position of the labor demand curve is given once we know capital and TFP. The intersection of the curves determines the real wage and level of employment.

Next we can determine output, given employment. Using our functional form assumptions, $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$. Let's log-linearize this:

$$\begin{aligned}\ln y_t &= \ln a_t + \alpha \ln k_t + (1-\alpha)\ln n_t \\ \frac{y_t - y^*}{y^*} &= \frac{a_t - a^*}{a^*} + \alpha \frac{k_t - k^*}{k^*} + (1-\alpha)\frac{n_t - n^*}{n^*}\end{aligned}$$

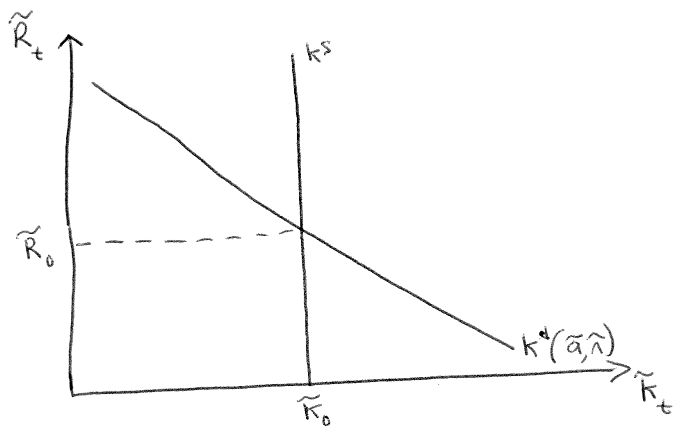
Using the "tilde" notation, we get:

$$\widetilde{y}_t = \widetilde{a}_t + \alpha \widetilde{k}_t + (1-\alpha)\widetilde{n}_t \tag{54}$$

Now let's log-linearize the capital demand equation:

$$\begin{aligned}\ln R_t &= \ln \alpha + \ln a_t + (\alpha - 1)\ln k_t + (1-\alpha)\ln n_t \\ \widetilde{R}_t &= \widetilde{a}_t + (\alpha - 1)\widetilde{k}_t + (1-\alpha)\widetilde{n}_t\end{aligned}$$

Capital supply is fixed, at least within period. Hence, the capital supply curve is vertical. Hence the intersection of capital demand and supply determines the real rental rate on capital as follows:



Once we have the rental rate, we can get the interest rate, because, in log-linear form, it must be the case that:

$$r_{t+1} = E_t R_{t+1} - \delta \tag{55}$$

Finally, let's determine investment. In equilibrium, $y_t = c_t + I_t$. In log-linear form, we can solve for the equilibrium amount of investment:

$$\widetilde{y}_t = \frac{c^*}{y^*}\widetilde{c}_t + \frac{I^*}{y^*}\widetilde{I}_t$$
$$\widetilde{I}_t = \frac{y^*}{I^*}\left(\widetilde{y}_t - \frac{c^*}{y^*}\widetilde{c}_t\right)$$

Since we have already determined output and consumption, this effectively determines the amount of investment. In some sense, in these models investment is "residually" determined. That isn't to say that investment isn't forward-looking: when firms in isolation are making investment decisions, they are explicitly looking to the future. But in equilibrium when markets are clearing, investment is determined from the accounting identity once the other stuff is determined.

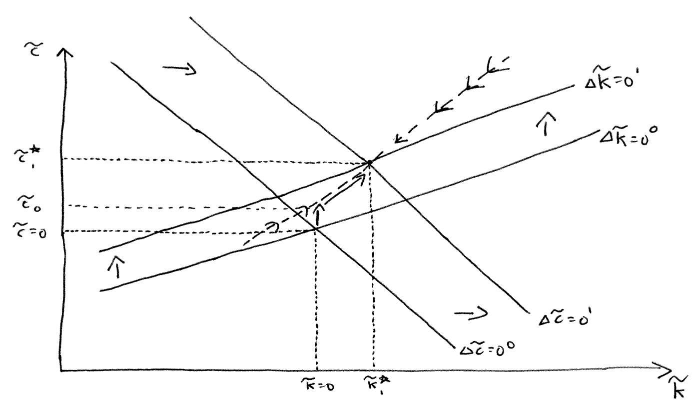
Hence, one can think about there being a "causal ordering". Given TFP and capital, determine consumption from the phase diagram. Given consumption, determine employment and the real wage. Given employment and TFP, determine the rental rate on capital. Given the rental rate, determine the interest rate. Then given employment, determine output. Then given output and consumption, determine investment:

$$\widetilde{a}_0 \ \& \ \widetilde{k}_0 \to \widetilde{c}_0 \to \widetilde{n}_0 \ \& \ \widetilde{w}_0 \to \widetilde{R}_0 \to \widetilde{r}_1 \to \widetilde{y}_0 \to \widetilde{I}_0$$

4 Dynamic Analysis in Response to TFP Shocks

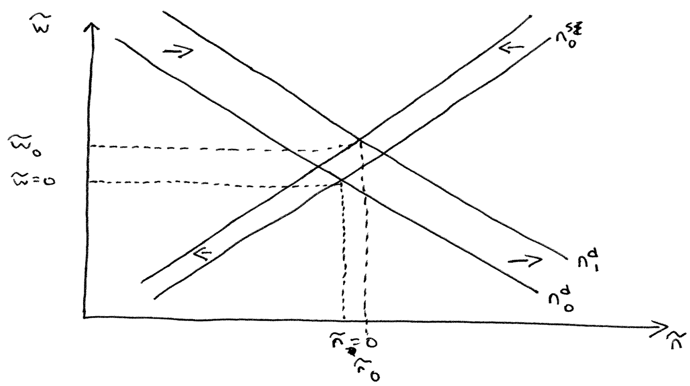
We want to qualitatively characterize the dynamic responses of the endogenous variables of the model to shocks to TFP (the only source of stochastic variation in the model as it currently stands).

Consider first an unexpected, *permanent* increase in a_t. This means we will end up in a new steady state, which means that the new steady state of the linearized variables will *not* be zero, as we linearized about the old steady state. From (50) and (51), we know that the two isoclines must both shift "up" when $\widetilde{a}_0$ suddenly increases. The new isoclines must cross at a point with a higher (relative to the initial) steady state capital and consumption (one can show this analytically). There will also be a new saddle path associated with the "new" system. At time 0, consumption must jump to the new saddle path, from which point it must be expected to "ride" it all the way to the new steady state. The initial jump in consumption turns out to be ambiguous – it could increase, decrease, or not change at all (similarly to the basic neoclassical case with fixed labor input). That being said, for plausible parameterizations, consumption will jump up on impact (permanent income intuition), and hence that's how I'm going to draw it. See the phase diagram below:

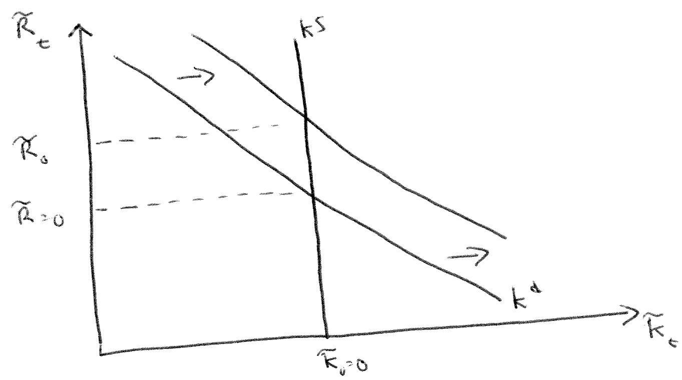


We can see from the picture that $\widetilde{c}$ jumps up on impact, from its initial steady state (with $\widetilde{c} = 0$), to $\widetilde{c}_0$. From thereafter it must ride the new saddle path to the new steady state, which in terms of these linearized variables will feature $\widetilde{k}^* > 0$ and $\widetilde{c}^* > 0$, since we linearized about the steady state associated with the old level of TFP.

Now that we know consumption, go to the labor market. Higher TFP shifts labor demand out. Higher consumption shifts labor supply in. The net effect is for the real wage to definitely be higher, but there is an ambiguous effect on employment – it could go up, down, or not change at all. I'm going to draw it as going up, as this seems to be the plausible case.



Next, go to the picture for capital demand. Higher TFP and higher employment shift the demand curve out; the supply is fixed. Hence the rental rate rises.

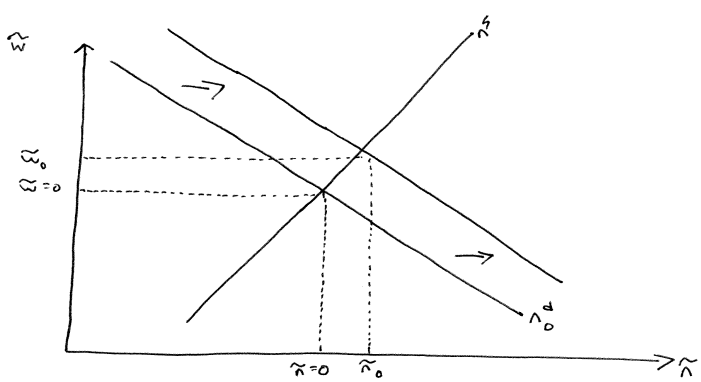


With employment and TFP higher, it must be the case that output is higher. What about investment? From the dynamics of the phase diagram, we know that we must be accumulating more capital; hence investment must rise on impact. What about the risk-free real interest, r_{t+1}? There are two ways to see what must happen here. One is to note that, from the phase diagram, consumption must be increasing. There is a positive relationship between consumption growth and the real interest rate in the Euler equation for bonds; hence r_{t+1} must go up. Another way to see it is to look at the (approximate) arbitrage condition between capital and bonds, and assume that $\widetilde{R}_{t+1}$ will be higher as well, so that r_{t+1} will go up too.

To summarize, following a permanent increase in TFP, we have an ambiguous initial jump in consumption (though likely up), an increase in the real wage, an ambiguous change in employment, an increase in output, an increase in investment, and an increase in the real interest rate. After these impact effects, we follow the dynamics of the phase diagram. In particular, consumption and the capital stock will grow (which means the real interest rate and investment will stay high). The real wage will continue to grow – this is because, as consumption grows, the labor supply curve will continue to shift in, and as capital grows, the labor demand curve will continue to shift out. Employment will end up going back to its original steady state at some point – this is because the expression for steady state employment is independent of a^* (more on this later).

Next, consider a completely transitory increase in $\widetilde{a}_t$. In other words, it only lasts today and the expected value of $\widetilde{a}_{t+1}$ is unchanged. In the phase diagram, this will *approximately* do nothing. In reality consumption must jump a little, ride unstable dynamics for one period, and then end up on the original saddle path, from which point it must be expected to return to its initial steady state. Since the change is only "in effect" for a very short period of time, we can approximate the jump in consumption as being zero. Thus, what approximately happens in the phase diagram is nothing at all.

Given that consumption doesn't jump, next go to the labor market. Labor demand immediately shifts out, and the outward shift is the same as in the case of a permanent change in TFP. Labor demand only depends on current TFP, and hence the persistence of the shock does not factor in at all. But since consumption doesn't change, labor supply doesn't shift. Hence, we observe that both hours and the real wage go up. Importantly, relative to the case of the permanent shock, hours rise by more (in the permanent case the effect on hours was ambiguous) and the real wage rises by less. See the figure below:



Once we know what happens to employment, then we know what happens to output. Since employment goes up by more in the case of the purely transitory shock, we can see that output rises by more to a TFP shock when it is transitory than when it is permanent.

Finally, go to the demand for capital curve. The demand for capital depends just on current conditions; hence it shifts out. It is important to note that it actually shifts out by more than in the case of a permanent technology shock, the reason being that employment increases by more here, so the marginal product of capital increases by more.

Now what happens to investment? Output goes up by more (relative to the permanent shock case) and consumption goes up by less – hence investment goes up and goes up by more than if the shock were permanent. It isn't easy to see, but the real interest rate must actually go down by a little. Why? Consumption today must actually jump by a little and be expected to fall, hence r_{t+1} must go down.

The two cases thus far considered are somewhat knife edge. Most of the time we are interested in looking at what happens in response to persistent – but transitory – shocks. The above exercises are helpful because they provided "bounding results". The more persistent the shock is (e.g. the bigger is ρ), the more the results look like the permanent case. The less persistent the shock (e.g. the smaller is ρ), the more the results like the purely transitory case.

We can thus make the following qualitative statements that can be verified quantitatively by numerically solving the model:

1. The more persistent the increase in TFP, the more consumption increases on impact (or falls by less). In the limiting case where the change in TFP is just one period, consumption will approximately not react.

2. The more persistent the increase in TFP, the less hours react and the more real wages increase.

3. The more persistent the increase in TFP, the less output reacts

4. The more persistent the increase in TFP, the less investment reacts and the real interest rate increases by more (or falls by less).