This document lays out a basic real business cycle model as the solution to a planner’s problem. The planner’s problem can be written:

$$\begin{aligned}
\max_{c_t, k_{t+1}, n_t} & \quad E_0 \sum_{t=0}^{\infty} \beta^t (\ln c_t + \theta \ln l_t) \\
\text{s.t.} & \quad k_{t+1} = a_t k_t^\alpha n_t^{1-\alpha} - c_t + (1 - \delta) k_t \\
& \quad 1 = n_t + l_t \\
& \quad k_0 \text{ given}
\end{aligned}$$

After substituting in the time constraint, I write the problem out as a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ (\ln c_t + \theta \ln (1 - n_t)) + \lambda_t \left(a_t k_t^\alpha n_t^{1-\alpha} - c_t + (1 - \delta) k_t - k_{t+1}\right)\right\}$$

The first order conditions are:

$$\begin{align*}
\frac{\partial \mathcal{L}}{\partial c_t} &= 0 \iff \frac{1}{c_t} = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial n_t} &= 0 \iff \frac{\theta}{1 - n_t} = \lambda_t (1 - \alpha) a_t k_t^\alpha n_t^{-\alpha} \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= 0 \iff \lambda_t = \beta E_t \lambda_{t+1} \left(\alpha a_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1 - \delta)\right) \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= 0 \iff k_{t+1} = a_t k_t^\alpha n_t^{1-\alpha} - c_t + (1 - \delta) k_t \\
& \quad k_0 \text{ given} \\
TV \lim_{t \to \infty} \beta^t \lambda_t k_{t+1} &= 0
\end{align*}$$

The first FOC can be eliminated, leaving three conditions (not counting the initial and terminal conditions):
\[
\frac{\theta}{1 - n_t} = \frac{1}{c_t} (1 - \alpha) a_t k_t^\alpha n_t^{1-\alpha}
\]
\[
\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} (\alpha a_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1 - \delta))
\]
\[
k_{t+1} = a_t k_t^\alpha n_t^{1-\alpha} - c_t + (1 - \delta) k_t
\]

The first step in solving the model is to (a) calibrate the parameters and (b) solve for the steady state. Calibration has proven to be highly controversial within the field. Loosely speaking, calibration involves picking the model’s parameters to match long run properties of the data – things like the average labor’s share of income over long periods, or the average amount of time spent working, etc. It can (loosely) be thought of as a kind of method of moments estimator with identity weighting matrix and no standard errors reported. Much of the RBC literature (especially the earlier literature) eschewed the econometric approach to coming up with the parameter values (i.e. formal estimation, inference, and hypothesis testing) in favor of calibration because this approach lends itself to over-fitting. The model above is stylized and simple. It is patently false. As such, the data will reject it in favor of a more complicated model (or a less theoretically founded econometric model). This does not mean that there is nothing to be learned from the model, however.

The parameter \( \alpha \) can be measured in the data as the share of total income paid to owners of capital. In the NIPA accounts this number comes out to be \( \frac{1}{3} \) and is fairly constant in the data (that’s in fact one of the main growth facts (Kaldor (1956))). The steady state gross real interest rate will be equal to the inverse of the discount factor, i.e. \( 1 + r^* = \frac{1}{\beta} \). Because we can measure the nominal interest rate on safe bonds (i.e. government debt) and inflation over long periods of time, we know that the average real interest rate (expressed at an annual frequency) is roughly 2%, or 0.02. This would imply a quarterly discount factor of about 0.995. The discount factor has to be adjusted for steady state growth, which amounts to roughly 2 percent per year (or 0.005) per quarter. This gives us a discount factor of \( .995/1.005 = 0.99 \). The depreciation rate can be measured by making use of the accounting identity and annual data on the capital stock and investment. The implied average depreciation rate in the data appears to be about 0.1, or about 0.025 at a quarterly frequency. In reality there are many different kinds of capital goods, some with very low depreciation rates (structures), and others with very high rates of depreciation (computers). The 10 percent at an annual frequency is roughly an average depreciation rate then, since the model only has one kind of capital good.

Calibrating the weighting parameter on leisure in the utility function, \( \theta \), is a little bit trickier. It turns out that we can use information on the average amount of time spent working to pin down a value for this. To see that, let’s move on to the calculation of the non-stochastic steady state. I begin my making an assumption about the stochastic process for technology, \( a_t \). I assume that it follows a mean zero AR(1) process in its natural log, with an iid disturbance with some variance, \( \sigma^2 \). Mean zero in the log implies a mean of unity in the level, so my steady state level of technology is \( a^* = 1 \), and I omit it from most of the derivations below.

\[
\ln a_t = \rho \ln a_{t-1} + \varepsilon_t
\]
In the steady state all variables here are constant (i.e. we’ve implicitly already removed population and deterministic technology growth). This means that \( c_{t+1} = c_t = c^* \), \( k_{t+1} = k_t = k^* \), and \( n_{t+1} = n_t = n^* \). Begin with the Euler equation for consumption. Imposing the steady state yields:

\[
\frac{1}{\beta} = \alpha k^* n^{1-\alpha} + (1 - \delta)
\]

It will be helpful to solve the model in terms of the capital to labor ratio. We can do that here:

\[
\alpha \left( \frac{k^*}{n^*} \right)^{\alpha-1} = \frac{1}{\beta} - (1 - \delta)
\]

\[
\left( \frac{k^*}{n^*} \right) = \left( \frac{\alpha}{\frac{1}{\beta} - (1 - \delta)} \right)^{\frac{1}{1-\alpha}}
\]

With the parameters as calibrated above, this implies that the steady state capital to labor ratio is 36.989.

Now I move to the intratemporal condition for labor supply. The FOC evaluated at the steady state is:

\[
\frac{\theta}{1 - n^*} = \frac{1}{c^* (1 - \alpha)} \left( \frac{k^*}{n^*} \right)^{\alpha}
\]

I can solve this for an expression for the steady state consumption to labor ratio:

\[
\frac{c^*}{n^*} = \left( \frac{1 - n^*}{n^*} \right) \left( \frac{1 - \alpha}{\theta} \right) \left( \frac{k^*}{n^*} \right)^{\alpha}
\]

Now look at the aggregate accounting identity evaluated in the steady state:

\[
\left( \frac{k^*}{n^*} \right)^{\alpha} n^* = c^* + i^* = c^* + \delta k^*
\]

\[
\left( \frac{k^*}{n^*} \right)^{\alpha} = \frac{c^*}{n^*} + \delta \left( \frac{k^*}{n^*} \right)
\]

\[
\frac{c^*}{n^*} = \left( \frac{k^*}{n^*} \right)^{\alpha} - \delta \left( \frac{k^*}{n^*} \right)
\]

We now have two equations in \( \frac{c^*}{n^*} \). Set them equal to one another:

\[
\left( \frac{k^*}{n^*} \right)^{\alpha} - \delta \left( \frac{k^*}{n^*} \right) = \left( \frac{1 - n^*}{n^*} \right) \left( \frac{1 - \alpha}{\theta} \right) \left( \frac{k^*}{n^*} \right)^{\alpha}
\]

In US data, people work about 20% of their time (i.e. \( n^* = 0.2 \)). Using this fact, we can now solve this equation for the \( \theta \) that makes this work:
\[
1 - \delta \left( \frac{k^*}{n^*} \right)^{1-\alpha} = \left( \frac{1 - n^*}{n^*} \right) \left( \frac{1 - \alpha}{\theta} \right) \\
\left( \frac{1 - \alpha}{\theta} \right) = \left( \frac{n^*}{1 - n^*} \right) \left( 1 - \delta \left( \frac{k^*}{n^*} \right)^{1-\alpha} \right) \\
\theta = (1 - \alpha) \left( \frac{n^*}{1 - n^*} \right) \left( 1 - \delta \left( \frac{k^*}{n^*} \right)^{1-\alpha} \right)^{-1}
\]

Using the parameterization above, \( \theta = 3.49 \). Given this, we can get the rest of the values. These turn out to be:

\[
y^* = 0.6163 \\
k^* = 5.8527 \\
n^* = 0.2 \\
c^* = 0.4700
\]

To solve the model I log-linearize the first order conditions about the steady state, using the policy function to eliminate the static variable \( n_t \) from the set of difference equations. I actually solve the model in Dynare, which does the linearization for you (it can also do a second order approximation).

In the RBC methodology, after the parameters have been picked and the steady state has been solved for, the tradition is to then simulate the data. The question is then how to impose some structure on the shocks I feed in. In the simple example, the only stochastic disturbance is the technology shock:

\[
\ln a_t = \rho \ln a_{t-1} + \varepsilon_t
\]

One possibility would be to just pick a value for \( \rho \) and a distribution for \( \varepsilon_t \) and feed that into the model. But we can actually do better. Recall the aggregate production function:

\[
y_t = a_t k_t^\alpha n_t^{1-\alpha}
\]

We can actually get a measurement of \( \ln a_t \) by calculating the Solow residual. As in the document “stylized business cycle facts”, I construct a measure of the capital stock using the perpetual inventory method (i.e. I start with a beginning, end of year, annual value of the capital stock, use the depreciation rate and capital’s share calibrations from above and data on investment to construct a quarterly capital stock series). Then using data on hours and output, I can construct a measure of \( a_t \), which clearly has an upward trend. The tradition is to linearly detrend it and then fit an autoregression to the linearly detrended series. This is somewhat problematic because it implies that output, etc.. should be trend stationary, while the data tend to indicate difference stationarity is the better description. Nevertheless, I proceed with the “traditional” approach here.

I first take my measure of \( a_t \) (which is logged) and estimate a linear time trend:
\[ \ln a_t = \alpha_0 + \alpha_1 t + u_t \]

Given estimates of the intercept and the slope coefficient, I get a measure of \( \hat{u}_t \). This is the empirical counterpart to the level of technology in the model. Then I estimate a univariate autoregression of this variable on its lag:

\[ \hat{u}_t = \gamma_0 + \rho \hat{u}_{t-1} + \varepsilon_t \]

\( \hat{\gamma}_0 \) should be very close to zero by construction, as \( u_t \) is, by construction since it is a regression residual, mean zero. I get the following estimates: \( \hat{\rho} = 0.965 \) and \( \hat{\sigma} = 0.0075 \), where the latter is the standard deviation of the residual \( \hat{\varepsilon}_t \). I also get a time series of \( \hat{\varepsilon}_t \).

There are two options for simulation. First, I could use the \( \hat{\rho} \) as my \( \rho \) and draw shocks from a distribution (such as a normal distribution) with standard deviation of \( \hat{\sigma} = 0.0075 \), or I could use \( \hat{\rho} \) as my \( \rho \) and feed in the actual time series of residuals from the regression above as my shocks.

I choose the latter – I feed in the actual shocks to measured TFP. I have data from the third quarter of 1949 to the fourth quarter of 2007. After running the simulation, here are the standard “business cycle moments” from the model (after first HP filtering):

<table>
<thead>
<tr>
<th>Variable</th>
<th>Volatility</th>
<th>Relative Volatility</th>
<th>Autocorrelation</th>
<th>Correlation with Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0.0161</td>
<td>1.0000</td>
<td>0.7550</td>
<td>1.0000</td>
</tr>
<tr>
<td>( c )</td>
<td>0.0101</td>
<td>0.6287</td>
<td>0.8146</td>
<td>0.9183</td>
</tr>
<tr>
<td>( i )</td>
<td>0.0518</td>
<td>3.2207</td>
<td>0.7459</td>
<td>0.9895</td>
</tr>
<tr>
<td>( n )</td>
<td>0.0088</td>
<td>0.5488</td>
<td>0.7441</td>
<td>0.9780</td>
</tr>
<tr>
<td>( y - n )</td>
<td>0.0076</td>
<td>0.4726</td>
<td>0.7850</td>
<td>0.9712</td>
</tr>
</tbody>
</table>

Below are plots of the simulated time series of these variables (HP filtered) along with their data counterparts:
The simulations are very good, subject to some caveats. In particular, the volatilities, relative volatilities, autocorrelations, and correlations with output are all “in the ballpark”. If we measure hours by average hours (i.e. the intensive margin, which is what the model has), then the model produces movements in hours that are too volatile. The correlations with output are all a little too high, and there’s a bit too much autocorrelation. But nevertheless, particularly as the plots of the data make clear, the model does a pretty amazing job.