When are Taylor Rules Contractionary and When are they Expansionary?

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Abstract

The New Keynesian model with price-stickiness and an interest rate-targeting central bank has become a workhorse for analyzing monetary policy and economic fluctuations. It is widely believed that Taylor (1993) type interest rate rules embody a “lean against the wind” feature since nominal interest rates respond more than one for one to inflation and positively to fluctuations in the output gap. I derive a useful analytical approximation that shows that Taylor rules move the real interest rate in the same direction as the natural rate of interest (the real rate that would obtain in the absence of price stickiness), but less than one for one. As such, Taylor rules are expansionary in response to shocks which would raise real rates in a flexible price model and contractionary in response to shocks which lower the natural rate of interest. I conclude with an application to technology shocks, and show that, in the baseline model, permanent technology shocks lead to positive output gaps and increases in hours worked, whereas transitory technology shocks lead to negative output gaps and impact declines in hours.

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1 Introduction

The New Keynesian model with price-stickiness has become a workhorse model for analyzing monetary policy and economic fluctuations. Most variants of the model from the last decade include an interest rate targeting rule as a description of central bank behavior. Called “Taylor rules” after John Taylor (1993), such rules are thought to provide both a good positive description of, and normative prescription for, monetary policy. The basic rule calls for central banks to adjust overnight interest rates more than one for one to deviations of inflation from target and positively to deviations of output from potential. It is thus widely believed and frequently asserted that Taylor rules embody a “lean against the wind” feature – the central bank figuratively puts its foot on the brake whenever the economy begins to soar but hits the gas pedal when things start to sour.1

Actually interpreting and gaining intuition for how Taylor rules work in a dynamic stochastic general equilibrium model is much more difficult than the simple “lean against the wind” interpretation, however. This is because the Taylor rule features nominal interest rates responding to endogenous variables – inflation and a measure of the output gap. Because movements in these variables are in turn functions of the exact form and parameters of the rule, it is not a well-conceived thought experiment to ask “what does the central bank do to interest rates if inflation or the output gap rise?” One needs to know what structural disturbance is generating those movements, and how those movements interact with the form of the rule, in order to be able to say much more.

In the context of a basic, textbook version of the New Keynesian model with a Taylor rule (e.g. Gali (2008)), I derive a useful analytical approximation that expresses the equilibrium real interest rate as a function of exogenous variables only. Doing so allows one to explicitly think about whether the stance of monetary policy is relatively “tight” or comparatively “loose”. In particular, the analytical approximation reveals that equilibrium real interest rates respond in the same direction as movements in the natural rate of interest, but less than one for one. The natural rate of interest (Woodford (2003), Wicksell (1898)), is defined as the real interest rate that would obtain in the absence of price rigidity. It responds to any structural shock which would obtain in the absence of price rigidity. It responds to any structural shock which would affect the variables of a flexible price DSGE model.

I define as “expansionary” a situation in which equilibrium real rates are less than the natural rate (so that the interest rate “gap” is negative), and “contractionary” when real rates are higher than the natural rate. Because a Taylor rule following central bank moves real rates in the same direction as fluctuations in the natural rate but less than one for one, monetary policy is therefore expansionary in response to structural shocks which raise the

1See, e.g., Carlstrom and Fuerst (2003).
natural rate and contractionary in response to shocks which lower the natural rate. Put differently, there is an inherent asymmetry in the way that Taylor rules work in conventional New Keynesian models – whether policy is relatively tight or relatively loose depends on which way the natural rate of interest is moving.

I conclude with a revealing application to technology shocks. A large empirical literature, perhaps best exemplified by Gali (1999), identifies technology shocks using a long run restriction in a vector autoregression. Although not without controversy, this literature often finds that positive productivity shocks lower hours of work on impact. Many have proposed that the sticky price, New Keynesian framework can account for this empirical finding.

Shocks which permanently increase the level of technology must temporarily raise the natural rate of interest in the New Keynesian model. As such, my approximation suggests that a Taylor rule following central bank pursues a relatively expansionary policy following such a shock. In the benchmark version of the model, a positive, permanent productivity shock should therefore lead to an impact increase in hours of work. I verify that this is exactly what happens in the basic New Keynesian model – when technology permanently increases, hours of work increase. Furthermore, the impact increase in hours of work is actually increasing in the amount of price rigidity. As such, price stickiness cannot be the source of impact decreases in hours of work to a positive, permanent technology shock if the central bank obeys a Taylor rule.³

The remainder of the paper is organized as follows. Section 2 lays out the basic New Keynesian model and derives an approximate expression for the equilibrium real interest rate. The Appendix provides a fuller and more complete derivation. Section 3 examines how technology shocks interact with the Taylor rule and price stickiness in the model. The final section concludes.

2 The New Keynesian Model

In its simplest form, the New Keynesian model can be reduced to three log-linear equations: a Phillips Curve, an IS/demand equation, and a monetary policy rule. For a more detailed discussion see the Appendix, Woodford (2003), or Gali (2008).

²In particular, see Christiano, Eichenbaum, and Vigfusson (2004) for an alternative viewpoint on the impact effect of technology shocks on hours.

³As I discuss in Section 3, transitory technology shocks – which lower the natural rate of interest in the model – are associated with decreases in hours in the model. The lack of permanence of such shocks is not consistent with the empirical identification.
\[
\pi_t = \gamma(y_t - y^f_t) + \beta E_t \pi_{t+1} \tag{1}
\]

\[
y_t = E_t y_{t+1} - \sigma (i_t - E_t \pi_{t+1}) \tag{2}
\]

\[
i_t = \phi_\pi \pi_t + \phi_y (y_t - y^f_t) + u_t \tag{3}
\]

\[
u_t = \rho_u u_{t-1} + \varepsilon_{i,t}
\]

The first equation is the Phillips Curve: \( \gamma \) is a reduced-form parameter reflecting the degree of price-stickiness and strategic complementarity, while \( \beta \) is a subjective discount factor.\(^4\)

The second equation is the “IS curve”, which is simply the consumption Euler equation imposing the accounting identity that consumption equal output (there is no capital in the benchmark model) and the Fisher relationship between nominal and real interest rates. \( \sigma \) is the inverse elasticity of intertemporal substitution. The third equation is a Taylor (1993) type nominal interest rate rule (with constants omitted); \( u_t \) is a potentially autocorrelated disturbance term, with \( \varepsilon_{i,t} \) an iid monetary policy shock.\(^5\) I restriction attention to cases in which \( 0 < \rho_u < 1 \) and \( \phi_\pi > 1 \) and \( \phi_y \geq 0 \). \( y^f_t \) is the level of output that would obtain if prices were flexible; it can be thought of as obeying an exogenous stochastic process, to be discussed in more detail below.

Define \( x_t \equiv y_t - y^f_t \), and let \( r^f_t \) be the “natural rate of interest” – the real interest rate that would obtain if prices were flexible. This can be found by imposing that output equal the flexible price equilibrium level in the IS equation:

\[
r^f_t = \frac{1}{\sigma} \left( E_t y^f_{t+1} - y^f_t \right)
\]

In other words, the natural rate is proportional to expected growth in the flexible price level of output. Using these facts, we can eliminate \( y_t \) and \( y^f_t \) as variables and can re-write the equations of the model more compactly as:

\(^4\)It is common to include a “cost-push” shock into the Phillips Curve (e.g. Woodford (2003)). Doing so affects none of the results below.

\(^5\)In the Appendix I consider the case where there is an explicit nominal interest rate smoothing motive. This makes the math somewhat more cumbersome, but does not alter conclusions. Also note that I impose a constant intercept in the policy rule. It is possible to derive an “optimal” rule in which the intercept is stochastic and tracks fluctuations in the natural rate of interest (Woodford (2003), Cochrane (2007), Sims (2008)). Such a rule would be difficult to implement in practice; as such I restrict attention to the empirically implemented constant intercept rule.
Equation (1.2) says that the expected growth rate of the output gap is positively proportional to the gap between the actual and natural rate of interest. It can be solved forward to express the current output gap as the sum of expected future interest rate gaps:

\[ x_t = \frac{1}{\sigma} E_t \sum_{j=0}^{\infty} (r_{t+j} - r_{t+j}^f) \]

Monetary policy can be thought of as expansionary if the sum of interest rate gaps is negative and contractionary if the sum of these gaps is positive. If the central bank sets real rates too low relative to the natural rate of interest, this will drive output above the flexible price level today and vice versa.

The model can be closed by assuming an exogenous stochastic process for the natural rate of interest; as shown in the Appendix, this is equivalent to assuming an exogenous process for the flexible price equilibrium level of output:

\[ r_{t}^f = \rho_r r_{t-1} + \zeta \varepsilon_{s,t} \]

\( \varepsilon_{s,t} \) can be thought of as a “supply shock” – any shock which would affect real interest rates in a flexible price model. There are no restrictions on the sign of \( \zeta \) – some “supply” shocks should raise the natural rate while others might lower it. I restrict attention to the case where \( 0 \leq \rho_r \leq 1 \).

### 2.1 Approximate Expression for the Real Interest Rate

Interpreting how exactly the Taylor rule works in the model is inherently difficult because it is assumed that nominal interest rates respond to endogenous variables – inflation and the output gap. Movements in both of these variables are in turn functions of the Taylor rule parameters. Here I derive a useful analytical approximation for the equilibrium behavior of
the real interest rate in the model that expresses the real interest rate as a function only of exogenous variables.

Begin by using the Fisher relationship (i.e. \( r_t = i_t - E_t \pi_{t+1} \)) to eliminate the current nominal interest rate from the Taylor rule:

\[
  r_t = -E_t \pi_{t+1} + \phi_{\pi} \pi_t + \phi_x x_t + u_t
\]

Now substitute in the Phillips Curve to eliminate current inflation and simplify:

\[
  r_t = -E_t \pi_{t+1} + \phi_{\pi} (\gamma x_t + \beta E_t \pi_{t+1}) + \phi_x x_t + u_t
\]

\[
  r_t = (\phi_{\pi} \gamma + \phi_x) x_t + (\beta \phi_{\pi} - 1) E_t \pi_{t+1} + u_t
\]

To simplify matters, I will use the approximation that \((\beta \phi_{\pi} - 1) E_t \pi_{t+1} \approx 0\). I will verify that this approximation is suitable below. Armed with this, I express the current output gap in terms of the current real interest rate and monetary policy disturbance:

\[
  x_t = \frac{1}{\phi_{\pi} \gamma + \phi_x} r_t - \frac{1}{\phi_{\pi} \gamma + \phi_x} u_t
\]

Plug this expression for the output gap into the IS equation and simplify:

\[
  r_t = r_t^f + \sigma E_t x_{t+1} - \sigma x_t
\]

\[
  r_t = r_t^f + \sigma E_t \left( \frac{1}{\phi_{\pi} \gamma + \phi_x} r_{t+1} - \frac{1}{\phi_{\pi} \gamma + \phi_x} u_{t+1} \right) - \sigma \left( \frac{1}{\phi_{\pi} \gamma + \phi_x} r_t - \frac{1}{\phi_{\pi} \gamma + \phi_x} u_t \right)
\]

\[
  r_t = \left( \frac{\phi_{\pi} \gamma + \phi_x}{\phi_{\pi} \gamma + \phi_x + \sigma} \right) r_t^f + \left( \frac{\sigma}{\phi_{\pi} \gamma + \phi_x + \sigma} \right) E_t r_{t+1} - \left( \frac{\sigma}{\phi_{\pi} \gamma + \phi_x + \sigma} \right) E_t u_{t+1} + \left( \frac{\sigma}{\phi_{\pi} \gamma + \phi_x + \sigma} \right) u_t
\]

To economize on notation, define the following auxiliary parameter:

\[
  \psi = \frac{\sigma}{\phi_{\pi} \gamma + \phi_x + \sigma}
\]

The expression for the real interest rate above can be solved forward using the terminal condition that \( E_t r_{t+j} \to 0 \) as \( j \to \infty \) (recall that this is the deviation of the real rate from its steady state), yielding:

\[
  r_t = \left( \frac{\phi_{\pi} \gamma + \phi_x}{\phi_{\pi} \gamma + \phi_x + \sigma} \right) E_t \sum_{j=0}^{\infty} \psi^j r_{t+j}^f + \psi u_t + (\psi - 1) E_t \sum_{j=1}^{\infty} \psi^j u_{t+j}
\]
Now, using the facts that $E_t r^f_{t+j} = \rho_f^j r^f_t$ and $E_t u_{t+j} = \rho_u^j u_t$, one can write the expression for the real interest rate as:

$$r_t = \left( \frac{\phi_x \gamma + \phi_x}{\phi_x \gamma + \phi_x + \sigma} \right) \frac{r^f_t}{1 - \rho_r \psi} + \left( \frac{\psi (1 - \rho_u)}{(1 - \rho_u \psi)} \right) u_t$$

Using the definition of $\psi$, one obtains:

$$r_t = \left( \frac{\phi_x \gamma + \phi_x}{\phi_x \gamma + \phi_x + \sigma (1 - \rho_r)} \right) r^f_t + \left( \frac{\sigma (1 - \rho_u)}{\phi_x \gamma + \phi_x + \sigma (1 - \rho_u)} \right) u_t \quad (5)$$

This expression has a very intuitive interpretation. Both the coefficient on the output gap and on the monetary policy disturbance are positive and less than unity. Put differently, both real and monetary shocks move the real interest rate, but less than one for one. As prices become flexible, the slope of the Phillips Curve becomes large ($\gamma \to \infty$), which drives the coefficient on the natural rate of interest to one and the coefficient on the monetary policy disturbance to zero. Conversely, if prices are very sticky, $\gamma \to 0$, the equilibrium real interest rate is rather unresponsive to changes in the natural rate of interest but very responsive to the monetary policy disturbance.

This approximate expression for the equilibrium behavior of the real interest rate is useful because it allows one to draw a couple of qualitative conclusions. First, real rates move in the same direction as shocks to the policy rule; the extent to which this move occurs depends on how sticky prices are and how responsive the Taylor rule is to inflation and the output gap. Secondly, the real rate moves in the same direction, but less in magnitude, than does the natural rate of interest. This means that positive shocks to the natural rate open up a negative interest rate gap, while negative shocks to the natural rate lead to a positive interest rate gap.

There is thus an asymmetry in terms of the stance of monetary policy under a Taylor rule. Monetary policy is expansionary in response to positive shocks to the natural rate and relatively contractionary when a shock lowers the natural rate. An example of a real shock which would raise the natural rate in the model is a positive government spending or preference shock; the Taylor rule following central bank would allow output to expand by more than it would to these shocks if prices were fully flexible. Conversely, in the model a positive temporary technology shock would lower the natural rate; the Taylor rule would be relatively contractionary in response to such a shock, allowing output to expand by less than it would if prices were flexible.
2.2 Quantitative Analysis

Figures 1 and 2 show impulse responses to both the real and monetary policy shocks in a quantitative version of the model. I use the following parameter configuration: $\sigma = 1$, $\gamma = 0.08$, $\beta = 0.99$, $\rho_r = 0.9$, $\phi_r = 1.5$, $\phi_y = 0.5$, $\zeta = 1$, and $\rho_u = 0.9$. The size of both shocks is set to 0.1 percent. A positive shock to the natural rate of interest leads to a persistent increase in the output gap, the actual real interest rate, and inflation. The real interest rate undershoots the response of the natural rate; it is this positive real interest rate gap that leads to the positive response of the output gap. A positive shock to the Taylor rule raises real interest rates and lowers the output gap and inflation. By construction, there is no effect of the monetary policy shock on the natural rate of interest.

Figure 3 provides evidence on the numerical quality of the analytic approximation for the behavior of the real interest rate. The upper panel shows the responses of the actual and approximated real interest rates and real interest rate gaps to the real shock, while the lower panel does the same for the monetary shock. It is evident that the approximation is quite good, both qualitatively and quantitatively. Given that the approximation makes use of the assumption that $(\beta \phi_r - 1) E_t \pi_{t+1} \approx 0$, it is quantitatively better the less persistent are shocks (so that $E_t \pi_{t+1}$ is smaller), the smaller is the response of nominal rates to inflation in the Taylor rule, and the more heavily households discount future utility flows. In spite of the small differences, the approximation remains quite good for any plausible combination of parameters.

3 Taylor Rules and Technology Shocks

A large literature, dating back to Shapiro and Watson (1988) and Blanchard and Quah (1989), uses long run restricted structural VARs to identify “supply” and “demand” shocks. More recently, Gali (1999) identifies technology shocks as the source of the unit root in average labor productivity. He finds that a positive technology shock is associated with an impact decrease in hours worked, which is at odds with sensibly parameterized real business cycle models. Basu, Fernald, and Kimball (2006) obtain similar results using alternative growth accounting techniques. These authors have argued that the empirical evidence in support of “contractionary” technology shocks lends credence to sticky price, New Keynesian models.

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As shown more formally in the Appendix, the calibrated value of $\gamma$ is consistent with a Calvo parameter of 2/3, which implies that firms change prices once every three quarters on average; a coefficient consistent with labor’s share of 2/3; and a steady state markup of price over marginal cost of 10 percent.

The model is solved using Dynare.
The intuition for why sticky prices could help explain “contractionary” technology shocks is most easily seen by assuming that the central bank obeys an exogenous money growth rule instead of a Taylor type rule. Assuming that the interest elasticity of money demand is near zero, equilibrium in the money market requires that a quantity type equation hold:

\[ M_t = P_t Y_t \]

If prices cannot fully adjust and the central bank keeps the money supply unchanged, then output must respond by less than it would if prices were flexible when the economy is hit by any external shock. If prices are sticky enough, then the response of output to a positive technology shock could be quite small, and employment might fall.

The quantity-theoretic intuition does not, in general, apply to the same economy when the central bank follows a Taylor rule. In such an economy, the money supply is set endogenously so as to achieve a given interest rate target. As described in the Appendix, assume that the production function is:

\[ Y_t = A_t N_t^{1-a} \]

If the conditions for balanced growth are satisfied, then it is straightforward to show that employment is constant in the flexible price equilibrium. Letting lowercase letters denote log-deviations from steady state, it then follows that:

\[ y^f_t = a_t \]  

(6)

In other words, in the flexible price equilibrium output is simply equal to the level of technology.

Assume first that technology follows a stationary autoregressive process \((0 \leq \rho_a \leq 1)\):

\[ a_t = \rho_a a_{t-1} + \varepsilon_{a,t} \]  

(7)

Plugging (6) into (7), and then using the definition of the natural rate of interest, one obtains:

\[ r^f_t = \rho_a r^f_{t-1} + (\rho_a - 1)\varepsilon_{a,t} \]  

(8)

In terms of the general parameters shown in equation (4), \(\rho_r = \rho_a\) and \(\zeta = (\rho - 1) < 0\). This means that the natural rate of interest falls in response to a positive, transitory technology shock; the amount by which the natural rate declines depends on how transitory the shock is.

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8With additive separability between consumption and leisure this requires a unit elasticity of intertemporal substitution in consumption. i.e. \(\sigma = 1\).
From the approximation of Section 2.1, one would then expect that a Taylor rule following central bank would also lower real interest rates in response to the shock, but not by as much as the decline in the natural rate. As such, the real interest rate gap would go positive, which would lead to a negative output gap and a decline in employment.

Figure 4 shows actual impulse responses of technology, hours, the output gap, and the real interest rate gap (actual, not approximated) from the model with $\rho_a = 0.9$ (the remaining parameters are as calibrated in Section 2.2). It is easy to see that the shock leads to a positive interest rate gap, a negative output gap, and a decline in hours. Following an almost identical exercise as to that shown in Figure 4, Gali (2008) states that “such a [downward] response of employment is consistent with much of the recent empirical evidence on the effects of technology shocks” (p. 55).

As noted above, however, the empirical evidence assumes that technology shocks permanently affect productivity and output. So as to be consistent with the empirical identification, suppose that neutral technology follows an AR(1) in its growth rate:

$$\Delta a_t = \rho_a \Delta a_{t-1} + \varepsilon_{a,t}$$

(9)

Using this process one can derive an expression for the natural rate of interest:

$$r^f_t = \rho_a r^f_{t-1} + \rho_a \varepsilon_{a,t}$$

(10)

When the technology shock is permanent ($\rho_a \geq 0$), the natural rate of interest increases in response to a positive shock. From the approximation result, this means that in equilibrium real rates should rise, but by less than the increase in the natural rate. This should lead to real interest rates that are too low relative to the natural rate and therefore to a positive output gap and an increase in hours.

Figure 5 shows responses to the permanent technology shock for a value of $\rho_a = 0.5$ (the remaining parameters are the same as in Figure 4). As predicted by the simple analytical approximation, the real interest rate gap goes negative. Hours increase on impact, as do the output gap and inflation.$^9$

From expression (5), one can say something about the interaction between price-stickiness and the response of hours to permanent technology shocks. The coefficient on the natural rate of interest is strictly increasing in $\gamma$, the slope coefficient of the Phillips Curve. As prices become stickier, $\gamma$ becomes smaller, and the equilibrium real interest rate responds by

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$^9$Gali, Lopez-Salido, and Valles (2003) show a similar result concerning the movement of hours following a permanent technology shock, but do not stress or seek to understand it. Their paper is more focused on identifying the structure of the monetary policy rule as opposed to the interaction between the rule and structural shocks.
less to fluctuations in the natural rate of interest. The real interest rate gap thus responds by more, in absolute value, to natural rate shocks, the stickier are prices. This means that the response of the output gap, and hence hours, should be more positive to a permanent technology shock the stickier are prices.

Figure 6 confirms this result quantitatively. It shows impulse responses of hours to a positive, permanent technology shock (using the assumed process above), for different values of the Calvo (1983) parameter governing price-stickiness: $\phi = 0.33$, $\phi = 0.66$, and $\phi = 0.98$. One clearly observes that the impact increase in hours is strictly increasing in the amount of price-stickiness; the impulse response of hours to a positive technology shock is also more persistent the stickier are prices.

So as to investigate the empirical implications of this model, I generate 2000 data sets from the model, each with 200 observations, using the benchmark parameterization described above. On each simulated data set, I estimate a two variable VAR (with four lags and a constant) with the growth rate of average labor productivity and the level of labor hours. I then orthogonalize the innovations following Gali (1999), using a long run restriction to identify a “supply” shock (which corresponds to the permanent technology shock in the model) and a “demand” shock (corresponding to the monetary policy shock).

The theoretical and simulated responses are shown in Figure 7. The average estimated responses across the simulation are given by the solid dark line, while the theoretical responses are given by the dashed line. The shaded gray area represents the 90% confidence region from the simulations. It is clear that the long run restriction does a very good job at identifying the two structural shocks. Importantly, labor hours clearly rise both in the model and in the model generated data in response to a positive productivity shock. In only about 6 percent of the simulated samples do hours decline on impact in response to the favorable productivity shock; further, the impact decline in these cases is never anywhere as large as has been frequently estimated in the data. This result is robust across different parameterizations of the model – with a Taylor rule and permanent technology shocks, hours always rise on impact in both the model and identified impulse responses from model generated data.

The result that hours rise following a permanent technology shock is quite robust to various substantive variations and/or extensions of the basic model, as long as the basic structure of the Taylor rule is kept the same. Inclusion of capital in the model, for example, makes it likely that hours will rise in response to even transitory technology shocks. This happens because capital in the production function ties the natural rate of interest to the marginal product of capital. Transitory but persistent technology shocks raise the marginal product of capital, and will thus lead to increases in the natural rate. Other features,
such as habit formation in consumption (see, e.g. Francis and Ramey (2005)), are capable of generating impact declines in hours to positive technology shocks, but it is never price stickiness per se that is responsible for this decline.

4 Conclusion

Obtaining intuition for how exactly Taylor type nominal interest rate rules work is difficult because the Taylor rule expresses an endogenous variable as functions of other endogenous variables. Put differently, engaging in the thought experiment of “if inflation and/or the output gap increase then what happens to nominal interest rates?” is not a well-defined exercise. This paper provides a useful analytical approximation for gaining intuition for how Taylor rules work in a simple New Keynesian model. In particular, I show that Taylor rules move the real interest rate in the same direction as fluctuations in the natural rate of interest, but less than one for one. This means that monetary policy is expansionary in response to shocks which raise the natural rate of interest and contractionary in response to shocks which lower the natural rate.

I conclude with an application to technology shocks, and show that permanent technology shocks necessarily raise hours of work on impact in both the model and in responses estimated from model-generated data. This result is important because a large empirical literature argues that the apparent “contractionary” effects of technology shocks in the data point to New Keynesian models with price-stickiness. I show, both analytically using the approximation and quantitatively, that the New Keynesian model with a conventional Taylor type interest rate rule cannot account for these empirical findings.
References


5 Appendix

The Appendix presents a fuller and more complete version of the linearized NK model of Section 2.

5.1 Households

Each period, the representative household choose consumption, employment, nominal money holdings, and holdings of a riskless one period bond to maximize expected discounted lifetime utility subject to a nominal flow budget constraint:

$$\max_{C_t, B_t, N_t, M_t} \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \frac{C_t^{1-1/\sigma} - 1}{1 - 1/\sigma} - \chi \frac{N_t^{1+1/\eta}}{1 + 1/\eta} \frac{(M_t/P_t)^{1-\nu}}{1 - \nu} \right\}$$

s.t.

$$P_t C_t + B_t + M_t \leq W_t N_t + (1 + i_{t-1}) B_{t-1} + \Pi_t + M_{t-1}$$

$P_t$ is the price of the composite consumption good, $\sigma$ is the elasticity of intertemporal substitution, $\eta$ is the Frisch labor supply elasticity, $\chi$ is a scaling parameter, $i$ is the riskless nominal interest rate, and $\Pi$ denotes lump sum transfers (i.e. dividends). The first order conditions are:

$$C_t^{-1/\sigma} = \beta E_t \left( C_{t+1}^{-1/\sigma} \frac{(1 + i_t) P_t}{P_{t+1}} \right)$$

$$N_t^{1/\eta} = C_t^{-1/\sigma} W_t$$

$$(M_t/P_t)^{-\nu} = C_t^{-1/\sigma} \left( \frac{1 + i_t}{i_t} \right)$$

Letting lower-case variables denote logs, the linearized first order conditions are:

$$E_t c_{t+1} = c_t + \sigma (i_t - E_t \pi_{t+1}) \quad (A.1)$$

$$\frac{1}{\eta} n_t = -\frac{1}{\sigma} c_t + w_t - p_t \quad (A.2)$$

$$-\nu m_t + v p_t = -\frac{1}{\sigma} c_t - \left( \frac{\beta}{1 - \beta} \right) i_t \quad (A.3)$$
5.2 Firms

The final composite consumption/output good is a CES aggregate of a continuum of intermediate goods indexed by $j$ over the unit interval:

$$Y_t = \left[ \int_0^1 Y_t(j)^{\frac{\xi-1}{\xi}} dj \right]^{\frac{1}{\xi-1}}$$

The price index is:

$$P_t = \left[ \int_0^1 P_t(j)^{1-\xi} dj \right]^{\frac{1}{1-\xi}}$$

These together imply a demand function for intermediate goods of the form:

$$Y_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\xi} Y_t$$

Each intermediate goods producer has a production function of the form:

$$Y_t(j) = A_t N_t(j)^{1-\alpha}$$

Cost minimization implies the static labor demand condition:

$$\frac{W_t}{P_t} = MC_t (1 - \alpha) A_t N_t(j)^{-\alpha}$$

In a symmetric equilibrium, the $j$ subscripts can be ignored and the log-linearized labor demand and production function are given by:

$$y_t = a_t + (1 - \alpha) n_t \quad (A.3)$$

$$w_t - p_t = mc_t + a_t - \alpha n_t \quad (A.4)$$

In the absence of price rigidity, firms would choose prices so that there is a constant markup of price over marginal cost. This steady state markup is given by:

$$\mu^* = \frac{\xi}{\xi - 1}$$

Firms are not freely able to adjust prices each period. In particular, they face a constant hazard of $1 - \theta$ of being able to update (equivalently a probability of $\theta$ each period of not being able to update). Firms with the opportunity to update their prices in any period
solve (in the symmetric equilibrium firms with the opportunity to update will all choose the same price, so I ignore the $j$ subscripts):

$$\max_{P_t} \sum_{k=0}^{\infty} E_t \left\{ \theta^k \beta^k \left( \frac{C_{t+k}}{C_t} \right)^{-1/\sigma} \left( P_t^* \left( \frac{P_t}{P_{t+k}} \right)^{-\xi} Y_{t+k} - \Psi \left( \left( \frac{P_t^*}{P_{t+k}} \right)^{-\xi} Y_{t+k} \right) \right) \right\}$$

$\beta^k \left( \frac{C_{t+k}}{C_t} \right)^{-1/\sigma}$ is the stochastic discount factor, $\left( \frac{P_t^*}{P_{t+k}} \right)^{-\xi} Y_{t+j}$ is expected output of the intermediate good firm in period $k$, and $\Psi(\cdot)$ is the total cost function. Solving this maximization problem, aggregating, and linearizing about the zero inflation steady state yields the New Keynesian Phillips Curve:

$$\pi_t = (1 - \theta)(1 - \theta \beta) \zeta mc_t + \beta E_t \pi_{t+1}$$

(A.5)

Where $\zeta = \frac{1 - \alpha}{1 - \alpha + \alpha \xi}$.

### 5.3 The Natural Rate

Combine the first order condition for labor supply with the firm’s static labor demand condition to eliminate the real wage:

$$N_{t}^{1/\eta} = C_{t}^{-1/\sigma} M C_t (1 - \alpha) A_t \sigma_t^{-\alpha} = C_{t}^{-1/\sigma} \frac{1}{\mu_t} (1 - \alpha) \frac{Y_t}{N_t}$$

The aggregate resource constraint is that $C_t = Y_t$. Imposing that $\sigma = 1$ (which is necessary to simultaneously have balanced growth with stationary labor hours and separability between consumption and leisure in the flow utility function, see King, Plosser, and Rebelo (1988)) and then eliminating $Y_t$ yields:

$$N_{t}^{1/\eta} = \frac{1}{\mu_t} (1 - \alpha) \frac{1}{N_t} \Rightarrow N_t = \left( \frac{1 - \alpha}{\mu_t} \right)^{1/\eta}$$

In the flexible price equilibrium, as noted above, the markup is constant: $\mu_t = \mu^*$, or equivalently, real marginal cost is constant. This implies that employment is constant in the flexible price equilibrium and does not respond to technology shocks. With constant employment, the flexible price equilibrium level of output is just equal to a constant times the level of technology, or, in logs:

$$y_t^f = a_t$$
Armed with this, one can substitute into the labor demand and supply relationships to derive a relationship between real marginal cost and the output gap:

\[ mc_t = \kappa(y_t - y_t^f) \]

Where \( \kappa = \frac{1}{\sigma} + \frac{1/\eta + \alpha}{1 - \alpha} \). As such, the coefficient on the output gap in the Phillips Curve can be written:

\[ \gamma = \frac{(1 - \theta)(1 - \theta\beta)}{\theta} \left( \frac{1 - \alpha}{1 - \alpha + \alpha\xi} \right) \left( \frac{1}{\sigma} + \frac{1/\eta + \alpha}{1 - \alpha} \right) \]

In Section 2, the calibrated value of \( \gamma \) is 0.083. This presumes values for the underlying structural parameters as follows: \( \theta = 0.67, \alpha = 0.33, \beta = 0.99, \xi = 11, \sigma = 1, \) and \( \eta = 1 \). This parameterization of \( \xi \) yields a steady state markup of 10 percent, while the value of \( \eta \) is set equal to the central estimate in Shapiro and Kimball (2003) of the Frisch labor supply elasticity.

The natural rate of interest, \( r^n_t \), can be found by solving from the log-linearized Euler equation after substituting in for the flexible price level of output:

\[ E_t a_{t+1} = a_t + \sigma r^n_t \Rightarrow r^n_t = \frac{1}{\sigma} (E_t a_{t+1} - a_t) \quad (A.6) \]

In other words, the natural rate of interest is simply proportional to expected growth in technology. When technology follows an exact random walk, the natural rate of interest is constant.

Note that the Taylor rule, the Phillips Curve, and the IS equation jointly determine the behavior of inflation, the output gap, and the nominal interest rate. From the Fisher relationship this then yields a time path for the real interest rate. The level of output plus the nominal interest rate is enough to determine the money supply from the demand for real balances without explicitly specifying a money supply relationship.

### 5.4 Alternative Real Interest Rate Approximation

As an alternative to equation (3) in the text, many authors (e.g. Clarida, Gali, and Gertler (1999)) assume an explicit interest rate smoothing desire on the part of central banks. Let the Taylor rule under such a specification be given by:

\[ i_t = \rho_i i_{t-1} + (1 - \rho_i)\phi \pi_t + (1 - \rho_i)\phi x_t + \varepsilon_{i,t} \quad (A.7) \]

Put differently, one can think of the current nominal rate as a convex combination of the
previous period’s nominal rate and the “target” nominal rate that would be given by the rule like (3). Here \( \varepsilon_{i,t} \) is assumed to be mean zero and iid. As in the main body of the text, use the Fisher relationship to eliminate the current nominal rate:

\[
    r_t = \rho_t \pi_{t-1} - E_t \pi_{t+1} + (1 - \rho_t) \phi_x \pi_t + (1 - \rho_t) \phi_x x_t + \varepsilon_{i,t}
\]

Use the Fisher relationship again to eliminate the lagged nominal rate:

\[
    r_t = \rho_t r_{t-1} - E_t \pi_{t+1} + ((1 - \rho_t) \phi_x + \rho_t) \pi_t + (1 - \rho_t) \phi_x x_t + \varepsilon_{i,t}
\]

Plug in the Phillips Curve to eliminate current inflation:

\[
    r_t = \rho_t r_{t-1} - E_t \pi_{t+1} + ((1 - \rho_t) \phi_x + \rho_t) \pi_t + (1 - \rho_t) \phi_x x_t + \varepsilon_{i,t}
\]

Similarly as in the text, use the approximation that \( \beta ((1 - \rho_t) \phi_x + \rho_t) = 1 ) E_t \pi_{t+1} \approx 0:

\[
    r_t = \rho_t r_{t-1} + ((1 - \rho_t) \phi_x + \rho_t) x_t + \varepsilon_{i,t}
\]

To economize on notation, let \( \vartheta = (1 - \rho_t) (\phi_x + \phi_x) + \rho_t \gamma \). Solve for the output gap in terms of the current and lagged real rate:

\[
    x_t = \frac{1}{\vartheta} r_t - \frac{\rho_t}{\vartheta} r_{t-1} - \frac{1}{\vartheta} \varepsilon_{i,t}
\]

Now use the IS equation, repeated here:

\[
    r_t = r_t^f + E_t x_{t+1} - x_t
\]

Plug the above in for the output gap in terms of the real rates:

\[
    r_t = r_t^f + \frac{1}{\vartheta} E_t r_{t+1} - \frac{\rho_t}{\vartheta} r_t - \frac{1}{\vartheta} E_t \varepsilon_{i,t+1} - \frac{1}{\vartheta} r_t + \frac{\rho_t}{\vartheta} r_{t-1} + \frac{1}{\vartheta} \varepsilon_{i,t}
\]

Note that \( E_t \varepsilon_{i,t+1} = 0 \). Solve for the current real interest rate and simplify:

\[
    r_t = r_t^f + \frac{1}{\vartheta} E_t r_{t+1} - \left( \frac{1 + \rho_t}{\vartheta} \right) r_t + \frac{\rho_t}{\vartheta} r_{t-1} + \frac{1}{\vartheta} \varepsilon_{i,t}
\]

\[
    r_t = \left( \frac{\rho_t}{1 + \vartheta + \rho_t} \right) r_{t-1} + \left( \frac{\vartheta}{1 + \vartheta + \rho_t} \right) r_t^f + \left( \frac{1}{1 + \vartheta + \rho_t} \right) E_t r_{t+1} + \left( \frac{1}{1 + \vartheta + \rho_t} \right) \varepsilon_{i,t}
\]
Simplify on notation again. Let $\lambda = \frac{1}{1 + \theta + \rho}$:

$$r_t = \rho_t \lambda r_{t-1} + \theta \lambda r_t^f + \lambda E_t r_{t+1} + \lambda \varepsilon_{i,t}$$

Now solve forward:

$$r_t = \rho_t \lambda r_{t-1} + \rho \lambda^2 \sum_{j=0}^{\infty} \lambda^j E_t r_{t+j} + \theta \lambda \sum_{j=0}^{\infty} \lambda^j E_t r_t^f + \lambda \varepsilon_{i,t}$$

Assume that all but the $j = 0$ term matters for the forward interest rate (this will be a pretty good assumption, as $\rho_t \lambda^3$, for example, is going to be pretty close to zero). Then we have:

$$r_t = \rho_t \lambda r_{t-1} + \rho_t \lambda^2 r_t + \theta \lambda \sum_{j=0}^{\infty} \lambda^j E_t r_t^f + \lambda \varepsilon_{i,t}$$

Using the process for the natural rate, we can eliminate the infinite sum:

$$r_t = \rho_t \lambda r_{t-1} + \rho_t \lambda^2 r_t + \theta \lambda \sum_{j=0}^{\infty} (\lambda \rho_r)^j + \lambda \varepsilon_{i,t}$$

$$r_t = \rho_t \lambda r_{t-1} + \rho_t \lambda^2 r_t + \frac{\theta \lambda}{1 - \lambda \rho_r} r_t^f + \lambda \varepsilon_{i,t}$$

Simplifying:

$$r_t = \frac{\rho_t \lambda}{1 - \rho_t \lambda^2} r_{t-1} + \frac{\theta \lambda}{(1 - \rho_t \lambda^2)(1 - \lambda \rho_r)} r_t^f + \lambda \varepsilon_{i,t}$$

It is relatively straightforward to show that $0 \leq \frac{\theta \lambda}{(1 - \rho_t \lambda^2)(1 - \lambda \rho_r)} \leq 1$. The non-negativity is trivially satisfied, as all of the parameters are positive but less than one. As prices become flexible, $\theta \to \infty$, $\lambda \to 0$, and $\theta \lambda \to 1$. We thus obtain a similar qualitative result to that in the text – as prices become more flexible, the equilibrium real interest rate responds more to the the natural rate and less to the monetary policy disturbance. As long as there is some price-stickiness, $\theta$ is finite and the coefficient on the natural rate is less than unity, so one again sees that real rates fluctuate less than one for one with the natural rate. Quantitative simulations reveal that this approximation is also quite good.
Figure 1
Impulse Responses to Natural Rate Shock

Figure 2
Impulse Responses to Monetary Policy Shock

Notes: impulse responses in Figures 1 through 5 are obtained from solving the NK model as described in text at parameter values as described in text.
Figure 3
Actual and Approximated Responses of Real Interest Rates

Real Rate Responses to Real Shock

Real Rate Gap Responses to Real Shock

Real Rate Responses to Monetary Shock

Real Rate Gap Responses to Monetary Shock

Figure 4
Impulse Responses to Transitory Technology Shock

Technology

Hours

Output Gap

Real Interest Rate Gap
Figure 5
Impulse Responses to Permanent Technology Shock

Figure 6
Impulse Responses of Hours to Technology for Different Amounts of Price Stickiness

Note: These are impulse responses of hours to a permanent technology shock for different values of price-stickiness, which impacts the slope coefficient, $\gamma$, of the Phillips Curve.
Figure 7
Simulated and Theoretical IRFs from NK Model

Note: the dashed lines are the theoretical impulse responses to the technology and monetary policy shocks as described in the text. The solid lines are the average responses obtained from 2000 simulations of data series with 200 observations from the model. The shaded gray regions are the 90% confidence intervals from the simulations.