Optimal Experimentation and the Perturbation Method.*

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Abstract: The perturbation method is used to approximate optimal experimentation problems. The approximation is in the neighborhood of the linear regulator problem which has a well-defined solution procedure. The first order perturbation of the optimal decision under experimentation is a combination of the linear regulator solution and a term that captures the impact of the uncertainty on the agent’s value function. An algorithm is developed to quickly implement this procedure on the computer. As a result, the impact of optimal experimentation on an agent’s decisions can be quantified and estimated for a large class of problems encountered in economics.

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1 Introduction

Economic agents generally find themselves in circumstances in which they have to maximize their welfare at the same time they must learn about fundamental relations which influence their payoff. Optimal experimentation arises when the actions of the agents impact their payoff. In these circumstances the agent must trade off optimal control with optimal experimentation. In general these problems are difficult to solve since the action of the agents affects the distribution of payoffs. This paper demonstrates how to use the perturbation method to approximate the solution to these problems.

In a recent paper Wieland (2000a) shows how to use value function iteration to examine the behavior of optimal experimentation problems. He introduced the optimal experimentation problem into otherwise static optimization problem. Within the context of these problems he was able to identify as much as a 52% increase in the agent’s value from experimentation.

Extending this procedure to more general problems is problematic. In the case of a continuous distribution in which the agent is learning about four parameters Wieland’s computer program takes as much as a week to find a solution. This limits the application of this procedure to calibration exercises. In addition optimal experimenting about multiple parameters or more complex dynamic settings is prohibitive. This paper provides an alternative procedure for approximating optimal experimentation problems.

This alternative procedure is based on the perturbation method of Judd and Gaspar (1997) and Judd (1998). The perturbation method is useful when a more general problem reduces to a simpler problem under some well-defined circumstances. In addition the simpler problem has a well-developed solution method. The perturbation method proceeds to introduce parameters such that the general problem reduces to the simpler problem when these parameters are zero. The more general problem is then approximated by taking a Taylor series approximation around
zero values for the perturbation parameters.

In most applied problems on optimal experimentation the objective is quadratic and the equations of motions are linear. In addition it is usually possible to introduce parameters which remove the optimal experimentation problem. For example in Weiland’s (2000b) optimal monetary policy problem the optimal experimentation issue would not be present when the central bank knows the impact of interest rates on inflation. Consequently, the optimal experimentation may be removed by attaching a parameter to the error term for this slope coefficient and setting the parameter to zero.⁴

Without the optimal experimentation these problems fall into the general rubric of the discounted stochastic regulator problem. The procedures for solving these problems have been developed by Hansen and Sargent (1998) and Anderson, Hansen, McGrattan and Sargent (1996). As a result we can use the perturbation method to approximate the optimal decision of an agent in the presence of experimentation. In this paper the second order perturbation to the experimentation problem is found. The optimal decision starts with the optimal decision found in Anderson et. al. Appended to this decision is a term which captures the effect of the conditional variance-covariance matrix on the optimal decisions.

This additional impact on the optimal decisions of the agent is akin to Ito’s lemma in continuous time stochastic control problems. The effect of optimal experimentation works through the second order effect of the uncertainty on the agent’s marginal valuation of each state. This effect may be decomposed into two parts based on the chain rule. Uncertainty first changes how each state variable impacts the equation of motion for the state variables. This part occurs because the change in uncertainty influences the Kalman gain. The second part consist of the marginal impact of the state variables on the agent’s valuation. The perturbation method develops a systematic way to measure this effect of optimal experimentation.⁵

There are several benefits to this procedure. First, it builds on a well-developed procedure for handling a large class of economic problems. This class of problems includes those found

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⁴A similar set up could be used to apply the perturbation method to Balvers and Cosimano (1990, 1994).
⁵Balvers and Cosimano (1990,1994) develop the qualitative characteristics in particular examples.
in the literature as well as problems with more complex dynamics. Second, the procedure can be implemented on a computer in a timely manner. Finally, by suitably modifying the estimation procedure of Anderson et. al. it is now feasible to estimate optimal decision rules of agents in the presence of optimal experimentation. For example Sargent (1999) develops an optimal monetary control problem which fits into the Anderson et. al. framework. This optimal monetary control problem is a more general version of Wieland (2000b). Thus, it would be feasible to estimate an optimal central bank reaction function in the presence of optimal experimentation.

The main drawback of this procedure is that it assumes that the value function and optimal decisions are differentiable. There are at least two reasons found in the literature when the value function is not differentiable. First, Balvers and Cosimano (1993) develop an example in which the objective of the agent is convex. This together with a result from Easley and Kiefer (1988), that the value function is convex in the conditional distribution of the shocks, means that the Bellman equation is convex. Thus, the optimal decision is a corner solution. Balvers and Cosimano show the value function is not differentiable when the agents switches between corner solutions. This problem would not occur when the one period reward function is sufficiently concave relative to the convex effect of experimentation. This condition may be checked in the perturbation method by examining whether a particular matrix is negative definite.

Wieland (2000a) shows that the value function is not differentiable when the agent has incorrect limit beliefs. Following Kiefer and Nyarko (1989) he identifies three properties of limit beliefs and optimal decisions. First beliefs must be self-reinforcing. Second, given the limit beliefs the reward function should be optimized. Third, if the control variable is held constant, the agent would learn the mean value of the latent variables. The possible solution to these conditions are incorrect when the expected values of the latent parameters do not converge to their true value. In the simulations Wieland finds that the value function and optimal decisions are not differentiable when the latent parameter corresponds to an incorrect limit belief. Thus, the procedure needs to avoid these possibilities.
Using the perturbation method in the neighborhood of the linear regulator problem to approximate optimal experimentation problems helps to mitigate the convergence to incorrect limit beliefs. Starting with Marcet and Sargent (1989) the literature on optimal learning has developed conditions under which the learning procedure converges to the true parameter values. Hansen and Sargent (1998) use the Kalman filtering procedure to represent optimal learning in stochastic linear regulator problems. They also show that the Kalman filtering procedure is the dual for the linear regulator problem. As a result, the conditions for the convergence of the Kalman filter are identical to those for stable linear regulator problems. In this paper the optimal experimentation problem is approximated by developing a parameterization which collapses the optimal experimentation problem to the optimal learning problem in linear regulator problems. When beliefs converge the expectation of the latent parameter will satisfy the mean prediction property. In addition, the linear regulator problem will be optimized. The difference is that the Kalman gain will be smaller so that the conditional variance-covariance matrix will converge faster to a larger value. Thus, the perturbation method applied to optimal experimentation problems in the neighborhood of the optimal learning problem will converge to correct beliefs as long as the linear regulator problem is stable.

The next section summarizes the procedures to solve augmented linear regulator problems following Anderson et. al. (1996). Section 3 develops the parameterization of the conditional variance-covariance matrix so that the optimal experimentation problem reduces to the Kalman filtering problem when the parameters are set equal to zero. Section 4 derives the formula for optimal conditions in the presence of a second order perturbation of the optimal experimentation problem. Section 5 illustrates the procedure. The analysis of Balvers and Cosimano (1990) is applied to a bank with some monopoly power that does not know the demand for loans or the supply of deposits. This example is a generalization of Balvers and Cosimano in that optimal experimentation for two separate relations is undertaken. In addition the state vector includes eleven variables which moves stochastically over time. The results of Balvers

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6 In optimal learning problems the agent’s actions do not affect the information used to update forecast, while it does under optimal experimentation. See Evans and Honkapohja (2001) for a detail survey and discussion of the work on optimal learning.
and Cosimano are quantified in that both the loan and deposits rates slowly adjust to changes in market conditions. The final section concludes the paper.

2 The Augmented Linear Regulator Problem

The perturbation method is used to solve a general experimentation problem by approximating the problem around a simpler problem with a known solution. In the optimal experimentation problem the augmented linear regulator problem is taken as the simpler problem. In this section we summarize the augmented linear regulator problem as well as its solution following Anderson, et. al. (1996). The agent is assumed to choose a sequence \( \{u_t\} \) to maximize

\[
-\mathbb{E} \left( \sum_{t=0}^{\infty} \beta^t \left[ u_t' R u_t + y_t' Q_{yy} y_t + 2 y_t' Q_{yz} x_t + 2 u_t' W_{yz} y_t + 2 u_t' W_{zz} x_t \right] | \mathcal{F}_0 \right)
\]

subject to

\[
x_{t+1} = \begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} A_{yy} & A_{yz} \\ 0 & A_{zz} \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix} + \begin{pmatrix} B_y \\ 0 \end{pmatrix} u_t + \begin{pmatrix} G_{yy} & G_{yz} \\ 0 & G_{zz} \end{pmatrix} \begin{pmatrix} w_{yt+1} \\ w_{zt+1} \end{pmatrix} = Ax_t + Bu_t + Gw_{1t+1},
\]

Here \( \{\mathcal{F}_t : t = 0, \cdots \} \) is an increasing sequence of information sets which is based on a martingale difference process \( w'_{1t+1} \equiv (w_{yt+1}, w_{zt+1})' \) such that \( E(w_{1t+1}|\mathcal{F}_t) = 0 \) and \( E(w_{1t+1}w_{1t+1}'|\mathcal{F}_t) = I \). \( u_t \) is the control vector, which may influence the endogenous state vector \( y_t \) but does not effect the exogenous state vector, \( z_t \). Each of the matrices are conformable to these vectors.

To solve this problem the cross product terms and the discount factor are eliminated by defining the selection matrices \( U_y \equiv [I, O] \) and \( U_z \equiv [0, I] \) such that \( U_y A U_y' = 0, U_z G U_y' = 0, \) and \( U_z B = 0 \). Next let

\[
y_t \equiv \beta^{t/2} U_y x_t, \quad z_t \equiv \beta^{t/2} U_z x_t, \quad v_t \equiv \beta^{t/2} (u_t + R^{-1} (W_y' W_z') x_t),
\]

(1)

\[
\begin{pmatrix} A_{yy} & A_{yz} \\ 0 & A_{zz} \end{pmatrix} = \beta^{1/2} \left( A - BR^{-1}W' \right) \quad \text{and} \quad \begin{pmatrix} Q_{yy} & Q_{yz} \\ Q_{yz} & Q_{zz} \end{pmatrix} = Q - WR^{-1} W'.
\]

The solution to this augmented regulator problem is given by

\[
v_t = -F_y y_t - F_z z_t,
\]

\(^7\)To avoid unnecessary notation I delete the superscript on \( A \) and \( Q \) below.
where $F_y \equiv \left[ R + B'_y P_y B_y \right]^{-1} B'_y P_y A_{yy}$ and $F_z \equiv \left[ R + B'_y P_y B_y \right]^{-1} B'_y [P_y A_{yz} + P_z A_{zz}]$. The solution is found in two steps. First, $P_y$ solves the Riccati equation

$$P_y = Q_{yy} + [A_{yy} - B_y F_y]' P_y [A_{yy} - B_y F_y] + F_y' R F_y,$$

and second, $P_z$ satisfies the Sylvester equation

$$P_z = Q_{yz} + [A_{yy} - B_y F_y]' P_y A_{yz} + [A_{yy} - B_y F_y]' P_z A_{zz}.$$

Reversing the definitions in (1) the solution to the discounted regulator problem is

$$u_t = -\left[ F_y + R^{-1} W_y \right] y_t - \left[ F_z + R^{-1} W_z \right] z_t.$$

Under the augmented linear regulator problem the agent can learn about the economy independent of their optimal decisions. This result follows from the certainty equivalence property. Certainty equivalence is dependent on quadratic objectives, linear constraints and independence among the distribution of shocks and the agents choice.

In learning problems the agent observes signals which are a linear combination of the hidden state, control and random error vectors. For simplicity only the endogenous state vector is hidden from the agent. The endogenous state vector follows

$$y_{t+1} = A_{yy} y_t + A_{yz} z_t + B_y u_t + G^1 w_{1t+1},$$

while the agent observes each period $t$ the signals

$$s_t = C_{yy} y_t + C_{yz} z_t + D u_t + H w_{2t}.$$ 

Assume

$$E\left( \begin{array}{c} w_{1t+1} \\ w_{2t} \end{array} \right) \left( \begin{array}{c} w'_{1t+1} \\ w'_{2t} \end{array} \right) = \left( \begin{array}{cc} I & 0 \\ 0 & V_2 \end{array} \right)$$

The agent is interested in forecasting the endogenous state vector $\hat{y}_t = E (y_t | u_t, z_t, s_t, s_{t-1}, \ldots, s_0, \hat{x}_0)$. The Kalman Filter updates the agent’s forecast according to

$$\hat{y}_{t+1} = A_{yy} \hat{y}_t + A_{yz} z_t + B_y u_t + K_t a_t$$

The superscript 1 refers to the column’s of $G$ associated with the endogenous state variables. Without loss of generality from now on I will delete the superscript on $G$.

Ljungqvist and Sargent (2000, pp. 643-649) derives the Kalman Filter in a similar circumstances.
where \( a_t = s_t - \hat{s}_t = C_{yy}(y_t - \hat{y}_t) + Hw_{2t} \). The Kalman Gain is defined by

\[
K_t = A_{yy} \Sigma_t C'_{yy} \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1}
\]

and the conditional variance-covariance matrix of the state is updated according to

\[
\Sigma_{t+1} = A_{yy} \Sigma_t A_{yy}' + GG' - A_{yy} \Sigma_t C'_{yy} \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1} C_{yy} \Sigma_t A_{yy}',
\]

In the optimal experimentation literature the agent has some ability to manipulate the flow of information. This means that \( H \) is a function of the agent’s decisions, so that the variance-covariance matrix for the signal is also a function of the control vector, \( u \). As a result the agent’s decision influences the distribution of the state vector. Thus, the certainty equivalence property no longer holds. This means that the agent’s optimization problem cannot be separated from their forecasting problem. In the next section the optimal experimentation problem is formulated.

3 Optimal Experimentation

In the optimal experimentation problem the agent chooses a sequence \( \{u_t\} \) to maximize

\[
- E \left( \sum_{t=0}^{\infty} \beta^t \left[ u_t' Ru_t + y_t' Q_{yy} y_t + 2 y_t Q_{yz} z_t + z_t Q_{zz} z_t + 2 u_t' W_{yy} y_t + 2 u_t' W_{zz} z_t \right] | F_0 \right)
\]

subject to

\[
x_{t+1} = A x_t + B u_t + G w_{1t+1},
\]

\[
\hat{y}_{t+1} = A_{yy} \hat{y}_t + A_{yz} z_t + B_{y} u_t + K_t a_t = F [u_t, \hat{y}_t, z_t, \Sigma_t, \tau],
\]

\[
K_t = A_{yy} \Sigma_t C'_{yy} \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1},
\]

\[
\Sigma_{t+1} = A_{yy} \Sigma_t A_{yy}' + GG' - A_{yy} \Sigma_t C'_{yy} \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1} C_{yy} \Sigma_t A_{yy}' = G [u_t, z_t, \Sigma_t, \tau],
\]

\[
z_{t+1} = A_{zz} z_t + G_{zz} w_{zt+1} = Z(z_t)
\]

and

\[
E \left( \sum_{t=0}^{\infty} \left[ |u_t|^2 + |y_t|^2 \right] | F_0 \right) < \infty.
\]
In the optimal experimentation problem, the variance-covariance of the signal, $HV_2H'$, is a function of the current control $u_t$ and state vector $z_t$. In particular, the uncertainty in the signals is a linear function of the control and state vectors. This effect may be represented by replacing $Hw_{2t}$ with

$$Hw_{2t} + \tau_1 u_t' e_{1t} + \tau_2 z_t' e_{2t}$$

where $w_{2t}$, $e_{1t}$, and $e_{2t}$ are not correlated. The variance covariance matrix, $HV_2H'$, is now

$$V_t = HV_2H' + \tau_1 u_t' V_3 \tau_1' u_t + \tau_2 z_t' V_4 \tau_2' z_t,$$

Where $V_3 = E_t [e_{1t} e_{1t}']$ and $V_4 = E_t [e_{2t} e_{2t}']$. The vectors $\tau_1'$ and $\tau_2'$ are perturbation vectors such that each element is equal to one under optimal experimentation.$^{10}$

In this case the Bellman equation becomes

$$V [\hat{y}_t, z_t, \Sigma_t, \tau] = E [\Pi [u_t, y_t, z_t, \tau] + \beta V [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] | \mathcal{F}_t]. \quad (2)$$

Here

$$\Pi [u_t, y_t, z_t, \tau] = - \left(u_t' Ru_t + y_t' Q_{yy} y_t + 2 y_t' Q_{yz} z_t + z_t' Q_{zz} z_t + 2 u_t' W_{yy} y_t + 2 u_t' W_{yz} z_t\right).$$

This dynamic programming problem incorporates two new effects which are not present in the augmented linear regulator problem. The first effect measures the effect of the choice on the Kalman gain which in turn influences the conditional expectation of $y_{t+1}$. The second effect deals with the optimal choice on the conditional variance-covariance matrix for $y_{t+1}$. To analyze these effects the following results, proved in the appendix, are useful.

**Lemma 1** The partial derivatives of the Kalman filter are

$$\frac{\partial E}{\partial u_t} = \text{vec} \left( [B_y', 0'] \right), \quad \frac{\partial E}{\partial y_t} = \text{vec} \left( [A_{yy}', 0'] \right) - \left( \left( \begin{array}{c} K_t \\ 0 \end{array} \right) \otimes I_q \right) \text{vec} (I_q), \quad \frac{\partial E}{\partial z_t} = \text{vec} \left( [A_{yz}', A_{zz}'] \right), \quad \frac{\partial^2 E}{\partial \Sigma_t} \neq 0, \quad \frac{\partial^2 E}{\partial \Sigma_t^2} = 0, \quad \frac{\partial^2 E}{\partial \sigma_{zz}^2} = 0, \quad \frac{\partial^2 E}{\partial \sigma_{zt} \sigma_{zt}} = 0, \quad \frac{\partial^2 F}{\sigma_{zt} \sigma_{zt}} = 0, \quad \frac{\partial^2 F}{\partial \sigma_{zt}^2} = 0, \quad \frac{\partial^2 F}{\partial \sigma_{zt}^2} = 0, \quad \frac{\partial^2 G}{\sigma_{zt}} = 0, \quad \frac{\partial^2 G}{\partial \sigma_{zt}^2} = 0, \quad \frac{\partial^2 G}{\partial \sigma_{zt}^2} = 0, \quad \frac{\partial^2 G}{\partial \sigma_{zt}^2} = 0, \quad \text{and} \quad \frac{\partial^2 G}{\partial \sigma_{zt}^2} \neq 0$$

when the perturbation parameters are zero.$^{11}$

$Ljungqvist and Sargent (2000, pp.643-649) allow for time varying variance-covariance matrix as long as the agent knows them.$

$^{11}$To cut down on notation the functions $F$ and $Z$ have been stacked together and is called $F$. vec$(A)$ stacks the columns of $A$ in a vector starting with the first column.
4 Perturbation Method

The optimal experimentation problem introduced in the previous section does not have an explicit solution. In this section the perturbation method is used to approximate this problem following the analysis of Judd (1998) and Judd and Gaspar (1997).

The tensor notation is used extensively in the perturbation method. This notation may be illustrated by writing the quadratic form $x'Ax$ as $a_{ij}x^iy^j$ which means $\sum_i \sum_j a_{ij}x^iy^j$. As a result, a summation occurs whenever a superscript and a subscript match. The partial derivatives $\frac{\partial F[y_t, z_t, \Sigma_t, \tau]}{\partial x_t}$ are represented by $F^i$ for each state vector. For example $F^i = a^{i,zz}_t$ for the exogenous state vectors $i$ and $j$. In a similar way $F^i$ would be the partial derivative of the $i^{th}$ state variable with respect to the $\alpha^{th}$ control. $F^i = 0$ for the exogenous state vectors. $F^i_I$ represents the partial derivative of the $i^{th}$ state variable with respect to the $I^{th}$ variance or covariance term. $F^i_I = 0$ for the exogenous state vectors. Finally, $F^i_I$ represents the partial derivative of the $i^{th}$ state variable with respect to the $I^{th}$ perturbation parameter. $F^i_I = 0$ for both state vectors.

Given this notation the Euler conditions may be written as

$$E \left[ \Pi_\alpha \left[ u \left[ \hat{y}_t, z_t, \Sigma_t, \tau \right], y_t, z_t, \tau \right] + \beta V_1 \left[ \hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] F^i \left[ u \left[ \hat{y}_t, z_t, \Sigma_t, \tau \right], y_t, z_t, \tau \right] + \beta V_1 \left[ \hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] G^I \left[ u \left[ \hat{y}_t, z_t, \Sigma_t, \tau \right], z_t, \Sigma_t, \tau \right] | F^i_I \right] \leq 0$$

for each control $\alpha$.

Solving the optimal learning problem (2) and (3) explicitly is problematic. The difficulty comes about because of the additional non-linearity introduced by the control variables influence on the Kalman Filter. However, the problem reduces to the augmented linear regulator problem when the perturbation vector, $\tau$, is set equal to zero. As a result the perturbation method of Judd (1998), and Judd and Gaspar (1997) may be applied to (2) and (3). Equation (3) implies an implicit function for the control vector, $u \left[ \hat{y}_t, z_t, \Sigma_t, \tau \right]$, so that equation (2) implies an implicit equation for $V \left[ \hat{y}_t, z_t, \Sigma_t, \tau \right]$. The perturbation method involves a taylor expansion of these functions around the known solution. In this case the expansion is around
\[ \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right], \text{ where the superscript LR refers to the linear regulator solution.} \]

\[
u^* [\hat{y}_t, z_t, \Sigma_t, \tau] \approx \nu^* \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] + u_i^* \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i \\
+ u^* \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \Sigma_t - \Sigma_t^{LR} \right]^i.
\]

(4)

\[
V [\hat{y}_t, z_t, \Sigma_t, \tau] \approx V \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] + V_t \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \Sigma_t - \Sigma_t^{LR} \right]^i + \frac{1}{2} V_{ij} \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^j \\
+ V_{II} \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i \left[ \Sigma_t - \Sigma_t^{LR} \right]^i + \frac{1}{2} V_{IJ} \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \Sigma_t - \Sigma_t^{LR} \right]^i \left[ \Sigma_t - \Sigma_t^{LR} \right]^J.
\]

(5)

These approximation may be done to a higher order, however, only the first order terms in the optimal control solution should be important for empirical work.

The first two terms in (4) are identical to the augmented linear regulator problem so that

\[
u^* \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] + u_i^* \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i = \\
- \left[ F_y + R^{-1} W_y \right] \left[ y_t - y_t^{LR} \right] - \left[ F_z + R^{-1} W_z \right] z_t.
\]

Next several terms in (5) are already known from the augmented linear regulator problem. First,

\[
V_t \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i + \frac{1}{2} V_{ij} \left[ \hat{y}_t^{LR}, z_t, \Sigma_t^{LR}, 0 \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^j = \\
\left[ \hat{x}_t - \hat{x}_t^{LR} \right]^i \left[ \begin{array}{c} P_y \\ P_z \\ P_{zz} \end{array} \right] \left[ \hat{x}_t - \hat{x}_t^{LR} \right]^j.
\]

Here \( P_{zz} \) was not needed for the solution to the linear regulator problem but can be found from the Riccati equation following Hansen and Singleton (1998, p. 162.)

\[
P_{zz} = Q_{zz} + A'_{zz} P_y A_{yz} + A'_{zz} P_z A_{yz} + A'_{yz} P_z A_{zz} - \left[ A'_{yz} P_y B_y + A'_{zz} P_z B_y \right] R^{-1} \left[ B'_y P_y A_{yz} + B'_y P_z A_{zz} \right] + A'_{zz} P_{zz} A_{zz},
\]

which is solved by iterating on \( P_{zz} \).

\(^{12}\)In the appendix it is shown that \( V_2 = 0 \) and \( u_2^* = 0 \).
Finally, the constant term in (5) is given by

\[ V \left[ \hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0 \right] = \rho \]

where \( \rho \) is found by iterating on

\[ \rho_{j+1} = \beta \rho_j + \beta \text{trace} (PGG'). \]

The rest of this section derives an expression for the last term in (4) \( u_t^0 \left[ \hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0 \right] \).

First the impact of the uncertainty on the value function is found by taking the total derivative of the value function (2) with respect to each of the \( \frac{q(q+1)}{2} \) variance-covariance terms in \( \Sigma_{t+1} \).

Here \( q \) is the number of endogenous state variables which are hidden from the agent.

\[ V_I [\hat{y}, z_t, \Sigma_t, \tau] = E_t \left[ \beta V_J [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F_l^I \left[ u [\hat{y}, z_t, \Sigma_t, \tau] , \hat{y}, z_t, \Sigma_t, \tau \right] + \beta V_J [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \right] . \] (6)

In this equation all the terms are known for the linear regulator problem except \( V_I \). As a result, these equations can be stacked into a vector of \( \frac{q(q+1)}{2} \) first order linear difference equations which can be iterated on to yield \( V_I \left[ \hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0 \right] \).

Next the second order effects on the value function are calculated by taking the total differentiation of (6) with respect to the \( q + r \) state variables to yield \( q + r \) difference equations for each variance-covariance term

\[ V_{lk} [\hat{y}, z_t, \Sigma_t, \tau] = \]

\[ E_t \left[ \beta V_{jl} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( F_l^k \left[ u_t, \hat{y}, z_t, \Sigma_t, \tau \right] + \right. \right. \]

\[ + \beta V_{jL} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( G_k^L \left[ u_t, \hat{y}, z_t, \Sigma_t, \tau \right] + \right. \]

\[ \right) \right] . \] (7)

If the perturbation vector is set equal to zero, then by Lemma 1 equation (7) becomes

\[ V_{lk} \left[ \hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0 \right] = E_t \left[ \beta V_j \left[ \hat{y}^{LR}_{t+1}, z_{t+1}, \Sigma^{LR}_{t+1}, 0 \right] \right] F_l^I + \beta V_{jL} \left[ \hat{y}^{LR}_{t+1}, z_{t+1}, \Sigma^{LR}_{t+1}, 0 \right] \left( F_l^I + F_l^I u_k^a \right) G_l^I . \] (8)
These second order partial derivatives are dependent on the variance-covariance matrix \( \Sigma \). Once this is complete, (9) may be stacked into a vector to yield a first order linear difference equation in the terms for \( V_{t} \left[ \hat{y}_{t}^{LR}, z_{t}, \Sigma_{t}^{LR}, 0 \right] \).

The second order partial derivatives of the value function with respect to the elements of the variance-covariance matrix satisfies

\[
V_{tJ} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] = E_{t} \left[ \beta V_{jl} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( F_{j}^{l} [u_{t}, \hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] + F_{l}^{\alpha} [u_{t}, \hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] u_{j}^{\alpha} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] \right) F_{j}^{l} [u_{t}, \hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] + \beta V_{jL} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( G_{j}^{l} [u_{t}, z_{t}, \Sigma_{t}, \tau] + G_{l}^{\alpha} [u_{t}, z_{t}, \Sigma_{t}, \tau] u_{j}^{\alpha} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] \right) F_{j}^{l} [u_{t}, \hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] + \beta V_{j} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( F_{j}^{l} [u_{t}, \hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] + F_{l}^{\alpha} [u_{t}, \hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] u_{j}^{\alpha} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] \right) \right] + \beta V_{KL} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( G_{j}^{l} [u_{t}, z_{t}, \Sigma_{t}, \tau] + G_{l}^{\alpha} [u_{t}, z_{t}, \Sigma_{t}, \tau] u_{j}^{\alpha} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] \right) G_{j}^{l} [u_{t}, z_{t}, \Sigma_{t}, \tau] + \beta V_{K} [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \left( G_{j}^{l} [u_{t}, z_{t}, \Sigma_{t}, \tau] + G_{l}^{\alpha} [u_{t}, z_{t}, \Sigma_{t}, \tau] u_{j}^{\alpha} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] \right) \right) \right].
\]

These second order partial derivatives are dependent on \( u_{j}^{\alpha} [\hat{y}_{t}, z_{t}, \Sigma_{t}, \tau] \), however, it will turn out that these partial derivatives can be calculated independent of (9) when the perturbation vector is zero. Once this is complete, (9) may be stacked into a \( \frac{q^{2}(q-1)^{2}}{4} \) vector to yield a first order linear difference equation in \( V_{tJ} [\hat{y}_{t}^{LR}, z_{t}, \Sigma_{t}^{LR}, 0] \).

The results of a change in elements of the variance-covariance matrix on the optimal control can now be calculated. While the calculations are long, the results are simplified since the function \( G \) is not influenced by changes in the optimal controls when the perturbation vector is zero. As a result,

\[
u_{j}^{\gamma} [\hat{y}_{t}^{LR}, z_{t}, \Sigma_{t}^{LR}, 0] = - E_{t} \left[ \Pi_{\alpha,\gamma} \left[ u_{t}^{LR}, \hat{y}_{t}^{LR}, z_{t}, 0 \right] + \beta V_{ik} [\hat{y}_{t+1}^{LR}, z_{t+1}, \Sigma_{t+1}^{LR}, 0] F_{i}^{\alpha} F_{k}^{j} \right]^{-1} \times E_{t} \left[ \beta V_{ik} [\hat{y}_{t+1}^{LR}, z_{t+1}, \Sigma_{t+1}^{LR}, 0] F_{i}^{\alpha} G_{j}^{k} \right],
\]

where the inverse refers to the inverse tensor matrix. When this matrix is negative definite, then the problem has an interior solution so that the bang-bang solution of Balvers and Cosimano (1993) is not present. By substituting (10) into (3) the linear approximation of the optimal controls is complete. Examination of (10) reveals that the variance-covariance matrix

\(^{13}\)Equation (10) is derived in the appendix.

\(^{14}\)See Judd (1998, p. 500).
for the hidden state variables influences the decisions of the agent through its influence on the value function for the agent’s problem. The chain rule implies that there are two parts to this effect. First, the change in uncertainty changes the variance-covariance matrix from the Kalman filter, $G^K_j$. Next the control vector leads to changes in the state vector, $F^i_\alpha$. Both of these effects are manifested through the change in uncertainty on the marginal value of the state vector, $V_{iK}$. These effects works through the impact on the value function in this respect the results are similar to Ito’s lemma in continuous time stochastic control.

5 An Example

Balvers and Cosimano (1990) use optimal experimentation to explain the slow adjustment of prices for a firm with some monopoly power. Subsequently, Cosimano, Emmons, Lee and Sheehan (2002) apply this argument to a bank to explain the slow adjustment of loan and deposit rates to changes in the treasury bill rate. The bank sees the demand for loans

$$L_t = l_{0,t} + l_1L_{t-1} - l_2,t [r_{L,i}^L - r_t^L] + \epsilon_{t,1}.$$  

and the supply of deposits

$$D_t = d_{0,t} + d_1^1D_{t-1} + d_2,t [r_{D,i}^D - r_t^D] + \epsilon_{t,2}.$$  

Here define $L_t$ as the demand for loans by the $ith$ bank at time $t$; $\epsilon_{t,1}$ is the random change in the demand for loans for the $ith$ bank at time $t$; $r_{L,i}^L$ is the $ith$ bank’s loan rate at time $t$; $r_t^L \equiv \frac{1}{N-1} \sum_{j=1,j\neq i}^{N} r_{t,j}^L$ is the average loan rate in the bank’s market at time $t$ excluding this institution, where $N$ is the number of competitor banks; $D_t$ represents the supply of deposits to the $ith$ bank at time $t$; $r_{t}^D$ is the $ith$ bank’s deposit rate at time $t$; $\epsilon_{t,2}$ is the random change in the supply of deposit for the $ith$ bank at time $t$; $r_t^D \equiv \frac{1}{N-1} \sum_{j=1,j\neq i}^{N} r_{t,j}^D$ is the average deposit rate in the bank’s market at time $t$ excluding this institution, where $N$ is the number of banks in the market;

The bank observes the quantity of loans and deposits but does not know the true slope and intercepts for the demand for loans and supply of deposits. The intercepts are autoregressive
to represent the consumers who are not sensitive to changes in interest rates. As a result the bank sees the two signals

\[ s_1 = l_{0,t} - \tau_L \epsilon_{t,5} (r_{t,i}^L - r_t^L) + \epsilon_{t,1} \quad \text{and} \quad s_2 = d_{0,t} - \tau_D \epsilon_{t,6} (r_{t,i}^D - r_t^D) + \epsilon_{t,2}. \]

The bank choose loan and deposit rates which maximizing profits

\[ \left( r_{t,i}^L - r_t - C_i^L \right) L_t + \left( r_t (1 - \alpha) - r_{t,i}^D - C_i^D \right) D_t \]

subject to the demand for loans and the supply of deposits. The \( r_t \) is the treasury bill rate; \( \alpha \) is the reserve ratio and \( C_i^L, C_i^D \) are the marginal resource cost of loans and deposits, respectively. The control vector is \( (r_{t,i}^L \quad r_{t,i}^D)' \), the endogenous state vector is \( (C_i^L \quad C_i^D \quad L_{t-1} \quad D_{t-1} \quad l_{0,t} \quad d_{0,t})' \) and the exogenous state vector is \( z_t \equiv (r_t^L \quad r_t^D \quad r_t \quad 1)' \). The matrices in the augmented linear regulator problem are

\[
R \equiv \begin{pmatrix} l_2 & 0 \\ 0 & d_2 \end{pmatrix};
W_x \equiv \begin{pmatrix} -l_2 & 0 \\ 0 & -d_2 \end{pmatrix};
W_y \equiv \begin{pmatrix} -l_2 & 0 \\ 0 & d_2 \end{pmatrix};
\]

\[
Q_{yy} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 \\ l_1 & 0 \end{pmatrix} \quad \text{is zero};
A_{zz} \quad \text{has roots less than one};
\]

\[
Q_{yy} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -d_2 & 0 \end{pmatrix};
Q_{zz} \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 \\ l_2 & d_2 (1 - \alpha) \end{pmatrix};
\]

\[
Q_{yz} \equiv \frac{1}{2} \begin{pmatrix} l_2 & 0 \\ 0 & 0 \end{pmatrix};
A_{yy} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix};
\]

\[
A_{yz} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_0 \end{pmatrix};
B_y \equiv \begin{pmatrix} 0 \\ -l_2 \\ 0 \\ 0 \end{pmatrix};
G_{yy} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{22} \end{pmatrix};
\]

\[
C_{yy} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D \text{ are zero.} \quad H \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad V_3 = V_4 \equiv \begin{pmatrix} \sigma_5 & 0 \\ 0 & \sigma_6 \end{pmatrix};
\]

\[^{15}_o \text{and} \; \sigma_{t,4} \text{ are random shocks to the intercepts} \; l_{0,t} \text{ and} \; d_{0,t}, \text{ respectively.} \]
The parameter values for this model are listed in Table 1. The parameter estimates are based on monthly data from 1993-1999 which was taken from a financial institution in a large metropolitan area. Loan commitments are used for the loan demand and savings accounts are used for deposits. Both accounts are highly persistent with the expected dependence on the spread between bank rates and market rates for the metropolitan area. The exogenous state variable, $z_t$, is represented by the VAR model in Tables 2 and 3. This state vector includes the market rates for loan commitments, savings deposit, interest bearing checking accounts and treasury bills. One month lag in each of the interest rate variables was sufficient to generate white noise errors.

The first step of the simulation procedure is to implement the doubling algorithm of Hansen and Sargent (1998) and Anderson et. al. (1996). This procedure generates the augmented linear regulator solution. The behavior of the bank’s loan and deposit rates are portrayed by the squares in Figures 1 and 2, respectively. For these simulations 11 random draws from the normal distribution each month were used to generate the movement in the state vector, $x_t$ from month to month. The first month is an initial value for the state vector which approximates a steady state. In Table 4 the state vectors for the first and sixth month are recorded for the linear regulator problem in columns 2 and 4. The optimal loan and deposit rate for the bank are listed in Table 5. The loan rate is about 24 basis points below the market average in the sixth month, while the deposit rate is about .87% above the market average.

In Table 6 the Kalman gain starts at about .5 for both rates and after six months drops to about .3 and .08 for the loan and deposit rate, respectively. The initial value of the Kalman gain can be manipulated by changing the variance of the constant relative to the variance in the regression. By lowering the variance of the constant the conditional variance of the constant decreases relative to the variance of the regression which leads to a decrease in the Kalman gain.

As the conditional variance of the constant for loans decreases over the six months, as seen in

$$V'_0 \equiv \begin{pmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 & 0 & 0 \end{pmatrix}.16$$

The stochastic specification is slightly different since there is a variance-covariance matrix, $V_5$, between $w_{t+1}$ and $w_{t+2}$. This causes the Kalman filter to change by replacing $A \Sigma_t C'$ with $A \Sigma_t C' + V_5$.

Rich Sheehan provided these estimates.
Table 7 row 2, the Kalman gain decreases which increases the convergence of this conditional variance.

In Table 8 the marginal value of the state variable, $V_i$ is listed. These results are consistent with intuition. An increase in cost reduces the value of the bank, while an increase in demand increases its value. In summary the linear regulator solution behaves in a consistent and intuitive way.

The conditional variance has a positive impact on the value of the bank in Table 9. Balvers and Cosimano (1990) show that the bank has an increasing return to uncertainty in the intercept. As a result, Jensen’s inequality implies that higher uncertainty leads to an increase in the value of the bank.

We can now examine the impact of optimal experimentation on the behavior of the bank. The simulation starts at the same initial values so that the first occurrence of experimenting is in month 2. This behavior is pictured in Figures 1 and 2 for the loan and deposit rates. The rhombus represents the optimal experimentation. The loan rate initially goes below the benchmark linear regulator solution by 35 basis points and goes about 7 basis points above by the sixth month. This increases the uncertainty in the regression which lowers the Kalman Gain to .08 in the second month and converges to the linear regulator value by the sixth month as seen in Table 6 row 1. As a result the conditional variance for the loan intercept converges faster to a higher value which can be seen in Table 7 column 3. The same experimentation occurs for the deposit rate except that the deposit rate is higher since it is a source of cost rather than revenue.

In Table 10 we can see the effect of the conditional uncertainty on the optimal decisions of the bank. An increase in the conditional variance of the demand for loan intercept tends to increase the loan rate while the increase in the condition variance for deposits lowers the deposit rate. This result works through the impact of the control variable on the state vector, $F_{ik}$, in equation (10). An increase in the bank’s loan rate decreases demand while the deposit rate increases supply. These effects follow from the marginal value of the state vector, $V_i$ in Table 8, interacting with, $F_{ik}$. The sign of these partial derivatives determines the negative influence
of $V_{tk}$ through equation (8), which in turn by equation (10) implies the partial derivative of the control variable with respect to its conditional variance.

The behavior of the loan rate in Figure 1 can now be understood. The conditional variance of the demand for loan intercept under experimentation is initially below the linear regulator case so that the loan rate is lower. By the sixth month this conditional variance is higher under optimal experimentation and its marginal impact on the optimal loan rate has also fallen. As a result the loan rate is now slightly above the linear regulator case. This continues till the 30th month when the loan rate is only 1 basis point above the linear regulator case.

The same basic pattern occurs for the deposit rate. The deposit rate first goes above the linear regulator case by 65 basis points. By the sixth month the spread is down to 7 basis points and goes below the linear regulator deposit rate by 3 basis points in the 8th month. The conditional variance of the supply of deposit intercept converges quicker to a smaller value.

These simulations confirm most of the qualitative results of Balvers and Cosimano (1990). The main new result is that after experimenting for a short period of time, the optimal loan rate goes above the loan rate under optimal experimenting. This new insight occurs since it is now possible to analyze optimal experimentation problems in a setting with changing state variables as well as more complex dynamic linear regulator problems. In addition, these simulations take less than a minute on a standard PC with a Pentium II 400 MHz chip. The estimation of the parameters of these models would involve the repeated solution of the algorithm as the parameters are changed to optimize a likelihood function. Thus, it is now feasible to estimate complex models of optimal experimentation.

6 Conclusion

This paper has developed a procedure for approximating optimal experimentation problems for the class of augmented linear regulator problems. This procedure uses the perturbation method of Judd (1998), and Judd and Gaspar (1997). The optimal learning problem within the context of the linear regulator problem is modified by introducing parameters into the conditional variance-covariance matrix. These parameters introduce the possibility that either
the control variables or the exogenous state vector can influence this variance-covariance matrix. This parameterization of the optimal experimentation problem includes all the examples seen in the literature such as Wieland (2000b), as well as more complex problems. When these parameters are zero, the optimal experimentation problem reduces to the optimal learning problem which has a well-defined solution. Thus, the perturbation procedure can be used to find the first order approximation of the optimal decision of the agents and the second order approximation of the value function.\footnote{This procedure could be used to find higher order approximations, however most empirical problems would only find second moments to be significant.}

The optimal decision under experimentation, (4), is a linear combination of the usual solution found by iterating on the Riccati equation and a term that captures the effect of uncertainty on the value function of the agent, (10). This second term uses four matrices as inputs which consist of derivatives of the equations of motion for the state vector and the conditional variance-covariance matrix from the Kalman Filter. The formula’s for these matrices are provided in the appendix. As a result optimal experimentation can be analyzed for any augmented linear regulator problem. To implement this program: first define the matrices for your particular problem as in Hansen and Sargent. Second, apply the formulas given in the appendix for the four matrices in equations (8) and (10). Third, iterate on the first order difference equation, (8), to measure the impact of the conditional variance-covariance matrix on the marginal value of each state variable. The Final step implements equation (10), which yields the effect of optimal experimentation on the optimal decision of the agent.

Implementation of this algorithm allows for the empirical evaluation of optimal experimentation on the optimal decisions of agents. Once the optimal decision for a particular problem is known, such as the optimal loan and deposit rate decisions found in section 5, the estimation procedure of Anderson \textit{et. al.} (1996) and Hansen and Sargent (1998) can be modified by replacing the linear regulator solution with the optimal experimentation solution. The estimates of the underlying parameters are found by optimizing the likelihood function built on this algorithm. It is feasible to estimate the effect of experimentation since each iteration on
this algorithm takes less than a minute on a standard PC. Thus, the impact of optimal ex-
perimentation on optimal decisions of agents can be accomplished for a large class of applied
economic problems.

7 Appendix

7.1 Derivatives of F.

Let $u_t$ have dimension $px1$, $y_t$ have dimension $qx1$, $z_t$ have dimension $rx1$, and $a_t$ have dimension $sx1$. The matrices have the dimensions so that product is well de

\[
\frac{\partial F}{\partial u_t} = vec\left(\begin{bmatrix} B'_y, 0' \end{bmatrix}\right) + vec\left(\begin{bmatrix} \frac{\partial K_t a_t}{\partial u_t}, 0' \end{bmatrix}\right).\]

$K_t a_t$ is dependent on $V_t$. As a result, look at

\[
\frac{\partial V_t}{\partial u_t} = \frac{\partial \tau_1 u'_t}{\partial u_t} V_3 u_t \tau'_1 + (\tau_1 u'_t \otimes I_p) (V_3 \otimes I_p) \frac{\partial u_t \tau'_1}{\partial u_t}
\]

which is zero for $\tau_1 = 0$. Thus, $\frac{\partial K_t a_t}{\partial a_t} = 0$, so that $\frac{\partial F}{\partial a_t} = vec\left(\begin{bmatrix} B'_y, 0' \end{bmatrix}\right)$.

\[
\frac{\partial F}{\partial y_t} = vec\left(\begin{bmatrix} A'_{yy}, 0' \end{bmatrix}\right) + vec\left(\begin{bmatrix} \frac{\partial K_t a_t}{\partial y_t}, 0' \end{bmatrix}\right)
\]

\[
\frac{\partial (K_t a_t)}{\partial y_t} = -\left(\begin{bmatrix} K_t \vline 0 \end{bmatrix} \otimes I_q\right) (C_{yy} \otimes I_q) vec(I_q).
\]

Here $0$ has dimension $rxs$. Thus,

\[
\frac{\partial F}{\partial y_t} = vec\left(\begin{bmatrix} A'_{yy}, 0' \end{bmatrix}\right) - \left(\begin{bmatrix} K_t \vline 0 \end{bmatrix} \otimes I_q\right) (C_{yy} \otimes I_q) vec(I_q).
\]

\[
\frac{\partial F}{\partial z_t} = vec\left(\begin{bmatrix} A'_{yz}, A'_{zz} \end{bmatrix}\right) + vec\left(\begin{bmatrix} \frac{\partial K_t a_t}{\partial z_t}, 0' \end{bmatrix}\right).
\]

$K_t a_t$ is dependent on $z_t$ through $V_t$, as a result look at

\[
\frac{\partial V_t}{\partial z_t} = \frac{\partial \tau_2 z'_t}{\partial z_t} V_4 z_t \tau'_2 + (\tau_2 z'_t \otimes I_r) (V_4 \otimes I_r) \frac{\partial z_t \tau'_2}{\partial z_t}
\]

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19 See Theorem 6.3 p.43 and item 4 p.50 of Rogers (1980).
20 See Theorem 6.4 from Rogers (1980, p. 43).
21 See item 16 Rogers (1980,p.53).
which is zero for \( \tau_2 = 0 \). Thus, \( \frac{\partial K_{at}}{\partial \tau_2} = 0 \), so that \( \frac{\partial F}{\partial \tau_2} = vec\left( \begin{bmatrix} A'_{yz} & A'_{z2} \end{bmatrix} \right) \).

\[
\frac{\partial F}{\partial \Sigma_t} = \begin{pmatrix} \frac{\partial K_{at}}{\partial \Sigma_t} \\ 0 \end{pmatrix}.
\]

where \( 0 \) has dimension \( q \times q \). \( K_{at} \) is the product of three matrices \( X \equiv A_{yy} \Sigma_t C_{yy}' \), \( Y \equiv C_{yy}' \Sigma_t C_{yy} + HV_2 H' \) and \( Z \equiv a_t \). By the product rule for differentiation of matrices

\[
\frac{\partial K_{at}}{\partial \Sigma_t} = \frac{\partial X}{\partial \Sigma_t} (Y \otimes I_q) (Z \otimes I_q) + (X \otimes I_q) \frac{\partial Y}{\partial \Sigma_t} (Z \otimes I_q).
\]

Next,

\[
\frac{\partial X}{\partial \Sigma_t} = \frac{\partial A_{yy} \Sigma_t C_{yy}'}{\partial \Sigma_t} = vec(A_{yy}') vec(C_{yy}'),
\]

which has dimension \( q^2 \times q \).

Next look at the effect of the perturbation vector

\[
\frac{\partial F}{\partial \tau_1} = \begin{pmatrix} \frac{\partial K_{at}}{\partial \tau_1} \\ 0 \end{pmatrix}.
\]

\( K_{at} \) is dependent on \( \tau_1 \) through \( V_t \), as a result look at

\[
\frac{\partial V_t}{\partial \tau_1} = \frac{\partial \tau_1 u_t'}{\partial \tau_1} V_3 u_t \tau_1' + (\tau_1 u_t' \otimes I_p) (V_3 \otimes I_p) \frac{\partial u_t \tau_1'}{\partial u_t}
\]

which is zero for \( \tau_1 = 0 \). Thus, \( \frac{\partial K_{at}}{\partial \tau_1} = 0 \), so that \( \frac{\partial F}{\partial \tau_1} = 0 \). It also follows immediately that

\[\frac{\partial F}{\partial \tau_2} = 0\]

\( \text{Note:} \) is not dependent on \( \Sigma_t \).

\[\text{See item 13, Rogers (1980, p.52).}\]

\[\text{See item 4 Rogers (1980, p.50).}\]
Now turn to the second order derivatives.

$$\frac{\partial^2 F}{\partial u_t \partial \Sigma_t} = \left( \frac{\partial^2 K_{t,q_{i,t}}}{\partial u_t \partial \Sigma_t} \right).$$

$\frac{\partial u_t}{\partial u_t}$ is independent of $\Sigma_t$ so that all the terms in this second order derivatives are dependent on this derivative. Thus, $\frac{\partial^2 F}{\partial u_t \partial \Sigma_t} = 0$.

$$\frac{\partial^2 F}{\partial y_t \partial \Sigma_t} = -\frac{\partial}{\partial \Sigma_t} \left[ (C_{yy} \otimes I_q)\text{vec}(I_q) \otimes I_s \right] =$$

$$- \left( \left( I_{(q,q)} \otimes I_q \right) \left( I_q \otimes \frac{\partial K_t}{\partial \Sigma_t} \right) \left( I_{(s,q)} \otimes I_q \right) \right) \left[ (C_{yy} \otimes I_q)\text{vec}(I_q) \otimes I_q \right],$$

where $I_{(s,q)}$ is the commutation matrix and $0$ has dimension $q^2 r \times q^2$. The partial derivative is

$$\frac{\partial K_t}{\partial \Sigma_t} = \left[ \text{vec}(A_{yy'}) - (K_t \otimes I_s) \text{vec}(C_{yy'}) \right] \text{vec}(C_{yy'})' \left( \left( C_{yy} \Sigma_t C_{yy'}^t + HV_2 H' \right)^{-1} \otimes I_s \right)$$

$$= \left[ \text{vec}(A_{yy'}) - \text{vec}(C_{yy} K_t) \right] \text{vec}(C_{yy'})' \left( \left( C_{yy} \Sigma_t C_{yy'}^t + HV_2 H' \right)^{-1} \otimes I_s \right).$$

The final second order derivative is

$$\frac{\partial^2 F}{\partial \Sigma_t^2} = \left( \frac{\partial^2 K_{t,q_{i,t}}}{\partial \Sigma_t^2} \right)$$

where $0$ has dimension $q^2 r x q^2$.

Let $S \equiv \left( I_{(q,q)} \otimes I_q \right) \left( I_q \otimes \frac{\partial (C_{yy} \Sigma_t C_{yy'} + HV_2 H')^{-1}}{\partial \Sigma_t} \right) \left( I_{(s,q)} \otimes I_q \right) \left( I_{(q,q)} \otimes I_q \right) \times \left( I_q \otimes \text{vec}(A_{yy'}) \text{vec}(C_{yy'})' \right) \left( I_{(s,q)} \otimes I_q \right)$ so that

$$\frac{\partial^2 K_{t,q_{i,t}}}{\partial \Sigma_t^2} = \left[ \text{vec}(A_{yy'}) \text{vec}(C_{yy'}') \otimes I_q \right] S \left( (a_t \otimes I_q) \otimes I_q \right)$$

$$- T \left[ \left( \left( C_{yy} \Sigma_t C_{yy'}^t + HV_2 H' \right)^{-1} \otimes I_q \right) \otimes I_q \right]$$

$$\times \left[ \left( \text{vec}(C_{yy'}) \text{vec}(C_{yy'})' \left( \left( C_{yy} \Sigma_t C_{yy'}^t + HV_2 H' \right)^{-1} a_t \otimes I_q \right) \right) \otimes I_q \right]$$

$$- \left[ \left( A_{yy} \Sigma_t C_{yy'} \otimes I_q \right) \otimes I_q \right] S$$

$$\times \left[ \left( \text{vec}(C_{yy'}) \text{vec}(C_{yy'})' \left( \left( C_{yy} \Sigma_t C_{yy'}^t + HV_2 H' \right)^{-1} a_t \otimes I_q \right) \right) \otimes I_q \right].$$

\[25\text{See Theorem 6.6 of Rogers (1980, p. 45).} \]

\[26\text{By Theorem 4.1 of Rogers (1980, p. 21), vec(XYZ) = (Z' } \otimes X \text{ )vec(Y).} \]
\[
- \left[ (A_{yy} \Sigma_t C'_{yy} \otimes I_q) \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \otimes I_q \right]
\]
\[
\left[ \text{vec}(C'_{yy}) \text{vec}(C'_{yy})' \otimes I_q \right] S \left[ (a_t \otimes I_q) \otimes I_q \right],
\]
where the partial derivative is calculated in \( \frac{\partial Y}{\partial \Sigma_t} \).

7.2 Derivatives of G

\( u_t, z_t, \) and \( \tau \) only effect \( G \) through the Kalman Gain which in turn is influenced by \( V_t \). As a result \( \frac{\partial G}{\partial u_t}, \frac{\partial G}{\partial z_t} \) and \( \frac{\partial G}{\partial \tau} \) are all zero. Next,

\[
\frac{\partial G}{\partial \Sigma_t} = \text{vec}(A'_{yy}) \text{vec}(A'_{yy})'
\]

\[
- \text{vec}(A'_{yy}) \text{vec}(C'_{yy})' \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \left[ C_{yy} \Sigma_t A'_{yy} \otimes I_q \right]
\]

\[
+ \left[ A_{yy} \Sigma_t C'_{yy} \otimes I_q \right] \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \text{vec}(C'_{yy}) \text{vec}(C_{yy})'
\]

\[
\times \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \left[ C_{yy} \Sigma_t A'_{yy} \otimes I_q \right]
\]

\[
- \left[ A_{yy} \Sigma_t C'_{yy} \otimes I_q \right] \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \text{vec}(C'_{yy}) \text{vec}(A'_{yy})'.
\]

Let \( U \equiv \left( I_{(q,s)} \otimes I_q \right) \left( I_q \otimes \text{vec}(C'_{yy}) \text{vec}(A'_{yy})' \right) \left( I_{(q,q)} \otimes I_q \right) \) so that the second order partial derivative is

\[
\frac{\partial^2 G}{\partial \Sigma_t^2} = - \left[ \text{vec}(A'_{yy}) \text{vec}(C'_{yy})' \otimes I_q \right] S \left[ (C_{yy} \Sigma_t A'_{yy} \otimes I_q) \otimes I_q \right]
\]

\[
- \left[ \left( \text{vec}(A'_{yy}) \text{vec}(C'_{yy})' \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \right) \otimes I_q \right] U
\]

\[
-T \left[ \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \text{vec}(C'_{yy}) \text{vec}(C_{yy})' \right]
\]

\[
\times \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \left[ C_{yy} \Sigma_t A'_{yy} \otimes I_q \right] \otimes I_q
\]

\[
+ \left[ A_{yy} \Sigma_t C'_{yy} \otimes I_q \right] \otimes I_q
\]

\[
\times \left\{ S \left[ \left( \text{vec}(C'_{yy}) \text{vec}(C_{yy})' \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \right) \otimes I_q \right] \right\}
\]

\[
+ \left[ \left( (C_{yy} \Sigma_t C'_{yy} + HV_2 H')^{-1} \otimes I_q \right) \text{vec}(C'_{yy}) \text{vec}(C_{yy})' \otimes I_q \right] S \}
\]

\[
\times \left[ C_{yy} \Sigma_t A'_{yy} \otimes I_q \right] \otimes I_q
\]

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\[ \begin{align*}
\frac{d}{dt} \left[ \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1} \otimes I_q \right] & \cdot \text{vec}(C_{yy}') \cdot \text{vec}(C_{yy})' \\
\times \left( \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1} \otimes I_q \right) \otimes I_q \right\} U \\
- T \left[ \left[ \left( C_{yy} \Sigma_t C_{yy}' + HV_2 H' \right)^{-1} \otimes I_q \right] \cdot \text{vec}(C_{yy}') \cdot \text{vec}(A_{yy}')' \otimes I_q \right] \\
- \left[ \left[ A_{yy} \Sigma_t C_{yy}' \otimes I_q \right] \otimes I_q \right] S \left[ \text{vec}(C_{yy}') \cdot \text{vec}(A_{yy}')' \otimes I_q \right] 
\end{align*} \]

7.3 Derivation of (10)

To find (10) first take the total differentiation of the Euler condition with respect to \( \Sigma_t \) for each control variable \( \alpha \)

\[ E_t \left[ \Pi_{\alpha, \gamma} \left[ u_t, y_t, z_t, \tau \right] U_{\alpha} \left[ \dot{y}_t, z_t, \Sigma_t, \tau \right] + \right. \]
\[ \beta V_{ik} \left[ \dot{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] F_{\alpha}^{i} \left[ u_t, \dot{y}_t, z_t, \Sigma_t, \tau \right] + \right. \]
\[ \beta V_{ik} \left[ \dot{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] G_{\alpha}^{i} \left[ u_t, \dot{y}_t, z_t, \Sigma_t, \tau \right] + \right. \]
\[ \beta V_{ik} \left[ \dot{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] \text{vec}(C_{yy}') \cdot \text{vec}(A_{yy}')' \otimes I_q \right] \\
- \left[ \left[ A_{yy} \Sigma_t C_{yy}' \otimes I_q \right] \otimes I_q \right] S \left[ \text{vec}(C_{yy}') \cdot \text{vec}(A_{yy}')' \otimes I_q \right] 
\]

The (11) can be solved for \( u_{\alpha}^{j} \) to yield

\[ u_{\alpha}^{j} \left[ \dot{y}_t, z_t, \Sigma_t, \tau \right] = - \left\{ E_t \left[ \Pi_{\alpha, \gamma} \left[ u_t, y_t, z_t, \tau \right] + \right. \right. \]
\[ \beta V_{ik} \left[ \dot{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] F_{\alpha}^{i} \left[ u_t, \dot{y}_t, z_t, \Sigma_t, \tau \right] + \right. \]
\[ \beta V_{ik} \left[ \dot{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] G_{\alpha}^{i} \left[ u_t, \dot{y}_t, z_t, \Sigma_t, \tau \right] + \right. \]
\[ \beta V_{ik} \left[ \dot{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau \right] \text{vec}(C_{yy}') \cdot \text{vec}(A_{yy}')' \otimes I_q \right] \\
- \left[ \left[ A_{yy} \Sigma_t C_{yy}' \otimes I_q \right] \otimes I_q \right] S \left[ \text{vec}(C_{yy}') \cdot \text{vec}(A_{yy}')' \otimes I_q \right] \right\}^{-1} X \]

Equation (10) in the text is found by using Lemma 1 to evaluate the partial derivatives at the linear regulator solution.
7.4 Proof of $V_I = 0$ and $u^I_\gamma = 0$.

For each of the perturbation parameters in the vector, $\tau$,

$$
V_I [\hat{y}_t, z_t, \Sigma_t, \tau] = E_t \left[ \Pi_I [u_t, y_t, z_t, \tau] + \beta V_j [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F^j_I [u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + \beta V_j [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] G^J_I [u_t, z_t, \Sigma_t, \tau] + \beta V_I [\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \right].
$$

The partial derivatives, $F^j_I$ and $G^J_I$, are zero by Lemma 1 when they are evaluated at the linear regulator solution. The partial derivatives of the variance-covariance terms $H(u_t, z_t) V_2 H(u_t, z_t)'$ in the Kalman Filter are zero when the perturbation parameters are zero. It follows that $u^I_\gamma (\hat{y}^{LR}_t, z_t, \Sigma^{LR}_t, 0) = 0$ since it is dependent on the partial derivatives $V_I, F^j_I$ and $G^J_I$. 

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References


Magnus, J. R. and H. Neudecker, 1988. Matrix differential calculus with applications in statis-


### Table 1: Parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Standard deviation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_0$</td>
<td>$4.8601 \times 10^4$</td>
<td>$\sigma_1$</td>
<td>$1.4436 \times 10^6$</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>0.9000</td>
<td>$\sigma_2$</td>
<td>1.2017 $\times 10^7$</td>
</tr>
<tr>
<td>$d_0$</td>
<td>$2.1029 \times 10^5$</td>
<td>$\sigma_3$</td>
<td>$5.9955 \times 10^5$</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>0.9000</td>
<td>$\sigma_4$</td>
<td>$7.4062 \times 10^5$</td>
</tr>
<tr>
<td>$l_1$</td>
<td>0.9946</td>
<td>$\sigma_5$</td>
<td>$1.8423 \times 10^8$</td>
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<td>$d_1$</td>
<td>0.9245</td>
<td>$\sigma_6$</td>
<td>$2.1667 \times 10^9$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>$9.0943 \times 10^8$</td>
<td>$\sigma_7$</td>
<td>$1.1708 \times 10^9$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$4.3335 \times 10^9$</td>
<td>$\sigma_8$</td>
<td>$8.9132 \times 10^9$</td>
</tr>
</tbody>
</table>

### Table 2: VAR for Exogenous State Vector

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$r^L$</th>
<th>$r^{D1}$</th>
<th>$r^{D2}$</th>
<th>$r$</th>
<th>Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^L$</td>
<td>0.5512</td>
<td>-0.5415</td>
<td>0.5000</td>
<td>0.2785</td>
<td>0.0470</td>
</tr>
<tr>
<td>$r^{D1}$</td>
<td>0.0172</td>
<td>0.6793</td>
<td>0.4354</td>
<td>0.0367</td>
<td>-0.00218</td>
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<tr>
<td>$r^{D2}$</td>
<td>-0.01459</td>
<td>0.0527</td>
<td>0.9075</td>
<td>0.0400</td>
<td>-0.0006</td>
</tr>
<tr>
<td>$r$</td>
<td>0.0352</td>
<td>-0.9238</td>
<td>1.2885</td>
<td>0.8467</td>
<td>0.0094</td>
</tr>
</tbody>
</table>

### Table 3: Variance-covariance Matrix for VAR

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$r^L$</th>
<th>$r^{D1}$</th>
<th>$r^{D2}$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^L$</td>
<td>$2.0177 \times 10^{-5}$</td>
<td>$5.5679 \times 10^{-7}$</td>
<td>$1.1708 \times 10^{-7}$</td>
<td>$4.4902 \times 10^{-9}$</td>
</tr>
<tr>
<td>$r^{D1}$</td>
<td>$1.2331 \times 10^{-7}$</td>
<td>$-1.1915 \times 10^{-8}$</td>
<td>$-5.5679 \times 10^{-10}$</td>
<td>$8.2302 \times 10^{-11}$</td>
</tr>
<tr>
<td>$r^{D2}$</td>
<td>$3.27 \times 10^{-6}$</td>
<td>$1.7462 \times 10^{-8}$</td>
<td>$1.5322 \times 10^{-6}$</td>
<td>$3.27 \times 10^{-6}$</td>
</tr>
<tr>
<td>$r$</td>
<td>$1.3108 \times 10^{-6}$</td>
<td>$2.8550 \times 10^{-6}$</td>
<td>$1.5322 \times 10^{-6}$</td>
<td>$1.1577 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

### Table 4: State Vector for 1st and 6th Month

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$x_1$</th>
<th>$x_{LR}^1$</th>
<th>$x_6$</th>
<th>$x_{LR}^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^L$</td>
<td>0.0600</td>
<td>0.0600</td>
<td>0.0600</td>
<td>0.0600</td>
</tr>
<tr>
<td>$C^D$</td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
<td>0.0100</td>
</tr>
<tr>
<td>$L$</td>
<td>$4.4424 \times 10^7$</td>
<td>$4.4424 \times 10^7$</td>
<td>$9.0006 \times 10^7$</td>
<td>$8.9132 \times 10^7$</td>
</tr>
<tr>
<td>$D$</td>
<td>$4.4062 \times 10^7$</td>
<td>$4.4062 \times 10^7$</td>
<td>$1.7462 \times 10^8$</td>
<td>$1.1577 \times 10^8$</td>
</tr>
<tr>
<td>$l_{0,t}$</td>
<td>$1.3108 \times 10^6$</td>
<td>$1.3108 \times 10^6$</td>
<td>$2.8550 \times 10^6$</td>
<td>$1.5322 \times 10^6$</td>
</tr>
<tr>
<td>$d_{0,t}$</td>
<td>$-5.9325 \times 10^7$</td>
<td>$-5.9325 \times 10^7$</td>
<td>$-3.3591 \times 10^7$</td>
<td>$-3.3592 \times 10^7$</td>
</tr>
<tr>
<td>$r^L$</td>
<td>0.1240</td>
<td>0.1240</td>
<td>0.1216</td>
<td>0.1216</td>
</tr>
<tr>
<td>$r^{D1}$</td>
<td>0.0260</td>
<td>0.0260</td>
<td>0.0247</td>
<td>0.0247</td>
</tr>
<tr>
<td>$r^{D2}$</td>
<td>0.0147</td>
<td>0.0147</td>
<td>0.0134</td>
<td>0.0134</td>
</tr>
<tr>
<td>$r$</td>
<td>0.0503</td>
<td>0.0503</td>
<td>0.0516</td>
<td>0.0516</td>
</tr>
<tr>
<td>Constant</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</table>

### Table 5: Control Vector for 1st and 6th Month

<table>
<thead>
<tr>
<th>Control Variable</th>
<th>$u_1$</th>
<th>$u_{LR}^1$</th>
<th>$u_6$</th>
<th>$u_{LR}^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^L$</td>
<td>0.12212</td>
<td>0.12212</td>
<td>0.11983</td>
<td>0.11917</td>
</tr>
<tr>
<td>$r^{D1}$</td>
<td>0.02542</td>
<td>0.02542</td>
<td>0.04131</td>
<td>0.04065</td>
</tr>
</tbody>
</table>

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### Table 6: Kalman Gain for 1st and 6th Month

<table>
<thead>
<tr>
<th>Kalman Gain</th>
<th>$K_1$</th>
<th>$K_1^{LR}$</th>
<th>$K_6$</th>
<th>$K_6^{LR}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{L}^L$</td>
<td>0.21628</td>
<td>0.49558</td>
<td>0.28141</td>
<td>0.29107</td>
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<tr>
<td>$r_{D1}^L$</td>
<td>0.006741</td>
<td>0.44867</td>
<td>0.0015415</td>
<td>0.077448</td>
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### Table 7: Variance for 1st and 6th Month

<table>
<thead>
<tr>
<th></th>
<th>$\Sigma_1^L$</th>
<th>$\Sigma_1^{D1}$</th>
<th>$\Sigma_6^L$</th>
<th>$\Sigma_6^{D1}$</th>
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</thead>
<tbody>
<tr>
<td>Optimal</td>
<td>6.0988*10^{11}</td>
<td>9.9114*10^{12}</td>
<td>8.8885*10^{11}</td>
<td>2.21946*10^{12}</td>
</tr>
<tr>
<td>LR</td>
<td>2.0474*10^{12}</td>
<td>1.1751*10^{13}</td>
<td>8.563*10^{11}</td>
<td>1.2122*10^{13}</td>
</tr>
</tbody>
</table>

### Table 8: Partial Derivative of Value Function with Respect to State Vector

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$V_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^L$</td>
<td>-1.3149 *10^9</td>
</tr>
<tr>
<td>$C^D$</td>
<td>-5.5258*10^9</td>
</tr>
<tr>
<td>$L$</td>
<td>0.1366</td>
</tr>
<tr>
<td>$D$</td>
<td>0.1107</td>
</tr>
<tr>
<td>$l_{0,t}$</td>
<td>0.7419</td>
</tr>
<tr>
<td>$d_{0,t}$</td>
<td>0.6506</td>
</tr>
<tr>
<td>$r_{L}^L$</td>
<td>9.075210^7</td>
</tr>
<tr>
<td>$r_{D1}^L$</td>
<td>-4.2130*10^9</td>
</tr>
<tr>
<td>$r_{D2}^L$</td>
<td>-1.0731*10^9</td>
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<tr>
<td>$r$</td>
<td>1.3256*10^9</td>
</tr>
<tr>
<td>constant</td>
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</table>

### Table 9: $V_I$ for 1st and 6th Month

<table>
<thead>
<tr>
<th></th>
<th>1st Month</th>
<th>6th Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_1^L$</td>
<td>1.2730*10^{-9}</td>
<td>7.5818*10^{-10}</td>
</tr>
<tr>
<td>$\Sigma_1^{D1}$</td>
<td>4.9437*10^{-10}</td>
<td>4.9505*10^{-10}</td>
</tr>
</tbody>
</table>

### Table 10: $u_j^*$ for 1st and 6th Month

<table>
<thead>
<tr>
<th></th>
<th>$\Sigma_1^L$</th>
<th>$\Sigma_1^{D1}$</th>
<th>$\Sigma_6^L$</th>
<th>$\Sigma_6^{D1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{L}^L$</td>
<td>3.6500*10^{-14}</td>
<td>0</td>
<td>1.2785*10^{-14}</td>
<td>0</td>
</tr>
<tr>
<td>$r_{D1}^L$</td>
<td>0</td>
<td>-4.2285*10^{-17}</td>
<td>0</td>
<td>-4.7765*10^{-16}</td>
</tr>
</tbody>
</table>