Solving Asset Pricing Models when the Policy Function is Analytic*

Stefano G. Athanasoulis
Department of Finance
Mendoza College of Business
University of Notre Dame
E-mail: sathanas@nd.edu
Phone: (574) 631-9055
Fax: (574) 631-5255

Ovidiu L. Calin
Department of Mathematics
Eastern Michigan University

Thomas F. Cosimano
Department of Finance
Mendoza College of Business
University of Notre Dame
E-mail: thomas.f.cosimano.1@nd.edu
Phone: (574) 631-5178
Fax: (574) 631-5255

and

Alex A. Himonas
Department of Mathematics
University of Notre Dame

January 31, 2003
Abstract: We construct a new method to solve for asset pricing models and more generally Euler equations when the policy function is analytic. Our method is to assume the policy function is analytic and then to derive a set of conditions that proves the policy function is analytic. We describe the general method and then solve for two specific asset-pricing models within the paper. We then use the solution to the asset pricing equations to price an European call option, the S&P 500 index option, to show the applicability of the methodology. While we apply this methodology to asset pricing, its application is more general and can be applied to any Euler equation when the policy function is analytic. In order to make the methodology operational, we describe how one can use these methods without proving analyticity for the particular case the researcher may be interested in. The user will be able to input the intertemporal marginal rate of substitution, and solve for the policy function, price-dividend ratio in this case, and have well defined measures that the solution is accurate.

Journal of Economic Literature Classification Numbers:

Key Words: Analyticity, Asset pricing.

*We would like to thank seminar participants at Oxford University Said School of business for helpful comments. Any remaining errors are our responsibility. Tom Cosimano received financial support from the Center for Research in Financial Services Industry at The University of Notre Dame.
1 Introduction

In this paper we construct a new methodology for solving asset pricing models and more generally Euler equations when the policy function is analytic. This work comes from our observation that most relevant asset pricing models do not offer closed form solutions and consequently are solved for numerically.\(^1\) While the numerical solution for much work is accurate near the steady state, this is not true the further away one gets from the steady state, see for example Collard and Juillard [2001]. Since we are always in the short run in any economic or financial system, the accuracy away from the steady state is very important in many situations, one of which is asset pricing. One can easily imagine situations where inaccuracy away from the steady state can cost institutional investors, firms or investment banks millions of dollars. The solution in this paper is the actual solution to the policy function and consequently it will be accurate globally for the domain of the policy function.

While we obtain our solutions to these asset pricing models using the Gaussian distribution for the random variable, it is important to realize that this is not necessary. This is one reason this methodology is so useful, it can handle many different types of distributions as well as many different asset pricing models with very different utility specifications. The key conditions necessary to use this methodology is that all the derivatives of the policy function exist and satisfy some bounds, the Cauchy estimates, to be defined in the paper, and that all the moments of the distribution exist. Note that while one may find closed form solutions by solving the equations forward, our method may be applicable even if one cannot prove that a closed form exists, that is the policy function is analytic. We offer advice on how to use our methodology even when the proof of analyticity is not obtained. This is important since we can find a measure of the error on the policy function, which is generally not available with numerical methods. Furthermore, our method can be generalized more easily to handle many

\(^1\)This is true for almost all models that use CRRA utility and subsequent utility functions that researchers find reasonable such as habit persistence and recursive utility functions. Though some closed form solutions with the CRRA utility function have been found, see for example Campbell [1986], Labadie [1989], Burnside [1998], Birdarkota and McCulloch [2003], and Tsionas [2003], among others, our method finds the solutions for general utility functions that offer closed forms without specific distributions for consumption and dividend.
assets and portfolio choice and heterogeneous agents. This is not true of the models that solve forward.

Obtaining the actual solution also gives us insight as to what is going on within the model. For example, when we plot out the price-dividend ratio versus consumption growth for the Mehra-Prescott [1985] model, we are able to gain insight why there is not enough volatility generated in stock prices to obtain an appropriate equity premium by observing the slope of the price-dividend function. There are therefore two additions within the asset pricing literature, which this methodology offers: better accuracy of the solution globally and better intuition from the solution. The key insight to solving these models analytically is to assume that the policy function, price-dividend ratio in this case, is analytic and then deriving a set of conditions to prove that it actually is.

We demonstrate this methodology solving for two asset pricing models, Mehra and Prescott [1985] and Abel [1990]. We choose the Mehra and Prescott [1985] model since this is the paper that began this whole equity premium puzzle literature. We choose the Abel [1990] model since it adds relative consumption in the model as well as (internal) habit persistence to the original Mehra and Prescott Model.\(^2\) We feel that solving for these two models in this order allows the reader to see how the methodology works and how additional more complicated assumptions added to the model affects the solution. Using more complicated asset pricing models only makes the insights from our methodology more difficult to comprehend.\(^3\) The Mehra and Prescott model is nested within the Abel model. In each case we solve out for the price-dividend ratio analytically and then parameterizing the model as in the literature, we also solve for this numerically so that the solution is ‘accurate’, i.e., some measure of the error is ‘machine zero’. A number is said to be ‘machine zero’ when the computer cannot tell

\(^2\)While the parameterization that includes habit persistence in the Abel [1990] model is problematic, it leads to negative marginal utility for some values of consumption growth with internal habit persistence, it demonstrates our methodology well. It also points out a problem in numerical work with bounding the support of the distribution of the normal when getting the solution numerically. The bounding of the distribution in numerical work matters to the numerical solution.

\(^3\)We can also do this for other asset pricing models such as for example Epstein and Zin [1989,1990,1991], Campbell and Cochrane [1999], Cecchetti, Lam and Mark [1990,2000], and Constantinides and Duffie[1996] among several others. The additional complexity of those models makes the proofs more involved though this methodology can handle these cases.
the difference between that number and zero.\footnote{Machine zero has changed over time, the accuracy of computations, and will change more in the future.} Since any analytic function can be represented as an infinite order polynomial, a convergent power series, we will use a polynomial of some finite order, so that the global solution is such that the remainder term, the tail of the power series, is machine zero, the remainder is less than $10^{-16}$. This is what we mean when we say we will also solve for the model numerically. Solving for these two cases is very insightful since in the Mehra and Prescott [1985] model as well as in Abel [1990], the global solution was not investigated. Consequently we will be able to analyze the global solution to the Mehra and Prescott [1985] model and the Abel [1990] model using our methodology and comparing it to state of the art numerical methods.

It is important to realize what this methodology can and cannot do. We can obtain an analytic solution to the policy function here for a certain range of convergence of the state variable. Thus while we do get the actual solution for a certain range of convergence, i.e., the domain of the policy function, one is unable to analyze the model outside of that range of convergence with this methodology. We feel this is a shortcoming only if the values of the state variable outside of the range of convergence are relevant. We find, in the cases we study, they are not, however, one will have to judge whether those values are important for the particular case they are studying.

Our conjecture is that as long as the pricing kernel, intertemporal marginal rate of substitution, is analytic and the dividend process is analytic, and all the moments of the stochastic process exist, then the policy function, price-dividend ratio, should be analytic. We do not offer a general proof of this conjecture but will be studied in future work. This would be an important theorem to prove and would be very useful in asset pricing as well as solving Euler equations in general.

While we use this methodology to solve for two specific models here as examples, one can use this for any Euler equation where the policy function is analytic. The amount of work required to solve these models and prove analyticity is extensive and many may find the time required to do so, prohibitive. In order to make this methodology operational, we are offering
direction on how to use this methodology without having to prove analyticity. In particular, as long as the first $k$ derivatives of the policy function exist, and the first $k$ moments of the random variable exist, then one can use this methodology to approximate the solution with a $k-1$ order polynomial and obtain a bound on the remainder term exploiting the mathematical properties of the Euler equation. This will give the researcher a bound on the error term of the policy function. One important question of interest in finance is which asset pricing kernel should we use to evaluate securities. This methodology will give the user the ability to test whether a particular specification of an asset pricing kernel will price assets appropriately. This has been a major endeavor in finance for several years now. This differs from numerical methods in that one cannot get a true estimate of the error of the solution to the policy function from the true policy function, see Judd [1998]. Consequently, this methodology gives us a good measure of the error in pricing that one obtains.

2 Asset Pricing Model

We consider a model of asset pricing which includes Abel [1990] as well as Mehra and Prescott [1985]. The investor is assumed to maximize:

\[
V(W_0, z_0) = \max_{\{c_t, \omega_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t \int_{Z} u(c_t(W_t), z_t) \mu^t(z_0, dz^t)
\]

subject to:

\[
W_{t+1} = w_t R_{t+1}(W_t - c_t).
\]

and we will let the state variable follow a process of the form:

\[
z_{t+1} = \xi(z_t, \nu_{t+1})
\]

\footnote{Santos [2000] shows that under some strict conditions, one can show that the Euler equation residual is of the same order of magnitude as the residual in the policy function. Even though this is the case, there is still some judgment which one will have to make. This is not the case with our methodology in getting an estimate of the policy function.}
where $\nu_{t+1}$ is an i.i.d. random variable, the function $\xi$ is defined such that $z_t$ is a markov process, $R_{t+1}$ is an $n$ dimensional column vector, $0 \leq \beta \leq 1$ and $\Omega \equiv \{ \omega : \omega'1 = 1 \text{ and } \omega \in R^n \}$. Here $u(.,.)$ is the one period reward function, $R$ is a vector of returns for each asset, $W_t$ is the wealth of an individual at time $t$ and also the endogenous state variable, and $c$ is consumption. $\omega$ is a vector of portfolio shares for each asset. $z$ is an exogenous state variable that follows a Markov process defined by the transition function $Q$ and $\xi$ is a Markov transition probability for the exogenous state variable. Given the transition function $Q$, one can define the probability measure $\mu^t(z_0,.)$ given $z_0$ on every $t$-fold product space comprising all partial histories of the form $z^t = (z_1, z_2, \ldots, z_t)$.

Following Santos [1999], let $(W, W)$ be the measure space of endogenous states and $(Z, Z)$ be the measure space of exogenous states. Both $W$ and $Z$ are Borel subsets of $R$. The state space $(S, S) \equiv (W \times Z, W \times Z)$ is convex.

Our goal is to show that the policy function is analytic. The function is analytic when it can be expressed as a convergent power series $\sum_{j=0}^{\infty} b_j(x-a)^j$ where $a$ is a real number. This power series converges uniformly to a function, $f(x)$, for a range of convergence $|x| < r$. This range of convergence is the domain of the function. It turns out, according to Cauchy’s theorem, that restrictions on the derivatives of the policy function are necessary for a function to be analytic. As a result, we use the analysis of Blume, Easley and O’Hara [1982] to assure that the policy function is $C^\infty$. They make the following assumptions to assure the differentiability property of the policy function.

1. Define the Markov transition probability by $\xi : Z \times Z \to Z$. The history $z^t = (z_1, \ldots, z_t) \in Z^t$. This results in a time invariant transition function $Q$ on $(Z, Z)$ so that for each $z_0 \in Z$, $\exists$ a probability measure $\mu^t(z_0,.)$ defined on $(Z^t, Z^t)$. In addition for all $Z \in Z$, $Q(\nu, Z) = \int_Z f(\nu, \tilde{\nu})d\tilde{\nu}$ and $f \in C^\infty(Z, Z)$. We will also require that $\int_Z \tilde{\nu}^n f(\nu, \tilde{\nu})d\tilde{\nu} \exists$ for all integers $n$.

2. The action space $C \times \Omega$ is compact and convex. The reward function is $u : C \times Z \to R_+$ is

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6Note that $W$ with a subscript is wealth and $W$ with no subscript is a measurable space.

bounded above with \( u \in C^\infty(C \times Z) \). In addition \( u(., z) \) is strictly concave and increasing on \( C \) for all \( z \in Z \).

3. The wealth accumulation equation (2) for the state variable \( W_t \) is \( g : W \times Z \times C \times \Omega \rightarrow W \).

\( g \) is concave on \( W \times C \times \Omega \). For all \( c \in C \) and \( \omega \in \Omega \) \( g(., c, \omega, z) \) is increasing on \( W \) for all \( z \in Z \). \( g(., ., ., z) \in C^\infty(W \times C \times \Omega) \) for all \( \omega \in \Omega \). In addition, \( g(., ., ., z) \) is invertible.

Following Santos [1999] for each \( z_0 \) the value function \( V(., z_0) \) is concave and satisfies the Bellman equation.

\[
V(W_0, z_0) = \max_{c_0, \omega_0} \left\{ u(c_0, z_0) + \beta \int_Z u(W_1, z_1)Q(z_0, dz_1) \right\} \tag{4}
\]

subject to (2) and (3). In addition the optimal policy functions satisfy the Euler Equations:

\[
u'(c_t, z_t) = \beta \int_Z u'(c_{t+1}, z_{t+1})w_t^iR_{t+1}Q(z_t, dz_{t+1}). \tag{5}\]

while for any asset \( i \):

\[
u'(c_t, z_t) = \beta \int_Z u'(c_{t+1}, z_{t+1})R_{t+1}^iQ(z_t, dz_{t+1}) \tag{6}\]

where the superscript \( i \) indicates the \( i^{th} \) security. The assumptions of Blume, Easley and O’Hara [1982] are satisfied so that the value function \( V \in C^\infty(S) \) for \( s \in \text{int}(S) \). In addition there exist policy functions \( c = c(s) \) and \( \omega = \omega(s) \) such that \( (c, \omega) \in C^\infty(S) \) for \( s \in \text{int}(S) \).

Given that the policy functions are continuously differentiable we can now proceed to a discussion of the equilibrium for the economy. Notice that we can guarantee by the analysis of Blume, Easley and O’Hara [1982] that all the derivatives for the policy function exist for the problems we will be solving. It remains to show that all the derivatives are bounded by the Cauchy estimates in order to prove analyticity, see Rudin [1974]. Notice, that if the policy function is not analytic, then we cannot represent the policy function by a convergent power series.

In general we have the return on an asset \( i \) is given by:
where $P^i$ is the price of asset $i$ and $D^i$ is the dividend for asset $i$. Suppose there are two assets, i.e., $i = 1, 2$, as in the models we solve. Let the first asset be a risk free one period bond so that the investor is guaranteed a fixed return $R^f_{t+1}$ independent of the state. Let the second asset be a share of stock, which has an uncertain dividend payment which is assumed to follow a Markov stochastic process. The dividend process is:

$$D^i_{t+1} = D^i_t \varphi\left(z^i_{t+1}\right) \quad (8)$$

where $\varphi \in C^\infty(Z)$. We also need the $\varphi(.)$ function to be invertible, that is $z^i_{t+1} = \varphi^{-1}\left(\frac{D^i_{t+1}}{D^i_t}\right)$ such that $D^i_{t+1} = D_t \varphi\left(\varphi^{-1}\left(\frac{D^i_{t+1}}{D^i_t}\right)\right)$. For the models we solve this guarantees the inverse of the $g$ function in Blume, Easley and O’Hara [1982].

In equilibrium the only source of income for the aggregate economy is the dividend from the stock. As a result $c_t = D_t$ for all $t$. We consider utility functions that are homogenous of degree $k$. As a result the functional equation for the price-dividend ratio, $\frac{P}{D}$, is a combination of (3), (5), (7) and (8) to yield:

$$\frac{P}{D}(z) = \beta \int \varphi(\xi(z, \nu), \xi(z, \nu)) u(1, z) \varphi(\xi(z, \nu)) \left(1 + \frac{P}{D}[\xi(z, \nu)]\right) f(\nu) d\nu. \quad (9)$$

We want to prove that this integral equation has a solution which is a function $\frac{P}{D}(z) \in C^\infty(Z)$. We also want to show this function to be analytic. It is well known that there exists a continuous price-dividend function that solves this equation, however, it has not been proved under what conditions the function is analytic. Rather than providing this proof, our approach is to posit that the price-dividend function for the particular problem is analytic which is equivalent to being able to express the function as a convergent power series:

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8 With our methodology, one could construct a more complicated economy with many sources of risk and consumption to be comprised of more income sources than just dividend, however this would make our derivations too tedious without adding anything to the main ideas.

9 See Altug and Labadie (1994, chapter 2) for a proof and survey of the literature.
\[
\frac{P}{D}(z) = \sum_{j=0}^{\infty} b_j (z - a)^j
\]  
(10)

where \(a\) is a real number. This power series converges in \(\{z : |z - a| < r\}\) for every \(r < r_c\) where \(r_c\) is the range of convergence.\(^{10}\)

The next step is to evaluate the integral equation (9) when the price-dividend function satisfies (10) for a large number of coefficients, \(b_j, j = 0, 1, \cdots, n\). The right hand side of this integral equation can then be manipulated using standard procedures such as the binomial theorem and Leibnitz’s rule to yield a power series which is comparable to (10). This analysis yields a system of linear equations in the \(n\) coefficients, the \(b_j\)’s, of the original system plus an error term, \(O(n + 1) \equiv \sum_{j=n+1}^{\infty} b_j (z - a)^j\). It is important to realize that the coefficients in the power series cannot be solved recursively, since the value of \(b_j\) is dependent on both higher and lower order coefficients. This means that the coefficients can and do change as the number of coefficients, \(n\), is increased. As a result we add additional coefficients until both the residual, \(O(n + 1)\) and the change in coefficients are very small, machine zero. As long as this error term is very small and the coefficients change by a small magnitude, the function can be approximated by the finite order power series since the value of the tail is small.

We first consider the Mehra-Prescott [1985] Model. In this case we are able to find and solve the system of equations in the coefficients so that we can represent the price-dividend function as a power series. In this case we are able to exploit the properties of the contraction mapping (9) to show that all the derivatives of the price-dividend function have an upper bound as long as the dividend growth rate is above \(-20\%\), which is needed for existence of the solution. In addition we can demonstrate that these derivatives satisfy the Cauchy estimate:

\[
\left| \frac{P^{(n)}}{D} (z) \right| \leq n! C^n M
\]  
(11)

in which \(M\) is an upper bound on the price-dividend function for dividend growth in the range of convergence \([-2, r_c]\). We need to satisfy these Cauchy estimates in order to prove that the

\(^{10}\)See Rudin (1974, chapter 10) for the development of the elementary properties of analytic functions.
function is analytic.\footnote{See Rudin (1974, pp. 229-230).}

We proceed to the model constructed by Abel [1990] where the model adds habit persistence and relative consumption in the utility function. Similar to the Mehra and Prescott case, we are able to exploit the properties of the contraction mapping (9) to show that all the derivatives of the price-dividend function have an upper bound as long as the dividend growth rate is above $-20\%$. In addition we can demonstrate that these derivatives satisfy the Cauchy estimate (11) in which $M$ is an upper bound on the price-dividend function for dividend growth in the range of convergence $[-.2, r_c]$. We need to satisfy these Cauchy estimates in order to prove that the function is analytic.

3 Mehra-Prescott Model

The classic case for the equity premium puzzle that was first studied is the Mehra and Prescott [1985] model. There are four assumptions to the Mehra and Prescott model: There is a representative agent that has constant relative risk aversion utility, financial markets are complete, financial markets are frictionless and individuals are free to trade the risky stock and one period bonds which are in zero net supply. Models with these assumptions and some variant of them are solved for numerically in the literature and we wish to get the actual analytic solution to the asset prices here.\footnote{While this model has also been solved for by Campbell [1986], Labadie [1989], Burnside [1998], Birdarkota and McCulloch [2003], and Tsionas [2003], our method is more easily extended to cases with far more securities than two as well as heterogeneous agents.} We will be working with the price-dividend ratio rather than expected returns though we are able to convert these price-dividend ratios into expected returns and compare the results in the literature to ours.

The model is as follows: Individuals have constant relative risk aversion utility as follows:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},$$

(12)

where in this case the only source of real income is the dividend from the risky security and thus $c_t = D_t$, and $D_t$ is the dividend. The dividend process for the risky security is:
\[ D_{t+1} = D_t G \exp(\phi u_t + \nu_{t+1}) \]  

where \( \nu_{t+1} \sim NIID(0, \sigma^2) \), \( G = \exp(u_0) \) where \( u_t \) is the continuously compounded growth rate of the dividend and follows an AR(1) process: \( u_{t+1} = u_0 + \phi u_t + \nu_{t+1} \). We wish to solve for the price-dividend ratio which we define as \( P_D(u) \), where \( u = u_t \). The Euler equation we want to solve for is:

\[
\frac{P_t}{D_t} = E_t \left[ M_{t+1} \frac{D_{t+1}}{D_t} \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \right].
\]

Here:

\[
M_{t+1} \equiv \beta u'(D_{t+1})/u'(D_t) = \beta \left( \frac{D_{t+1}}{D_t} \right)^{-\gamma}
\]

All the assumptions of Blume, Easley and O’Hara [1982] are satisfied so that we may write equation (14) in the form of equation (9), and it becomes the following linear integral equation:

\[
\frac{P}{D}(u) = \frac{K_0}{\sqrt{2\pi}\sigma} \exp(K_1 u) \int_{-\infty}^{+\infty} \exp\{((1-\gamma)x - (x^2/2\sigma^2))\}[1 + \frac{P}{D}(u_0 + \phi u + x)]dx
\]

where \( K_0 \equiv \beta G^{1-\gamma} \) and \( K_1 \equiv (1-\gamma)\phi \). In many asset pricing papers, this particular Euler equation, (16), is solved forward and a transversality condition is imposed to insure the solution exists, and obtain an explicit solution to this equation, see for example Burnside [1998]. This is how many representative agent models are solved for. In contrast we do not solve this equation forward but rather solve for the above functional equation. In doing so, we derive a set of conditions in order for the solution to be analytic. It turns out that the condition for the solution to be analytic in the above problem is identical to that in models that solve for this equation forward. The problem we face is to solve for the function \( \frac{P}{D}(u) \). To do this we begin by defining the two integrals:

\[ \text{Notice that } u \text{ is the exogenous state variable } z \text{ and } x \text{ is the random variable } \nu. \]
\[ I_1 = \int_{-\infty}^{+\infty} \exp\{(1 - \gamma)x - \frac{x^2}{2\sigma^2}\} \, dx \]  
(17)

and:

\[ I_2(u) = \int_{-\infty}^{+\infty} \exp\{(1 - \gamma)x - \frac{x^2}{2\sigma^2}\} \frac{P}{D}(u_0 + \phi u + x) \, dx. \]  
(18)

Then we can write the integral, equation (16), as:

\[ \frac{P}{D}(u) = \frac{K_0}{\sqrt{2\pi\sigma}} \exp(K_1u)[I_1 + I_2(u)]. \]  
(19)

If we normalize today’s dividend to one dollar, then the first integral multiplied by the coefficient outside the bracket in equation (19) represents the value to the security of the dividend next period. Similarly normalizing today’s dividend to one dollar, the second integral multiplied by the coefficient outside the bracket in equation (19) represents the value to the security today of the ex-dividend price of the security next period, that is the value of all future dividends from period two on. Thus these two integrals give us a per dollar decomposition of the value of the security today, in terms of the per dollar dividend and per dollar price. One can always do this by multiplying the equation through by today’s dividend. One will notice that solving for \( I_1 \) is far more simple than solving for \( I_2(u) \) since the first integral is evaluating next periods dividend and the second integral is evaluating all future dividends from the second period on.

The solution to \( I_1 \) which is independent of the state variable, \( u \), is:

\[ I_1 = \sqrt{2\pi\sigma} \exp(P_0) \]  
(20)

where \( P_0 \equiv \frac{\sigma^2}{2}(\gamma - 1)^2. \)  
\(^{14}\) In order to solve for \( I_2(u) \) we guess that a solution for \( \frac{P}{D}(u) \) has the functional form:

\[ \frac{P}{D}(u) = \exp(K_1u)(b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + \ldots b_n(u - \bar{u})^n + O((u - \bar{u})^{n+1}), \]  
(21)

\(^{14}\)This is straightforward since we know how to solve for the expected value of a lognormal random variable.
where $\bar{u}$ is steady state dividend growth. Consequently we may write $\frac{P}{D}(u_0 + \phi u + x)$ as:

$$\frac{P}{D}(u_0 + \phi u + x) = \exp(K_1(u_0 + \phi u + x)) (b_0 + b_1(u_0 - \bar{u} + \phi u + x) + b_2(u_0 - \bar{u} + \phi u + x)^2 + \ldots + b_n(u_0 - \bar{u} + \phi u + x)^n + O((u - \bar{u})^{n+1}))$$

Using the Binomial theorem\textsuperscript{15} we may rewrite equation (22) as:

$$\frac{P}{D}(u_0 + \phi u + x) = \exp(K_1(u_0 + \phi u + x)) \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i + O((u - \bar{u})^{n+1}) \right)$$

and substituting this into (18) we obtain:

$$I_2(u) = \int_{-\infty}^{\infty} \exp((1 - \gamma)x) \exp(K_1(u_0 + \phi u + x))$$

$$\left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i \right) \exp \left( -\frac{x^2}{2\sigma^2} \right) dx.$$ 

In calculating the integral $I_2(u)$, we can rewrite equation (24) as:

$$I_2(u) = \exp(K_1(u_0 + \phi u)) \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} \gamma_i$$

where:

$$\gamma_i = \int_{-\infty}^{\infty} \exp((1 - \gamma)x) \exp(K_1 x) x^i \exp \left( -\frac{x^2}{2\sigma^2} \right) dx.$$ 

In the evaluation of this integral, $I_2(u)$, see the appendix, we use the moments of the distribution of dividend growth in $\gamma_i$. We use the Gaussian distribution in this model however, if an alternative distribution is used, this step can still be taken as long as all the moments of the distribution exist. For the case of the Gaussian distribution we obtain:

$$\gamma_i = \exp(Q_0) \sqrt{2\pi} a_i$$

\textsuperscript{15}The binomial theorem states that we may write $(u_0 - \bar{u} + \phi u + x)^k = \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i$. 


where \( Q_0 \equiv \frac{\sigma^2}{2} [1 - \gamma + K_1]^2 \) and \( a_i \equiv \sum_{j=0}^{i} \binom{i}{j} (\sigma^2[1-\gamma+K_1])^{i-j}(\sqrt{2\sigma})^j(1+(-1)^j)(1/2)\Gamma[i+1/2] \).

We can now write \( I_2(u) \) as:

\[
I_2(u) = \exp(K_1(u_0 + \phi u)) + Q_0)\sqrt{2\sigma} \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i}a_i \right). \tag{28}
\]

Since we have solved for the integrals \( I_1 \) and \( I_2(u) \), we can now proceed to solving for our price-dividend ratio \( \frac{P}{D}(u) \). We do this by solving for the coefficients \( b_i \), \( i = 0, \ldots n \) in the guessed solution for \( \frac{P}{D}(u) \) in equation (21).

### 3.1 Solving for the Coefficients \( b \)’s

Now that we have representations for \( I_1 \) and \( I_2(u) \), we go back to equation (19) and using equation (21) we write:

\[
\exp(-K_1 u) \frac{P}{D}(u) \equiv \frac{K_0}{\sqrt{2\sigma}} (I_1 + I_2(u)) \equiv
\]

\[
\frac{K_0}{\sqrt{2\pi \sigma}} \left[ \sqrt{2\pi \sigma} \exp(P_0) + \exp(K_1(u_0 + \phi u)) + Q_0)\sqrt{2\sigma} \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i}a_i \right) \right]
\]

\[
= b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + \ldots + b_n(u - \bar{u})^n + O((u - \bar{u})^{n+1}).
\]

That is we equate \( \exp(-K_1 u) \frac{P}{D}(u) \) to the guessed solution. We would like to solve for the \( b_i \)’s, \( i = 0, \ldots, n \), in the above equation, (29). In order to do this, we collect all the functions of \( u \) on the left hand side of this equation and use Taylor’s theorem to take an \( n^{th} \) order Taylor expansion around the steady state. We then obtain an \( n + 1 \) system of linear equations in the \( b_i \)’s. Define the following functions:

\[
g(u) \equiv K_1(u_0 + \phi u),
\]

\[
r_{i,k}(u) \equiv (u_0 - \bar{u} + \phi u)^{k-i},
\]

13
and

\[ w_{i,k}(u) \equiv \exp(q(u))r_{i,k}(u) \approx \sum_{l=0}^{n} \frac{1}{l!}w_{i,k}^{(l)}(\bar{u})(u - \bar{u})^l. \]

Then we substitute \( w_{i,k}(u) \) into equation (29) to obtain:

\[
\frac{K_0}{\sqrt{2\pi}} ![\sigma \sqrt{2\pi} \exp(P_0) + \exp(Q_0) \sqrt{2\sigma} \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \sum_{l=0}^{n} \frac{1}{l!}w_{i,k}^{(l)}(\bar{u})(u - \bar{u})^l a_i \right) ] \]

\[ = b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + ... + b_n(u - \bar{u})^n + O((u - \bar{u})^{n+1}) \]

and consequently we solve for the \( b_i \)'s, ignoring the term \( O((u - \bar{u})^{n+1}) \), as:

\[
b_0 = K_0 \exp(P_0) + \frac{K_0}{\sqrt{\pi}} \exp(Q_0) \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \frac{1}{l!}w_{i,k}^{(l)}(\bar{u}) a_i \right) \]

(31)

and:

\[
b_l = \frac{K_0}{\sqrt{\pi}} \exp(Q_0) \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \frac{1}{l!}w_{i,k}^{(l)}(\bar{u}) a_i \right), \]

(32)

where \( l = 1, \ldots, n \). Equations (31) and (32) are an \( n + 1 \) system of linear equations in the \( b_i \)'s.

We are further able to show that for some range of convergence for \( u \in [a, b] \) the bound on the remainder term is given by:

\[
|R_n| \leq \nu_{n+1} u_{max}^{n+1} \]

(33)

where \( \nu_{n+1} \equiv \sup_u \left\{ \left( \exp(-K_1 u) \frac{d}{du} (u) \right)^{n+1} \right\} \) and \( u_{max} \equiv \max \{ |a|, |b| \} \). We also have:

\[
\nu_n = \frac{K_0 B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} \nu_k \phi^k}{1 - K_0 \phi^n B_0},
\]

(34)

where:

\[
\nu_0 = \frac{\exp(P_0)}{1 - K_0 B_0}, \]

(35)

and:

14
\[ B_0 \equiv \exp(K_1[u_0 + \phi u] + Q_0). \] (36)

We show how to obtain this bound on the remainder term and the proof of analyticity in the appendix. The condition for analyticity is that \(|K_0 B_0| < 1\).\(^\text{16}\)

### 3.2 Numerical Application

We parameterize this model the same as we find in the literature and see how the numerical solutions in the literature compare to our solution method.\(^\text{17}\) It is important to realize that our solution is the actual solution to the problem while the numerical work gives an estimate of the solution. When we solve for the model, we need a stopping rule in order to choose the number of coefficients we will solve for. We will use the rule where we stop when \(\max|\sup_i |b_{n-1,i} - b_{n,i}|, R_n| < \epsilon\) where \(\epsilon\) is a machine zero number and \(b_{n,i}\) is the \(i^{th}\) coefficient of the solution with \(n + 1\) coefficients and \(R_n\) is the bound on the remainder with \(n + 1\) coefficients. We say a number is machine zero when it is less than \(10^{-16}\). We will also compare our solution to a state of the art numerical procedures, Gaussian quadrature procedure, to see how it fares against that technique.\(^\text{18}\)

We parameterize our model as in Lucas [1994] and compare our results to hers. The parameter values are as follows, \(\sigma = .036, \gamma = 2.5, \beta = .95, \bar{u} = .017, \) and \(\phi = -.14\). With this parameterization, we find that the supremum of the changes in the coefficients is machine zero when we include 9 coefficients. We report the nine coefficients in Table 1 in the appendix.

\(^{16}\)This is identical to the discounting property in Blackwell’s Theorem.

\(^{17}\)We can obtain all these moments from the price-dividend ratio and the bond price converting them to returns as in equation (7) and the interest rate and then integrating over the random variable to obtain expected values and standard deviations.

\(^{18}\)For the arguments in favor of using quadrature procedures in asset pricing see Tauchen and Hussey [1991]. Since we are using a normal distribution, we could use Gauss-Hermite quadrature methods which allows integration over the entire support of the distribution. We choose Gaussian quadrature instead because in several asset pricing models, such as Abel’s internal habit persistence, one cannot use the entire support of the distribution. Dividend growth must be bounded from above and below. With the Gaussian quadrature procedure, we can bound the support of the distribution. Secondly, even if there were no bounds on the support of the distribution, we could still get an accurate solution with the Gaussian quadrature procedure integrating over many standard deviations as we can make the limits of integration as large as we like. Lastly, in order to approximate the \(\frac{C}{P}(u)\) function with Chebyshev polynomials, we need a maximum and minimum growth rate for the dividend and consequently, we need to bound the support of the distribution.
It is interesting to note that the coefficient $b_8$ is machine zero itself and thus will have no
effect on the numerical solution unless the state variable, $u$, is very far from the steady state,
i.e., $u - \bar{u} > 1$, which in our case means that consumption growth is above 101.7%. We also
report the bound on the remainder for the cases of where the range of convergence, $r_c$, is .02
and when it is .5. The bounds on the remainders are in Table 2 in the appendix.

With $r_c = .02$ we have the bound on the remainder is machine zero with 5 coefficients
and with $r_c = .5$ we have the remainder is machine zero with 8 coefficients. According to
our measure, we choose nine coefficients. In order for us to choose more coefficients than nine
according to our measure, we will need $r_c > 2$ which implies a consumption growth rate above
200%. Below is the graph of the $P_D(u)$ function with 1 through 4 coefficients with a range of
convergence of .02. One will notice that whether we use three or four coefficients, the solution
is the same within the range of convergence of .02. Note that while our measure tells us to use
9 coefficients, according to the remainder we should use five. However, whether we use three
or four coefficients, the bound on the remainder term is still very small, \( -2.77 \times 10^{-10} \) with
three coefficients and \( 3.62 \times 10^{-14} \) with four coefficients. It is important to remember these are
bounds on the remainder term and not the actual remainder term. These values are too small
for the naked eye to see the difference.

[Insert Figure 1]

Lastly we report in Tables 3 and 4 the moments, means and variances, we generate from
this model and compare them to those found in the literature. Since we use Lucas [1994] to
compare our results and she uses a simulation method, we will also solve this model numerically
using Gaussian quadrature methods since we solve for the price dividend functions and Lucas
solves for steady state moments with simulation methods.\(^{19}\) With the numerical method we
use, Gaussian quadrature methods, we bound the distribution of the random error term when
\(^{19}\) See for example Judd [1998] for an explanation of Gaussian quadrature methods. For a precise explanation
of our procedure, see the appendix.
applying this methodology. We solve with the numerical method by bounding the error term with three, four and five standard deviations to the right and left of the mean.\textsuperscript{20} Lucas [1994] uses a two state process for consumption growth and simulates 1000 years of data 1500 times. The individuals follow their optimal investment strategy and she reports the cross section of returns and standard deviations for those generated by the model between year 999 and 1000.

We see in Table 3 that the moments generated by Lucas [1994] comes close to matching the moments from our solution. Thus we can conclude that the simulation exercise that Lucas [1994] ran is very accurate in obtaining estimates of the moments she is interested in. The one statistic she does not report is the price-dividend ratio. We are interested in this statistic since one reason investors will be interested in this methodology is to identify mispricing. We have also solved the model by using Gaussian quadrature methods. It is important to note that our methodology does not rest on the assumption that the random variable is normally distributed but rather that all the moments of the distribution exist. In this respect, the methodology is very powerful. In several models, the bounding of the error term arises naturally from the model as we will see in the habit persistence model of Abel [1990]. Consequently, we run the Gaussian quadrature procedure with three different bounds on the error term, three, four and five standard deviations. When no bound is necessary, we can put arbitrarily large bounds on the limits of integration, the support of the distribution, and the solution will be a good approximation of the truth. An additional reason we do this is that when thinking of a normal distribution, it seems reasonable that integrating over three standard deviations will give an accurate answer to any problem since 99\% of the probability lies within the three standard deviations. However, we will see for the asset pricing results here it is not the case and we need five standard deviations. Even though the methodology gives a good solution to the policy function, there still may be errors in judgement in running the procedure, such as the bounding of the support of the distribution.\textsuperscript{21}

\textsuperscript{20}For a more detailed explanation, see the appendix. The reason we do this is because for many models, bounds on the distribution of the random variable arise naturally from the model as for example the habit persistence model in Abel [1990]. This gives us an idea of the accuracy of the solution, and how such judgements affect the solution.

\textsuperscript{21}We will see in the habit persistence case of Abel [1990], a natural bound arises from the model which may
We see in Table 4 that when using three standard deviations, most of the moments match the analytic solution except for the price-dividend ratio which is off by 47 cents. This is a large deviation from the truth since that large a difference in an asset price generated by the model can change an investor’s decision. Suppose that the dividend today is $1.00 and one was looking to find a mispriced security to buy or sell off. Suppose that the security is trading at $12.50. Then one would sell or even sell short the security according to the numerical solution while one would buy the security according to the analytic solution. This is of interest since the error to that numerical procedure, measured as the deviation of the Euler equation from zero divided by the price-dividend ratio, is very small and one may conclude that this solution is accurate. We do see however, that as we increase the number of standard deviations we integrate over, the price-dividend ratio is the same as the analytic solution in Table 4 at the steady state.

Below we graph three curves, the price-dividend ratio from the numerical solution with bounds of four standard deviations, five standard deviations and the analytic solution. The Analytic solution and the numerical solution with five standard deviations overlap. The graph of the numerical solution with four standard deviations lies below the other two. The plots of the numerical solution jump upward until they reach the truth.

[Insert Figure 2]

It is instructive to note that while in this case we can obtain a very good solution with the numerical method, the analytic solution gives the precise results with an actual error term as to how much the price-dividend ratio may be off from the truth for some range of convergence. One cannot provide this with the numerical methods. Secondly, except for algebraic errors, there are really no errors to be made as judgement errors. That is once we have the solution, and an estimate of the bound on the remainder term, then the procedure is straightforward and we know we will have the solution. An exception to this is the result of Santos [2000] where he finds under certain conditions, the residual on the Euler equation is of the same order of magnitude as the error in the policy function. Even there however, when estimating the policy cause us to obtain a bad numerical answer.
function numerically, one still needs to decide which numerical procedure to use. Consequently, one still needs to use good judgement in order to obtain an accurate solution. There are several researchers with this expertise but it is always easier to use the analytic solution when it exists.

4 Abel [1990] Model

The asset pricing model of Abel [1990], which includes habit persistence, both internal and external, fits into the general specification of section 2 as well.\footnote{For a recent paper that uses these preferences, see Chan and Kogan [2001].} Under a certain parameterization, the model becomes the same as in Mehra and Prescott. Individuals have the following utility function:

\[
u(c_t, v_t) = \left(\frac{c_t}{v_t}\right)^{1-\gamma}, \tag{37}\]

where:

\[
v_t = \left(c_{t-1}^{D} C_{t-1}^{1-D}\right)^{\alpha} \tag{38}\]

where \(\alpha \geq 0, D \geq 0, c_{t-1}\) is the consumer’s own consumption in period \(t-1\) and \(C_{t-1}\) is aggregate per capita consumption in period \(t-1\). Abel parameterizes the model by setting \(\alpha\) and \(D\) to zero and one depending on the case. Note that when \(\alpha = 0\) we are in the Mehra and Prescott case. When \(\alpha = 1\) and \(D = 0\) we are in the case of relative consumption of catching up with the Joneses also called external habit in which the habit is external to the individual’s choice. When \(\alpha = 1\) and \(D = 1\) we are in the internal habit case and individuals own lagged consumption affects their choice of consumption.

The only source of real income is the dividend from the risky security so that \(c_t = C_t = D_t\). The dividend process for the risky security is:

\[
D_{t+1} = D_t G \exp(\phi u_t + \nu_{t+1}) \tag{39}\]
where \( \nu \sim NIID(0, \sigma^2) \), \( G = \exp(u_0) \) where \( u \) is the continuously compounded growth rate of the dividend and follows an AR(1) process \( u_{t+1} = u_0 + \phi u_t + \nu_{t+1} \). We wish to solve for the price-dividend ratio which we define as \( \frac{P}{D}(u) \).\(^{23}\)

The Euler equation we wish to solve for in this case is:

\[
\frac{P_t}{D_t} = E_t[M_{t+1} \frac{D_{t+1}}{D_t} (1 + \frac{P_{t+1}}{D_{t+1}})]
\]

where:

\[
M_{t+1} = \beta u'(D_{t+1}, v_{t+1})/u'(D_t, v_t) = \beta \left( \frac{D_{t+1}}{D_t} \right)^{-\gamma} \left( \frac{v_{t+1}}{v_t} \right)^{\gamma-1} \frac{H_{t+2}}{E_t[H_{t+1}]}.
\]

We also have:

\[
H_{t+2} = 1 - \beta\alpha D(D_{t+2}/D_{t+1})^{1-\gamma}(D_{t+1}/D_t)^{(-\alpha(1-\gamma))} \quad \alpha \geq 0, \quad D \geq 0
\]

and:

\[
v_{t+1} = D_t^\alpha.
\]

We can rewrite the Euler equation, (40), as:

\[
\frac{P}{D}(u) = \beta E_t[(1 + \frac{P}{D}(u_0 + \phi u + x))(D_{t+1}/D_t)\frac{H_{t+2}}{E_t[H_{t+1}]} \left( \frac{D_{t+1}}{D_t} \right)^{-\gamma} \left( \frac{v_{t+1}}{v_t} \right)^{\gamma-1}].
\]

Define \( K_0 \equiv \beta G^{-\gamma+1}, K_1 \equiv (1-\gamma)(\phi-\alpha), K_2 \equiv \beta\alpha DG_0^{1-\gamma} \exp(P_0) \), and \( K_3 \equiv \beta\alpha DG_0^{(1+\phi-\alpha)(1-\gamma)} = \beta\alpha DG_0^{1-\gamma} \exp(K_1u_0) \). Note also that \( E_t(H_{t+1}) = 1 - \beta\alpha DG_0^{1-\gamma} \exp((\phi-\alpha)(1-\gamma)u) \exp(P_0) \equiv 1 - K_2 \exp(K_1u) \). Thus we can write the asset pricing equation as:

\[
\frac{P}{D}(u) = \frac{K_0}{\sqrt{2\pi}\sigma} \frac{\exp((K_1u))}{1 - K_2 \exp(K_1u)} \int_{-\infty}^{+\infty} \exp\{(1-\gamma)x - (x^2/2\sigma^2)\}(1 + \frac{P}{D}(u_0 + \phi u + x))dx - \frac{K_0}{\sqrt{2\pi}\sigma} \frac{\exp((K_1(1+\phi)u))}{1 - K_2 \exp(K_1u)} \int_{-\infty}^{+\infty} \exp\{(1-\gamma+K_1)x - (x^2/2\sigma^2)\}(1 + \frac{P}{D}(u_0 + \phi u + x))dx.
\]

\(^{23}\)All the assumptions of Blume, Easley and O’Hara [1982] are satisfied.
We rewrite this equation as follows:

$$\frac{P}{D}(u) = \frac{K_0}{\sqrt{2\pi} \sigma} \frac{\exp(K_1u)}{1 - K_2 \exp(K_1u)} [I_1(u) + I_2(u)]$$  \hspace{1cm} (46)$$

where:

$$I_1(u) = \int_{-\infty}^{+\infty} \exp\{((1 - \gamma)x - (x^2/2\sigma^2))(1 - \exp((K_1\phi u))K_3 \exp(P_0) \exp(K_1x))\} dx,$$  \hspace{1cm} (47)$$

and:

$$I_2(u) = \int_{-\infty}^{+\infty} \exp\{(1 - \gamma)x - (x^2/2\sigma^2)\}(1 - \exp(K_1\phi u)K_3 \exp(P_0) \exp(K_1x))Q(u_0 + \phi u + x) dx.$$  \hspace{1cm} (48)$$

It can be show that solving for $I_2(u)$ is identical to the Mehra and Prescott case, see the appendix. We can solve for $I_1(u)$ explicitly as:

$$I_1(u) = \sqrt{2\pi} \sigma \exp(P_0)(1 - K_3 \exp(K_1\phi u + Q_0))$$  \hspace{1cm} (49)$$

where we define $P_0 \equiv \frac{\sigma^2}{2}(\gamma - 1)^2$, and $Q_0 \equiv \frac{\sigma^2}{2}(1 - \gamma + K_1)^2$.\footnote{We can do this since we know the expected value of a lognormally distributed random variable.} Following a similar methodology as in the Mehra and Prescott case, we can solve for the $b_i$'s as:

$$b_0 = K_0 \left( \exp(P_0) - K_2 \exp(K_1(u_0 + \phi \bar{u}) + Q_0) \frac{1}{0!} \right) +$$  \hspace{1cm} (50)$$

$$\frac{K_0}{\sqrt{\pi}} \exp(Q_0) \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \frac{1}{0!} w_{i,k}(\bar{u})a_i \right)$$

and:

$$b_l = -K_0 K_2 \exp(K_1(u_0 + \phi \bar{u}) + Q_0)(K_1\phi)^l +$$  \hspace{1cm} (51)$$

$$\frac{K_0}{\sqrt{\pi}} \exp(Q_0) \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \frac{1}{l!} w_{i,k}(\bar{u})a_i \right)$$
for \( l = 1, \ldots, n \). We are further able to show that for some range of convergence for \( u \in [a, b] \) the bound of the remainder term is given by:

\[
|R_n| \leq \frac{\nu_{n+1} u_{\max}^{n+1}}{(n+1)!}
\]

where \( \nu_{n+1} \equiv \sup_u \left\{ \left( (1 - K_2 \exp(K_1 u)) \exp(-K_1 u) \right)^{(n+1)} \right\} \) and \( u_{\max} \equiv \max \{ |a|, |b| \} \).

We also have

\[
\nu_n = \frac{-K_0 B_0 K_2 (K_1 \phi)^n + K_0 B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} \nu^k \phi^k}{1 - K_0 \phi^n B_0}.
\]

Here:

\[
\nu_0 \equiv \frac{\exp(P_0) - K_2 B_0}{1 - K_0 B_0}
\]

and:

\[
B_0 \equiv \exp(K_1 [u_0 + \phi u] + Q_0).
\]

We show how to obtain this bound on the remainder term and the proof of analyticity in the appendix. We require that \(|K_0 B_0| < 1\) for the existence of an analytic function.\(^{25}\)

### 4.1 Numerical Application

We parameterize this model the same as we find in Abel [1990]. Abel solves for the case where \( \phi = 0 \). With this parameterization we, as well as Abel, obtain closed form solutions when \( \alpha = 0 \) and when \( \alpha = 1 \) and \( D = 0 \) and so in our solution method, only \( b_0 \) needs to be solved for and there is a zero remainder term. Consequently we obtain the same results for the cases where \( \alpha = 0 \) and in the case where \( \alpha = 1 \) and \( D = 0 \) as in Abel [1990]. We report these results in Table 5 in the appendix.\(^{26}\) The case where Abel does not obtain a closed form solution is when \( \alpha = D = 1 \). He uses a two point distribution to obtain his numerical results. Note that he does not get closed forms when \( \phi \neq 0 \) so we will investigate these cases as well.

\(^{25}\)This condition is identical to the discounting property in Blackwell’s Theorem.

\(^{26}\)Abel reports results for both a two point distribution and a lognormal distribution.
When solving for the habit persistence case, $\alpha = 1$ and $D = 1$, Abel’s model has one unfortunate property which is that for certain values of consumption growth, marginal utility is negative. The way to get around this is to choose values of $\gamma$ so that marginal utility is always positive. Abel does this by choosing a two state i.i.d. process and appropriate values of $\gamma$. The parameter values associated with $\sigma$ and $\bar{u}$ in his parameterization are $\sigma = 0.0354$ and $\bar{u} = 0.0172$. In our final solution, we cannot allow an unbounded support for the growth rate since this will mean marginal utility will be negative for some of those values. A sufficient condition in order for the marginal utility to always be positive is:

$$1 + \frac{\ln(\beta)}{\max(u) - \min(u)} \leq \gamma \leq 1 + \frac{\ln(\beta)}{\min(u) - \max(u)}$$

(54)

Here is one instance where a natural bound on the support of the distribution for the random variable, dividend growth, arises from the model so that marginal utility is never negative.\(^{27}\)

We solve for the model analytically with a normal distribution for the shock term and for the numerical application, we bound the support of the distribution.\(^{28}\) Once we bound the support of the distribution, we then can see what values of $\gamma$ are admissible. It is well known that the lognormal distribution is a good approximation to the two state i.i.d. process. Thus given the standard deviation, $\sigma$, and the mean growth, $\bar{u}$, once we decide where to bound the support of the distribution, say three standard deviations, then we can obtain the admissible values of $\gamma$. Bounded at three standard deviations, we have $\max(u) = 0.1234$ and $\min(u) = -0.089$. With these values we obtain $0.953 \leq \gamma \leq 1.047$. Abel obtains returns with $\gamma = 0.86, 0.94, 1.00, 1.06, 1.14$. He does so by choosing the two point process such that the sufficient condition is satisfied. The only solution we are able to check with our lognormal process with the support bounded by three standard deviations is $\gamma = 1$. This points out the value of obtaining the solution in this paper. One can see exactly how the distribution is bounded and what the solution is. We are unable to check the Abel results with\(^{27}\)If we use the quadrature procedure we would choose as the bounds values of the random variable so that marginal utility is always positive, see Judd [1998] p. 258.

\(^{28}\)We do this by choosing over what values of the random variable we integrate. Once that is chosen, we integrate using the Nintegrate command in Mathematica which is a numerical integration package.
a lognormal distribution and bounding the support of the distribution with three standard deviations. Consequently one may question the numerical results in Abel with $\gamma \neq 1$. The numerical answers obtained by Abel are correct, but one may question what their meaning is especially if one is thinking of the two state i.i.d. process as an approximation to the lognormal distribution.

We solve for the price-dividend ratio and then integrate to obtain the expected unconditional return as in equation (7). Note that with $\gamma = 1$ we can integrate over the entire support of the distribution and this will give us a good idea of how many standard deviations we would like to integrate over to obtain an accurate solution. When we integrate over three standard deviations we find the unconditional expected return to stocks and one period bonds is 2.27% and 2.42% respectively. We find that if we integrate over 5 standard deviations for the error term we match our results with those of Abel with $\gamma = 1$, that is the unconditional expected return to stocks and one period bonds is 2.83% and 2.70% respectively. When we integrate over 4 standard deviations we obtain 2.82% and 2.69% respectively. When we integrate over 5 standard deviations, we can only obtain unconditional expected returns for $0.972 < \gamma < 1.028$ so that marginal utility is greater than zero for all values of the distribution. We next change the parameterization in order to allow for a larger interval of values for $\gamma$. To do this we set $\beta = .95$ then with the same values of $\bar{u}$ and $\sigma$ we obtain $0.86 \leq \gamma \leq 1.14$. We know from the equity premium literature that a lower $\beta$ will make it more difficult to match the risk free interest rate but it is an interesting exercise. We also report the results of Abel where $\beta = .99$. The results reported for Abel in the first two panels uses the lognormal while the last panel with Habit formation uses his two point distribution.

Broadly speaking, when comparing Abel’s results to ours, in Table 5, $\beta = .99$ versus $\beta = .95$, the equity premia are similar for the time separable case and the relative consumption case. The big difference is the level of the returns which is driven by the value of $\beta$. This is not the case for the last panel with habit formation. The equity premia generated in Abel [1990] are far larger in general than they are in our exercise here. Thus a benefit from obtaining a solution like the one here is we can see exactly the implication of the consumption/dividend process on
equity premia. Our equity premia are similar with $\gamma = 1$ where we are free to integrate over the entire support of the distribution. We obtain almost the exact equity premium in that case. In all the other cases, the equity premium is far larger in the Abel [1990] model than it is in our solution. This points out the value of solving out the model explicitly with particular distributions. It also shows how these bounds on solving out the model explicitly with particular distributions. It also shows how these bounds on solving out the model explicitly with particular distributions. If one were to solve for this using Gauss Hermite quadrature procedures, they would get a much different answer with the normal distribution.

Below we investigate this same model with $\phi = -.14$, $\beta = .95$, $\gamma = 2.5$, $\alpha = 1$, $D = 0$ and $\bar{u} = .017$, the parameterization of Lucas [1994] with the exception that we will be looking at the relative consumption model, $\alpha = 1$ and $D = 0$. With this parameterization, we find that the supremum of the changes in the coefficients is machine zero, that is it is smaller than $1 \times 10^{-16}$, when we include 10 coefficients, however the remainder is machine zero after we include eleven coefficients and thus we report the results from the analytic solution with eleven coefficients.

Once again when we solve for the model using the Gaussian quadrature procedure with five standard deviations, we obtain the same solution at the steady state. One interesting result is that with four standard deviations, all the moments are matched with the analytic solution except for the price-dividend ratio. In particular, from the error in this procedure, the Euler equation residual divided by the pride-dividend ratio, one would conclude that the solution is very accurate with a four standard deviation bound however, in terms of pricing, the solution is wrong by three cents per dollar of dividend.

Below is the graphs of the $P_D(u)$ function with 1 through 4 coefficients with a range of convergence of .02. One will notice that whether we use three or four coefficients, the solution is the same within the range of convergence of .02.

[Insert Figure 3]

Below we graph three curves, the price-dividend ratio from the numerical solution with a four standard deviation bound, five standard deviation bound and the analytic solution. The

\[29\text{We do not analyze the internal habit persistence case since we have the problem of negative marginal utility.}\]
Analytic solution and the numerical solution with a five standard deviation bound overlap. The graph of the numerical solution with a four standard deviation bound lies below the other two. Once again, the plots of the numerical solution jump upward until they reach the truth. We find that with four standard deviations the error in the price dividend ratio is three cents. It remains at three cents far away from the steady state, \( u = .5 \), and with five standard deviations the two solutions are the same far from the steady state. This numerical procedure does very well in obtaining this solution.

[Insert Figure 4]

### 5 Option Prices

To show how one can take this methodology and apply it to pricing other securities, we will price an European Call option with both the Mehra and Prescott model and the Abel relative consumption model. The motivation for this section is to show how one can use this solution in order to price additional securities. We do not take this section to be a serious study of option pricing but only to offer an example of the versatility of the methodology. One place where a similar empirical exercise is conducted is Engle and Rosenberg [2002] where they estimate the pricing kernel empirically using the S&P 500 index returns and the S&P 500 index option data. They come up with empirical pricing kernels given the data. Using our methodology, once one has what they believe is an appropriate pricing kernel, one can then check with our methodology whether it will price other securities appropriately given the theoretical distribution of the return to the security the researcher wishes to price.

An area that has received enormous attention in asset pricing is option pricing, see for example Bakshi, Cao and Chen [1997] for an empirical comparison of alternative option pricing models.\(^{30}\) Though the models they compare are solved in continuous time, one can always solve a discrete time equilibrium version of them.\(^{31}\) The three general assumptions made in the models compared in Bakshi, Cao and Chen [1997] are: i) the underlying price process, ii)\(^{30}\) They compare the performance of several alternative models, such as stochastic volatility, stochastic interest rate, jump-diffusion models and mixtures of these models.

\(^{31}\)For an example of discrete time option pricing see Boyle and Vorst [1992].
the interest rate process and iii) the market price of factor risk. In our models, we obtain the price process, and the interest rate process from equilibrium and the market price of risk from the pricing kernel. One result we obtain from our model is that when the growth process for consumption and dividend is i.i.d., it turns out that the ex-dividend price grows at that same i.i.d. rate. This means that prices, dividend and consumption all have a lognormal distribution. Consequently, we obtain the Black Scholes [1973] pricing formula since dividend, consumption and the ex-dividend stock index price are all lognormal and the utility function is constant relative risk aversion.\textsuperscript{32} In the Abel model, of which the Mehra and Prescott model is nested within, the formula is as follows:

\[
Ca(t) = E_t \left[ M_{t+1} \max \left[ \frac{P_{t+1}}{D_{t+1}} D_{t+1} - K, 0 \right] \right]
\]  
(55)

where \( K \) is the strike price of the option. We know the functional form of the pricing kernel, \( M_{t+1} \), is:

\[
M_{t+1} = \beta \frac{u'(D_{t+1}, v_{t+1})}{u'(D_t, v_t)} = \beta \left( \frac{D_{t+1}}{D_t} \right)^{-\gamma} \left( \frac{v_{t+1}}{v_t} \right)^{\gamma-1} \frac{H_{t+2}}{E_t(H_{t+1})}.
\]  
(56)

We can write the call pricing equation as:

\[
C_a(t) = \frac{1}{\sqrt{2\pi}\sigma} \beta G^{-\gamma} \exp(-\gamma\phi + \alpha(\gamma - 1)u) \left[ \frac{P}{D} (u_0 + \phi u + x) D_t \exp(u_0 + \phi u + x) - K, 0 \right] \exp(\frac{\sigma^2}{2}) dx
\]  
(57)

\[
\int_{-\infty}^{+\infty} \exp(-\gamma x - (x^2/(2\sigma^2))) \left[ \frac{P}{D} (u_0 + \phi u + x) D_t \exp(u_0 + \phi u + x) - K, 0 \right] dx
\]

Since we will solve for the case where \( \phi = 0 \) and \( D = 0 \), we can rewrite equation (57) as:

\textsuperscript{32}It is well known that when consumption and the price process are lognormal and the utility function is CRRA, then the Black-Scholes formula obtains, see for example Huang and Litzenberger [1988]. Furthermore, when going to the relative consumption model, the Black-Scholes equation will still obtain since the utility function will still be of the power form, see Gali [1994] and Campbell [2001].
\[ C_a(t) = \frac{1}{\sqrt{2\pi}\sigma} \beta G^{-\gamma} \exp(\alpha(\gamma - 1)u) \exp\left(\frac{\sigma^2}{2\gamma}\right) \] (58)

\[ \int_{-\infty}^{+\infty} \exp(-\gamma x - (x^2/(2\sigma^2))) \max\left[\frac{P}{D}(u_0 + x)D_t \exp(u_0 + x) - K, 0\right] dx \]

We will solve for the option prices using data and statistics reported by Bakshi, Cao and Chen [1997]. They collect data on the S&P 500 European call options from June 1988 to May 1991. We report some of their statistics in Table 7. We report the prices for the S&P 500 call options with different maturities. Since we obtain the Black-Scholes equation, we will use their calculated implied volatilities from the Black-Scholes model, input them into these models, Mehra Prescott and relative consumption, and see how close our option prices are to the actual ones in the data. We do this since we know the models we solved for in this paper will not price options correctly and one way to make pricing more appropriate is to increase the volatility in the pricing kernel which using the implied volatility does. We calculate the option price given today’s stock price index is $1 and we then multiply this by the average S&P 500 index price over the months June 1988 to May 1991.\(^{33}\) This should give us a rough estimate of how well these models can price these call options. Note that by using the implied volatilities, we are using a far higher volatility than that of consumption growth so that we should price better than in the equity premium exercises. Here we just do this as an example and do not take this exercise as a serious study in option pricing. It is only meant to be instructive.

In Table 8 we report the call option prices we obtain when we input the implied volatilities reported in the third panel of Table 8.\(^{34}\) From Tables 7 and 8 we see that the Mehra and Prescott model tends to underprice the call option however, the relative consumption model does a pretty good job of pricing. This is what we would expect since the relative consumption

\(^{33}\)We obtain this daily data from the Security Price Index Record of Standard and Poor’s and use the monthly averages of the S&P 500 index to get an average index price over this time period. The average price over these three years is $321.31.

\(^{34}\)Note for the relative consumption model, the implied volatility we used is \(\frac{\sigma}{\gamma}\), where \(\sigma\) is the reported implied volatility, since in that model, this would give us the appropriate implied volatility to the underlying asset.
model prices better in general than the Mehra-Prescott model.\textsuperscript{35} Thus once one finds a model that prices correctly, one can then go on to try and price other securities appropriately. Note that one would input the pricing kernel into the model and then the price process for the underlying asset and obtain the price from the method above.

6 Discussion

There are some main points to take from this solution methodology we develop in this paper along with how to use this methodology to solve out more general problems in the future. First, if an analytic solution exists, then it will always be more accurate than numerical solutions. This can be significant in some cases, one example above is the Abel relative consumption model solved with the Gaussian quadrature procedure with a four standard deviations bound on the random variable. We observed that the price was off by 3 cents per dollar of dividend, which is substantial in asset pricing. Consequently it is our opinion that when an analytic solution exists, it should always be the preferred solution method since it is free from any judgement that one has to make in terms of the accuracy of the solution. However, there are instances when a solution may not be analytic or the proof of analyticity may be very involved and a researcher may not wish to go through the entire process of the proof.

Our suggestion in these cases is the following. For any continuously differentiable policy function and random variable for which the policy function is a function of dividend growth, if the first \( k \) derivatives exist, and the first \( k \) moments of the distribution exist, then one can solve for the coefficients \( b_0, b_1, \ldots, b_{k-1} \) and from the mathematical properties of the Euler equation, one can estimate a bound on the remainder term, \( R_k \). Thus one can simply solve for the coefficients and the remainder term and then use the measure \( \max[\sup_i |b_{n_1-1,i} - b_{n,i}|, R_n] < \epsilon \) as a stopping rule. If this measure is not satisfied with the first \( k \) coefficients, then it may be that the policy function cannot be approximated well with this method.

\textsuperscript{35}One should not take this exercise as a serious option pricing study since we have inputted the implied volatility in the model to obtain an appropriate implied volatility of the underlying asset. This implies that the volatility of dividend growth is higher than we find in the data and the interest rate is no longer the same as in the earlier sections in the paper since it is priced with this new volatility. We still satisfy the necessary condition for existence with this new volatility. With this higher volatility, the pricing of bonds and stocks changes.
One can program this procedure and have a computer solve for this without going through all the algebra as we have in this paper. Mathematica and Maple are all equipped to take analytic derivatives. In general one can input the IMRS and a guess for the functional form of the policy function \( P \). Once that is done, the computer can then solve for the Euler equation with the guess of the policy function and equate this solution to the guess of the policy function. At that point the coefficients, \( b_i \) for \( i = 0, \ldots, n \) are solved for. One then reports the supremum of the changes in the coefficients as one adds more coefficients. One can also write a program to find the bound on the remainder term in a similar fashion.

One numerical method that is similar to this method is the perturbation method used in Julliard and Collard [2002] and by Judd and Goo [2002]. The basic idea of the perturbation method is to start with the Euler equation, take an \( n^{th} \) order Taylor expansion of the policy function on the right hand side of the Euler equation. Then take the derivatives of the Euler equation until one has a system of \( n \) equations, similar to what we have. Then one solves for the coefficients in the Taylor expansion similar to ours. One difference is that in the perturbation method, they do not obtain a closed form solution as we do and consequently, they cannot obtain a bound on the remainder term. Julliard and Collard [2002] show how the perturbation method with a fourth order Taylor expansion does not do a good job of obtaining the true solution away from the steady state as \( \phi \) approaches one. We can answer why this is true from our methods. The reason is if \( \phi = 1 \) then the policy function is no longer analytic and thus as \( \phi \) approaches one, it takes more and more coefficients for the remainder to become machine zero.

7 Conclusion

In this paper we constructed a new methodology that solves Euler equations when the policy function is analytic. The attractiveness of this methodology over some of the numerical procedures is that we can obtain a bound on the remainder term where numerical methods cannot.

\[\text{Their Taylor expansion differs in that they take it around a known solution such as the case where } \sigma = 0. \text{ We actually obtain the global solution.}\]
There is very little judgment necessary in order to conclude whether the solution is accurate or not. We saw in the asset pricing models we solved that the solution generated by our method is very accurate and was matched by the numerical method only after enough standard deviations were included for the random variable.

One direction for future research is to extend this method to multidimensional cases. There are several problems of interest in multidimensional cases such as portfolio choice over many asset. Jin and Judd [2002] does this for the perturbation method and it would be easy to extend this methodology to those cases. Another important problem is to prove or disprove our conjecture that if the intertemporal marginal rate of substitution is analytic, the dividend process is analytic, and all the moments of the stochastic process exist, then the policy function is analytic. Such a theorem would be very useful to applied work with Euler equations.
### Table 1: Coefficients for Mehra Prescott Case

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0$</td>
<td>12.680</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$-3.06 \times 10^{-1}$</td>
</tr>
<tr>
<td>$b_2$</td>
<td>$3.99 \times 10^{-3}$</td>
</tr>
<tr>
<td>$b_3$</td>
<td>$-3.47 \times 10^{-5}$</td>
</tr>
<tr>
<td>$b_4$</td>
<td>$2.27 \times 10^{-7}$</td>
</tr>
<tr>
<td>$b_5$</td>
<td>$-1.19 \times 10^{-9}$</td>
</tr>
<tr>
<td>$b_6$</td>
<td>$5.19 \times 10^{-12}$</td>
</tr>
<tr>
<td>$b_7$</td>
<td>$-1.95 \times 10^{-14}$</td>
</tr>
<tr>
<td>$b_8$</td>
<td>$6.43 \times 10^{-17}$</td>
</tr>
</tbody>
</table>

*Notes:* Each coefficient $b_i$ multiplies $(u - \bar{u})^i$ in the power series, $\frac{u}{\bar{u}}(u) = \exp(Ku)(b_0 + b_1(u - \bar{u}) + \ldots + b_8(u - \bar{u})^8$. The parameterization used is $\sigma = 0.036$, $\gamma = 2.5$, $\beta = 0.95$, $\bar{u} = 0.017$, and $\phi = -0.14$.

### Table 2: Bound on Remainder for Mehra Prescott Case

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>$r_c = .02$</th>
<th>$r_c = .5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-6.11 \times 10^{-3}$</td>
<td>$-1.28 \times 10^{-1}$</td>
</tr>
<tr>
<td>2</td>
<td>$1.59 \times 10^{-6}$</td>
<td>$8.35 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>$-2.77 \times 10^{-10}$</td>
<td>$-3.64 \times 10^{-6}$</td>
</tr>
<tr>
<td>4</td>
<td>$3.62 \times 10^{-14}$</td>
<td>$1.19 \times 10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>$-3.79 \times 10^{-18}$</td>
<td>$-3.13 \times 10^{-11}$</td>
</tr>
<tr>
<td>6</td>
<td>$3.32 \times 10^{-22}$</td>
<td>$6.87 \times 10^{-14}$</td>
</tr>
<tr>
<td>7</td>
<td>$-2.49 \times 10^{-26}$</td>
<td>$-1.29 \times 10^{-16}$</td>
</tr>
<tr>
<td>8</td>
<td>$1.64 \times 10^{-30}$</td>
<td>$2.14 \times 10^{-19}$</td>
</tr>
<tr>
<td>9</td>
<td>$-9.65 \times 10^{-35}$</td>
<td>$-3.15 \times 10^{-22}$</td>
</tr>
<tr>
<td>10</td>
<td>$5.12 \times 10^{-39}$</td>
<td>$4.20 \times 10^{-25}$</td>
</tr>
</tbody>
</table>

*Notes:* The range of convergence tells us the domain over which we are analyzing the price-dividend function. With a range of convergence, $r_c$, of .02 and steady state growth of .17, the domain of the function is $[0.15, 0.19]$. Machine zero, a value less than $10^{-16}$, implies according to the remainder term that we should use five coefficients with $r_c = .02$ and eight coefficients with $r_c = .5$. The parameterization used is $\sigma = 0.036$, $\gamma = 2.5$, $\beta = 0.95$, $\bar{u} = 0.017$, and $\phi = -0.14$.
Table 3: Comparison of Results

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Lucas [1994]</th>
<th>Analytic Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Nine Coefficients</td>
</tr>
<tr>
<td>$E(R_s)$</td>
<td>0.097</td>
<td>0.098</td>
</tr>
<tr>
<td>$\sigma(R_s)$</td>
<td>0.047</td>
<td>0.046</td>
</tr>
<tr>
<td>$E(R_B)$</td>
<td>0.095</td>
<td>0.094</td>
</tr>
<tr>
<td>$\sigma(R_B)$</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>$E(R_s - R_B)$</td>
<td>0.003</td>
<td>0.004</td>
</tr>
<tr>
<td>$\sigma(R_s - R_B)$</td>
<td>0.045</td>
<td>0.044</td>
</tr>
<tr>
<td>$E(C_{t+1})/C_t$</td>
<td>0.017</td>
<td>0.017</td>
</tr>
<tr>
<td>$\sigma((C_{t+1})/C_t)</td>
<td>0.036</td>
<td>0.036</td>
</tr>
</tbody>
</table>

Notes: $R_s$ is the return to the stock and $R_B$ is the return to the bond. $C_t$ is per capita consumption at time $t$ and $P_D$ is the price dividend ratio. $E$ represents an expectation and $\sigma$ is a standard deviation. $P_D(0.17)$ is the value of the price dividend ratio at the steady state. All the evaluations of the statistics with the analytic solution are at steady state consumption growth, 0.017. The parameter values used are $\beta = 0.95$, $\sigma = 0.36$, $\gamma = 2.5$, $\bar{u} = 0.017$, and $\phi = -0.14$. We obtain these moments by using the price-dividend ratio and the price of the bond, convert them to returns and then integrate to obtain the expected returns and standard deviations.

Table 4: Comparison of Results

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Analytic Solution</th>
<th>Gaussian</th>
<th>Gaussian</th>
<th>Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nine Coefficients</td>
<td>Three Std.</td>
<td>Four Std.</td>
<td>Five Std.</td>
</tr>
<tr>
<td>$E(R_s)$</td>
<td>0.098</td>
<td>0.098</td>
<td>0.098</td>
<td>0.098</td>
</tr>
<tr>
<td>$\sigma(R_s)$</td>
<td>0.046</td>
<td>0.052</td>
<td>0.046</td>
<td>0.046</td>
</tr>
<tr>
<td>$E(R_B)$</td>
<td>0.094</td>
<td>0.094</td>
<td>0.094</td>
<td>0.094</td>
</tr>
<tr>
<td>$\sigma(R_B)$</td>
<td>0.014</td>
<td>0.014</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>$E(R_s - R_B)$</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$\sigma(R_s - R_B)$</td>
<td>0.044</td>
<td>0.044</td>
<td>0.044</td>
<td>0.044</td>
</tr>
<tr>
<td>$E(C_{t+1})/C_t$</td>
<td>0.017</td>
<td>0.017</td>
<td>0.017</td>
<td>0.017</td>
</tr>
<tr>
<td>$\sigma((C_{t+1})/C_t)</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
</tr>
<tr>
<td>$P_D(0.17)$</td>
<td>12.73</td>
<td>12.26</td>
<td>12.72</td>
<td>12.73</td>
</tr>
</tbody>
</table>

Notes: $R_s$ is the return to the stock and $R_B$ is the return to the bond. $C_t$ is per capita consumption at time $t$ and $P_D$ is the price dividend ratio. $E$ represents an expectation and $\sigma$ is a standard deviation. $P_D(0.17)$ is the value of the price dividend ratio at the steady state. All the evaluations of the statistics with the analytic solution are at steady state consumption growth, 0.017. The parameter values used are $\beta = 0.95$, $\sigma = 0.36$, $\gamma = 2.5$, $\bar{u} = 0.017$, and $\phi = -0.14$. We obtain these moments by using the price-dividend ratio and the price of the bond, convert them to returns and then integrate to obtain the expected returns and standard deviations.
Table 5: Unconditional Expected Returns

<table>
<thead>
<tr>
<th>γ</th>
<th>Stock</th>
<th>Bond</th>
<th>Stock</th>
<th>Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Separable Preferences, α = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>6.22</td>
<td>6.16</td>
<td>1.93</td>
<td>1.87</td>
</tr>
<tr>
<td>1.0</td>
<td>7.16</td>
<td>7.02</td>
<td>2.83</td>
<td>2.70</td>
</tr>
<tr>
<td>6.0</td>
<td>14.98</td>
<td>14.12</td>
<td>10.33</td>
<td>9.51</td>
</tr>
<tr>
<td>10.0</td>
<td>18.94</td>
<td>17.46</td>
<td>14.13</td>
<td>12.72</td>
</tr>
<tr>
<td>Relative Consumption, α = 1, D = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>7.13</td>
<td>7.09</td>
<td>2.80</td>
<td>2.76</td>
</tr>
<tr>
<td>1.0</td>
<td>7.16</td>
<td>7.02</td>
<td>2.83</td>
<td>2.70</td>
</tr>
<tr>
<td>6.0</td>
<td>11.05</td>
<td>6.36</td>
<td>6.72</td>
<td>2.06</td>
</tr>
<tr>
<td>10.0</td>
<td>19.28</td>
<td>5.83</td>
<td>14.95</td>
<td>1.55</td>
</tr>
<tr>
<td>Habit Formation, α = 1, D = 1</td>
<td></td>
<td></td>
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<tr>
<td>0.86</td>
<td>7.82</td>
<td>7.4</td>
<td>33.56</td>
<td>4.53</td>
</tr>
<tr>
<td>0.94</td>
<td>7.19</td>
<td>7.19</td>
<td>6.83</td>
<td>3.48</td>
</tr>
<tr>
<td>1.00</td>
<td>7.16</td>
<td>7.02</td>
<td>2.83</td>
<td>2.70</td>
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<tr>
<td>1.06</td>
<td>7.50</td>
<td>6.86</td>
<td>8.43</td>
<td>1.93</td>
</tr>
<tr>
<td>1.14</td>
<td>8.55</td>
<td>6.65</td>
<td>38.28</td>
<td>0.93</td>
</tr>
</tbody>
</table>

Notes: The columns labeled Abel with β = .99 are from Abel [1990]. For the Abel [1990] model with α = 0 and α = 1, D = 0, we report his numbers with the lognormal distribution. For the case with α = 1, D = 1, we report his numbers with the two point distribution since these are the numbers he reports. We solve this with β = .95 using the lognormal distribution for all cases. In the habit formation case, we bound the distribution with five standard deviations and solve for the returns numerically using the numerical integration command in mathematica, Nintegrate. The parameterization used is σ = 0.0354, γ = 2.5, β = 0.95, u = 0.0172, and φ = 0.
Table 6: Comparison of Results, Relative Consumption. $\alpha = 1$, $D = 0$

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Analytic Solution</th>
<th>Gaussian Eleven Coefficients</th>
<th>Gaussian Three Std.</th>
<th>Gaussian Four Std.</th>
<th>Gaussian Five Std.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(R_s)$</td>
<td>0.075</td>
<td>0.075</td>
<td>0.075</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_s)$</td>
<td>0.094</td>
<td>0.091</td>
<td>0.094</td>
<td>0.094</td>
<td></td>
</tr>
<tr>
<td>$E(R_B)$</td>
<td>0.069</td>
<td>0.068</td>
<td>0.069</td>
<td>0.069</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_B)$</td>
<td>0.071</td>
<td>0.070</td>
<td>0.071</td>
<td>0.071</td>
<td></td>
</tr>
<tr>
<td>$E(R_s - R_B)$</td>
<td>0.006</td>
<td>0.007</td>
<td>0.006</td>
<td>0.006</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_s - R_B)$</td>
<td>0.0844</td>
<td>0.0835</td>
<td>0.0844</td>
<td>0.0844</td>
<td></td>
</tr>
<tr>
<td>$E(C_{t+1})/C_t$</td>
<td>0.017</td>
<td>0.017</td>
<td>0.017</td>
<td>0.017</td>
<td></td>
</tr>
<tr>
<td>$\sigma((C_{t+1})/C_t)$</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
<td></td>
</tr>
<tr>
<td>$\frac{P}{D}(0.17)$</td>
<td>19.03</td>
<td>18.05</td>
<td>19.00</td>
<td>19.03</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>$-1.96 \times 10^{-16}$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: $R_s$ is the return to the stock and $R_B$ is the return to the bond. $C_t$ is per capita consumption at time $t$ and $\frac{P}{D}$ is the price dividend ratio. $E$ represents an expectation and $\sigma$ is a standard deviation. $\frac{P}{D}(0.17)$ is the value of the price dividend ratio at the steady state. All the evaluations of the statistics with the analytic solution are at steady state consumption growth, 0.17. The parameter values used are $\beta = .95$, $\sigma = .036$, $\gamma = 2.5$, $\bar{u} = 0.017$, and $\phi = -0.14$. We obtain these moments by using the price-dividend ratio and the price of the bond, convert them to returns and then integrate to obtain the expected returns and standard deviations.
Table 7: Sample Properties of S&P 500 Index Options, Call.
From Bakshi, Cao and Chen [1997].

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Days to Expiration</th>
<th>Days to Expiration</th>
<th>Days to Expiration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P/K$</td>
<td>&lt; 60</td>
<td>60 – 180</td>
<td>≥ 180</td>
</tr>
<tr>
<td>&lt; 0.94</td>
<td>$1.68$</td>
<td>$4.38$</td>
<td>$8.58$</td>
</tr>
<tr>
<td>0.94 – 0.97</td>
<td>$2.35$</td>
<td>$8.02$</td>
<td>$15.12$</td>
</tr>
<tr>
<td>0.97 – 1.00</td>
<td>$4.83$</td>
<td>$12.79$</td>
<td>$20.17$</td>
</tr>
<tr>
<td>1.00 – 1.03</td>
<td>$10.42$</td>
<td>$18.72$</td>
<td>$26.44$</td>
</tr>
<tr>
<td>1.03 – 1.06</td>
<td>$17.77$</td>
<td>$25.52$</td>
<td>$33.00$</td>
</tr>
<tr>
<td>≥ 1.06</td>
<td>$39.40$</td>
<td>$48.06$</td>
<td>$58.12$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P/K$</th>
<th>Implied Volatility.</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 0.94</td>
<td>18.27</td>
</tr>
<tr>
<td>0.94 – 0.97</td>
<td>16.64</td>
</tr>
<tr>
<td>0.97 – 1.00</td>
<td>16.95</td>
</tr>
<tr>
<td>1.00 – 1.03</td>
<td>18.80</td>
</tr>
<tr>
<td>1.03 – 1.06</td>
<td>21.40</td>
</tr>
<tr>
<td>≥ 1.06</td>
<td>28.72</td>
</tr>
</tbody>
</table>

Notes: The statistics reported here are from Bakshi, Cao and Chen [1997]. Moneyness, the price of the underlying asset divided by the strike price, measures how much the option is in or out of the money. The other columns give the average option price with those days to maturity, less than 60 days, between 60 and 180 days and greater than 180 days. The lower panel gives the implied volatilities for the above panel from the Black-Scholes model.
Table 8: European Call Option Prices from the Models.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Days to Expiration</th>
<th>Days to Expiration</th>
<th>Days to Expiration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P/K$</td>
<td>90</td>
<td>180</td>
<td>360</td>
</tr>
<tr>
<td>0.94</td>
<td>$0.49$</td>
<td>$3.40$</td>
<td>$8.99$</td>
</tr>
<tr>
<td>0.97</td>
<td>$1.93$</td>
<td>$5.92$</td>
<td>$12.27$</td>
</tr>
<tr>
<td>1.00</td>
<td>$5.63$</td>
<td>$9.95$</td>
<td>$15.71$</td>
</tr>
<tr>
<td>1.03</td>
<td>$11.55$</td>
<td>$15.07$</td>
<td>$19.92$</td>
</tr>
<tr>
<td>1.06</td>
<td>$18.64$</td>
<td>$20.66$</td>
<td>$24.13$</td>
</tr>
</tbody>
</table>

Mehra-Prescott Model.

| $P/K$     | 90                 | 180                | 360                |
| 0.94      | $2.64$             | $10.04$            | $23.37$            |
| 0.97      | $5.17$             | $13.26$            | $28.02$            |
| 1.00      | $9.59$             | $18.10$            | $32.32$            |
| 1.03      | $15.37$            | $23.69$            | $37.73$            |
| 1.06      | $21.83$            | $29.38$            | $42.98$            |

Abel Relative Consumption Model.

| $P/K$     | 90                 | 180                | 360                |
| 0.94      | 17.25              | 17.25              | 16.58              |
| 0.97      | 16.89              | 16.89              | 17.30              |
| 1.00      | 17.76              | 17.76              | 17.72              |
| 1.03      | 18.95              | 18.95              | 18.83              |
| 1.06      | 20.04              | 20.04              | 19.91              |

Implied Volatility

Notes: These are the option prices from the Mehra and Prescott model and the relative consumption model. We parameterize the model as we did in the paper except we input the implied volatility from the Balck-Scholes model instead of the volatility of consumption growth. The parameterization is $\beta = .95$, $\bar{u} = .17$, $\gamma = 2.5$, $\sigma =$implied volatility, and $\phi = 0$. For the Abel relative consumption case we have $\sigma =$implied volatility/$(\gamma - 1)$ in order for the implied volatility of the underlying asset to be correct.
Appendix

Integration of $\gamma_i$ in the Mehra-Prescott case.

We now calculate the integral in $I_2(u)$:

$$\gamma_i = \int_{-\infty}^{\infty} \exp\{(1 - \gamma + K_1)x - \frac{x^2}{2\sigma^2}\}x^i \, dx. \quad (59)$$

We need to complete the square and use a change of variable, $\sqrt{2}\omega = \frac{x}{\sigma} - \sigma[1 - \gamma + K_1]$, so that:

$$\gamma_i = \exp\{Q_0\sigma\sqrt{2}\int_{-\infty}^{\infty} \exp\{-\omega^2\} \left[\sqrt{2}\sigma\omega + \sigma^2[1 - \gamma + K_1]\right]^i \, d\omega. \quad (60)$$

Next use the binomial theorem again to obtain:

$$(\sqrt{2}\sigma\omega + \sigma^2[1 - \gamma + K_1])^i = \sum_{j=0}^{i} \binom{i}{j} (\sigma^2[1 - \gamma + K_1])^{i-j}(\sqrt{2}\sigma)^j \omega^j. \quad (61)$$

As a result substituting (61) into (60) we obtain:

$$\gamma_i(u) = \exp\{Q_0\sqrt{2}\sigma\sum_{j=0}^{i} \binom{i}{j} (\sigma^2[1 - \gamma + K_1])^{i-j}(\sqrt{2}\sigma)^j \int_{-\infty}^{\infty} \exp\{-\omega^2\}\omega^j \, d\omega. \quad (62)$$

This integral is known to be:

$$\int_{-\infty}^{\infty} \exp\{-\omega^2\}\omega^j \, d\omega = (1 + (-1)^j)(1/2)\Gamma\left[j + \frac{1}{2}\right] \quad (63)$$

which is independent of $u$. We also see that for any distribution in which all the moments exist this integral can be calculated following similar procedures.
The bound on the error term in Mehra-Prescott case.

The general proof of existence of an equilibrium price-dividend function is dependent on a distribution to dividend growth which has a finite support.\textsuperscript{37} In this case we use a normal distribution which has an infinite support. To circumvent this problem we look for a solution of the form, $\chi(u) \equiv \exp\{-K_1 u\} P\{\omega\}$ so that the price dividend function does not grow faster than dividend growth. In this case the second integral is now

$$I_2(u) = \int_{-\infty}^{\infty} \exp\{[1 - \gamma]x - x^2/2\sigma^2\} \frac{P\{\omega\}}{\exp\{K_1\omega\}} \exp\{K_1(u_0 + \phi u + x)\} dx$$  \hspace{1cm} (64)

where we define:

$$\omega(u) \equiv u_0 + \phi u + x.$$  \hspace{1cm} (65)

As a result, we use the following integral in the analysis below:

$$h(u) \equiv \int_{-\infty}^{\infty} \exp\{(1 - \gamma + K_1)x - x^2/2\sigma^2\} \chi(\omega) dx.$$  \hspace{1cm} (66)

We can now write the integral equation (19) using equation(66) as:

$$\chi(u) = \frac{K_0}{\sqrt{2\pi}\sigma}\left(I_1 + I_2\right) = \frac{K_0}{\sqrt{2\pi}\sigma}\left(\sigma\sqrt{2\pi}\exp\{P_0\} + \exp\{K_1(u_0 + \phi u)\}h(u)\right).$$  \hspace{1cm} (67)

We will use this integral equation to establish the bounds on the error term. The bounds are based on the properties of this contraction mapping which will satisfy the monotonicity and discounting properties of Blackwell’s contraction mapping theorem.\textsuperscript{38}

Next we know:

$$\int_{-\infty}^{\infty} \exp\{(1 - \gamma + K_1)x - x^2/2\sigma^2\} dx = \sigma\sqrt{2\pi}\exp\left(\frac{\sigma^2}{2}(1 - \gamma + K_1)^2\right) \equiv \sigma\sqrt{2\pi}\exp\{Q_0\}.$$  \hspace{1cm} (68)

Assume:

\textsuperscript{37}See Altug and Labadie (1994, chapter 2).
\textsuperscript{38}See Lucas and Stokey with Prescott (1989).
\[ |\chi(\omega)| < M_0 \]  

for some constant \( M_0 \). Substitute (69) into (66) to obtain:

\[ h(u) < M_0 \sigma \sqrt{2\pi} \exp\{Q_0\}. \]  

Substituting (70) into (67), the integral equation has a bound:

\[ |\chi(u)| < K_0 \left( \exp\{P_0\} + \exp\{K_1(u_0 + \phi u) + Q_0\} M_0 \right). \]  

Next we let:

\[ A_0 \equiv \exp\{P_0\} \]  

and:

\[ B_0 \equiv \exp\{K_1(u_0 + \phi u) + Q_0\}. \]  

Consequently, \( M_1 \equiv K_0(A_0 + B_0 M_0) \) is a new upper bound. We now define \( M_{n+1} \equiv K_0(A_0 + B_0 M_n) \). As a result a sequence of upper bounds may be expressed as:

\[ M_n = K_0 A_0 (1 + K_0 B_0 + K_0^2 B_0^2 + \ldots + K_0^{n-1} B_0^{n-1}) + K_0^n B_0^n M_0 \]  

\[ = K_0 A_0 \frac{1 - K_0^n B_0^n}{1 - K_0 B_0} + K_0^n B_0^n M_0 \]

Thus, we get:

\[ \nu_0 \equiv \sup_u \{|\chi(u)|\} = \lim_{n \to \infty} M_n = \frac{K_0 A_0}{1 - K_0 B_0} \]  

provided \( |K_0 B_0| < 1 \). This condition assures that the contraction mapping satisfies the discounting property. We may write this condition as:
\[ |\beta G^{-\gamma+1}\exp\{K_1(u_0 + \phi u) + Q_0\}| < 1. \]  (76)

Thus, the range of convergence for \( u \) must be:

\[ u > -\ln(\beta) + (\gamma - 1)\ln(G) - K_1u_0 - \frac{a^2}{2}(1 - \gamma + K_1)^2. \]  (77)

We will need the following derivative in the following analysis:

\[ h^{(n)}(u) = \int_{-\infty}^{\infty} \exp\{(1 - \gamma + K_1)x - x^2/2\sigma^2\} \chi^{(n)}(\omega)\phi^n dx. \]  (78)

We can now look at the bound on the first derivative:

\[ \chi^{(1)} = \frac{K_0}{\sqrt{2\pi}\sigma}(\exp\{K_1(u_0 + \phi u)\}\phi K_1h(u) + \exp\{K_1(u_0 + \phi u)\}h^{(1)}(u)). \]  (79)

As before assume:

\[ |\chi^{(1)}(\omega)| < M_0^1. \]  (80)

Substituting (80) into (78) we obtain:

\[ h^{(1)} < M_0^1\phi\sigma\sqrt{2\pi}\exp\{Q_0\}. \]  (81)

We have already proved that:

\[ h(u) < \nu_0\sigma\sqrt{2\pi}\exp\{Q_0\}. \]  (82)

Substituting (82) and (81) into (79) we obtain:

\[ \chi^{(1)} < K_0\left(\exp\{K_1(u_0 + \phi u) + Q_0\}\phi K_1\nu_0 + \exp\{K_1(u_0 + \phi u) + Q_0\}M_0^1\phi\right). \]  (83)

We now define:

\[ A_0^1 \equiv B_0K_1\phi\nu_0 \]  (84)
and:

\[ B_0^1 \equiv \phi B_0. \]  

(85)

For a bound we need:

\[ |K_0 \phi B_0| < 1 \]  

(86)

which is true given the bound on \( |K_0 B_0| < 1 \). As a result the bound \( \nu_1 \) on \( \chi^{(1)} \) exist. The bound is given by:

\[ \nu_1 \equiv \lim_{n \to \infty} M_n^1 = \frac{K_0 A_0^1}{1 - K_0 B_0^1}. \]  

(87)

We know from Blume Easley and O’Hara [1982] that the policy function has additional derivatives, it is \( C^\infty \). Thus, we may proceed by induction to the \( n^{th} \) derivative. We assume:

\[ |\chi^{(n)}(\omega)| < M_0^n \]  

(88)

and substituting (88) into (78) we obtain:

\[ h^{(n)}(u) < M_0^n \phi^n \sigma \sqrt{2\pi} \exp\{Q_0\}. \]  

(89)

We assume that the first \( n - 1 \) steps have been completed so that:

\[ h^{(n-1)}(u) < \nu_{n-1} \phi^{n-1} \sigma \sqrt{2\pi} \exp\{Q_0\}. \]  

(90)

Next let \( f(u) \equiv \exp\{K_1(u_0 + \phi u)\} h(u) \). By Leibnitz’s rule:

\[ f^{(n)}(u) = \exp\{K_1(u_0 + \phi u)\} \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} h^{(k)}(u) < \]  

\[ \sigma \sqrt{2\pi} \exp\{K_1(u_0 + \phi u) + Q_0\} \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} \nu_k \phi^k + \sigma \sqrt{2\pi} \exp\{K_1(u_0 + \phi u) + Q_0\} M_0^n \phi^n. \]  

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As a result, using equation (67), we have:

\[ \chi^{(n)}(u) < K_0 \left( B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} \nu_k \phi^k + B_0 M_0^n \phi^n \right). \]  

(92)

In this case we define:

\[ A_0^n \equiv B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} \nu_k \phi^k \]  

(93)

and:

\[ B_0^n \equiv \phi^n B_0. \]  

(94)

For a bound we need:

\[ |K_0 \nu^n B_0| < 1, \]  

(95)

which is true for all \( n \). Thus there is a bound to all the derivatives of \( \chi \) as long as limit as \( n \to \infty \), then \( |A_n| \) is bounded. The bound is given by:

\[ \nu_n = \frac{K_0 A_0^n}{1 - K_0 B_0^n} = \frac{K_0 B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1 \phi)^{n-k} \nu_k \phi^k}{1 - K_0 \nu^n B_0}, \]  

(96)

which provides a bound for \( \chi^{(n)}(u) \) for all \( n \).

Remember that

\[ \chi(u) = \exp\{-K_1 u\} P_D(u). \]  

(97)

The Taylor expansion of \( \chi \) around \( u = \bar{u} \) is

\[ \chi(u) = \exp\{-K_1 u\} P_D(u) = b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + \cdots + b_n(u - \bar{u})^n + R_n. \]  

(98)

It is known that:

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\[ |R_n| \leq \frac{\nu_{n+1}(u_{\text{max}} - \bar{u})^{n+1}}{(n + 1)!} \]  

where \( u_{\text{max}} \) is the range of convergence for \( |K_0B_0| < 1 \). That is if we have \( u \in [a, b] \) then \( u_{\text{max}} = \max(|a|, |b|) \). We also have:

\[ \nu_{n+1} = \sup_{|u| \leq u_{\text{max}}} |\chi^{(n+1)}(u)|. \]

We thus have a bound on the error term for an \( n^{th} \) order expansion program.
The price-dividend function is analytic in the Mehra-Prescott case.

To prove that the solution is analytic we must show that there is a $C$ such that for $u \in [a, b]$:

$$|\chi^{(n)}(u)| \leq n!C^n M. \quad (101)$$

See Rudin (1974, pp. 229-230). This condition is called the Cauchy estimate.

We will use a proof by induction. First for $n = 0$:

$$|\chi^{(0)}(u)| \leq 0!C^0 M = M \quad (102)$$

which we know is true for $M = \nu_0$.

Now assume:

$$|\chi^{(k)}(u)| \leq k!C^k M \forall u \in [a, b] \quad (103)$$

for $k = 0, 1, \cdots, n-1$. From the previous section we have bounds on the $n^{th}$ derivative given by:

$$|\chi^{(n)}(u)| < K_0B_0 \left( \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \phi^k k! C^k M + \phi^n \frac{K_0B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \phi^k k! C^k M}{1 - K_0\phi^n B_0} \right) \quad (104)$$

$$< K_0B_0 \left( \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \phi^k C^k + \phi^n \frac{K_0B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \phi^k C^k}{1 - K_0\phi^n B_0} \right) (n-1)! M$$

$$= K_0B_0 \left( [K_1\phi + \phi C]^n - [\phi C]^n + \phi^n \frac{K_0B_0 \left( [K_1\phi + \phi C]^n - [\phi C]^n \right)}{1 - K_0\phi^n B_0} \right) (n-1)! M$$

$$= K_0B_0 \left( \frac{1 - K_0\phi^n B_0}{1 - K_0\phi^n B_0} \left[ [K_1\phi + \phi C]^n - [\phi C]^n \right] + \phi^n \frac{K_0B_0 \left( [K_1\phi + \phi C]^n - [\phi C]^n \right)}{1 - K_0\phi^n B_0} \right) (n-1)! M$$

As a result we need to prove there exist a $C$ such that:
\[
\frac{K_0B_0}{1 - K_0\phi^nB_0} \left[ [K_1\phi + \phi C]^n - [\phi C]^n \right] < nC^n. \tag{105}
\]

For \( n=1 \) we have:

\[
\frac{K_0B_0}{1 - K_0\phi B_0} K_1\phi < C. \tag{106}
\]

So we can make \( C \) big enough to make this true.

We proceed to the \( n^{th} \) case:

\[
K_0B_0 [K_1\phi + \phi C]^n < [n - (n - 1)K_0\phi^nB_0] C^n. \tag{107}
\]

We may write this as:

\[
\left[ \frac{K_0B_0}{n - (n - 1)K_0\phi^nB_0} \right]^\frac{1}{n} \left[ \frac{K_1\phi + \phi C}{n} \right] < C. \tag{108}
\]

As a result, we have to choose \( C \) such that:

\[
\frac{[K_0B_0]^\frac{1}{n} K_1\phi}{[n(1 - K_0\phi^nB_0) + K_0\phi^nB_0]^\frac{1}{n} - \phi [K_0B_0]^\frac{1}{n}} < C. \tag{109}
\]

Now note that the denominator is positive as long as:

\[
n [1 - K_0\phi^nB_0] > 0. \tag{110}
\]

which is true for \( u \) within the range of convergence. We also have \( K_1 \leq 0 \) as long as \( \gamma \geq 1 \). As long as \( x \) is positive, \( x^{\frac{1}{n}} \to 1 \) as \( n \to \infty \).\(^{39}\) Consequently, we have \( n^{\frac{1}{n}} \to 1 \) as \( n \to \infty \). As a result, we choose \( C > \frac{|K_1\phi|}{[1 - \phi]} \) to assure the Cauchy estimate. Thus, there exist a \( C \) such that the Cauchy Estimate is satisfied so that the \( \mathcal{F}_D(u) \) function is analytic for \( u \in [a, b] \).

\(^{39}\)See Scott and Tims (1966, pp. 94-95).
The integral equation in the Abel case.

Since the logic to deriving these bounds is the same as the Mehra and Prescott case, we will not go through every step as in the Mehra and Prescott case. The integral equation (45) in the Abel case may be written as:

\[
P_D(u) = \frac{K_0}{\sqrt{2\pi\sigma}} \frac{\exp(K_1u)}{1 - K_2\exp(K_1u)} \int_{-\infty}^{+\infty} \exp\{(1-\gamma)x - (x^2/2\sigma^2)\}(1 + \frac{P_D(u_0+\phi u+x)}{P_D})dx - (111)
\]

\[
\frac{K_0K_2\exp(K_1(1+u_0+\phi)u)}{\sqrt{2\pi\sigma}} \frac{\exp(K_1u)}{1 - K_2\exp(K_1u)} \int_{-\infty}^{+\infty} \exp\{(1-\gamma+K_1)x - (x^2/2\sigma^2)\}(1 + \frac{P_D(u_0+\phi u+x)}{P_D})dx
\]

We can split this up into two integrals as follows:

\[
\frac{P_D(u)}{P_D} = \frac{K_0}{\sqrt{2\pi\sigma}} \frac{\exp(K_1u)}{1 - K_2\exp(K_1u)} [I_1(u) + I_2(u)] \quad (112)
\]

where:

\[
I_1(u) = \int_{-\infty}^{+\infty} \exp\{(1-\gamma)x - (x^2/2\sigma^2)\}(1 - K_2\exp\{K_1(u_0+\phi u+x)\})dx \quad (113)
\]

and:

\[
I_2(u) = \int_{-\infty}^{+\infty} \exp\{(1-\gamma)x - (x^2/2\sigma^2)\}(1 - K_2\exp\{K_1(u_0+\phi u+x)\}) \frac{P_D(u_0+\phi u+x)}{P_D} dx. \quad (114)
\]

For \(I_1(u)\) we know the solution to be:

\[
I_1(u) = \sqrt{2\pi\sigma} (\exp\{P_0\} - K_2\exp\{K_1(u_0+\phi u) + Q_0\}) \quad (115)
\]

where \(P_0\) and \(Q_0\) are identical to the Mehra-Prescott case except that the new definition of \(K_1\) applies.

For \(I_2(u)\) we have:
\[ I_2(u) = \int_{-\infty}^{+\infty} \exp\{(1 - \gamma)x - (x^2/2\sigma^2)\} (1 - K_2 \exp\{K_1(u_0 + \phi u + x)\}) \frac{P}{D}(u_0 + \phi u + x)dx. \] (116)

We assume a functional form of \( \frac{P}{D} \) as follows:

\[ \frac{P}{D}(u_0 + \phi u + x) = \frac{\exp\{K_1(u_0 + \phi u + x)\}}{1 - K_2 \exp\{K_1(u_0 + \phi u + x)\}} \] (117)

\[ (b_0 + b_1(u_0 - \bar{u} + \phi u + x) + b_2(u_0 - \bar{u} + \phi u + x)^2 + \ldots + b_n(u_0 - \bar{u} + \phi u + x)^n + O((u_0 - \bar{u} + \phi u + x)^{n+1})) \]

Here \( \bar{u} \) is the steady state consumption growth. The binomial theorem yields:

\[ (u_0 - \bar{u} + \phi u + x)^k = \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i \] (118)

As a result:

\[ \frac{P}{D}(u_0 + \phi u + x) = \frac{\exp\{K_1(u_0 + \phi u + x)\}}{1 - K_2 \exp\{K_1(u_0 + \phi u + x)\}} \]

\[ \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i + O((u_0 - \bar{u} + \phi u)^{n+1}) \right) \]

The integral \( I_2(u) \) becomes:

\[ I_2(u) = \int_{-\infty}^{+\infty} \exp\{(1 - \gamma)x - (x^2/2\sigma^2)\} (1 - K_2 \exp\{K_1(u_0 + \phi u + x)\}) \frac{P}{D}(u_0 + \phi u + x)dx \] (120)

\[ \frac{\exp\{K_1(u_0 + \phi u + x)\}}{1 - K_2 \exp\{K_1(u_0 + \phi u + x)\}} \left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i + O((u_0 - \bar{u} + \phi u)^{n+1}) \right) dx \]

Note we will define and bound the error \( O((u_0 - \bar{u} + \phi u + x)^{n+1}) \) below. We can now cancel common terms to Yield:

\[ I_2(u) = \int_{-\infty}^{+\infty} \exp\{(1 - \gamma)x - (x^2/2\sigma^2)\} \] (121)
\[
\exp\{K_1(u_0 + \phi u + x)\}\left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} x^i + O((u_0 - \bar{u} + \phi u)^{n+1}) \right) dx.
\]

It is clear that this expression for \( I_2(u) \) is identical to the Mehra-Prescott case so we can jump to the result:

\[
I_2(u) = \exp\{K_1(u_0 + \phi u) + Q_0\} \sqrt{2} \sigma
\]

(122)

\[
\left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} \sum_{j=0}^{i} \binom{i}{j} (\sigma^2[1 - \gamma + K_1])^{i-j} (\sqrt{2} \sigma)^j \Gamma[j + \frac{1}{2}] (1/2)(1 + (-1)^j) \right)
\]
Solving for the $b_i$'s in the Abel case.

Now go back to the integral equation:

$$\frac{1 - K_2 \exp\{K_1 u\}}{\exp\{K_1 u\}} \frac{P}{D}(u) = \frac{K_0}{\sqrt{2\pi} \sigma} (I_1 + I_2) \quad (123)$$

$$= \frac{K_0}{\sqrt{2\pi} \sigma} \left[ \sqrt{2\pi} \sigma \left( \exp\{P_0\} - K_2 \exp\{K_1(u_0 + \phi u) + Q_0\} \right) + \exp\{K_1(u_0 + \phi u) + Q_0\} \sqrt{2}\sigma \right]$$

$$\left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} (u_0 - \bar{u} + \phi u)^{k-i} \sum_{j=0}^{i} \binom{i}{j} (\sigma^2 [1 - \gamma + K_1])^{i-j} (\sqrt{2}\sigma)^j \Gamma\left[ \frac{j}{2} \right] (1/2)(1 + (-1)^j) \right)$$

$$= b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + ... + b_n(u - \bar{u})^n + O((u - \bar{u})^{n+1}).$$

We let:

$$q(u) \equiv K_1(u_0 + \phi u)$$

as in the case of Mehra-Prescott.

It turns out that $w_{i,k}(u)$ is identical to the Mehra-Prescott case since the integral $I_2(u)$ is the same. Thus, we can write the integral equation as:

$$\frac{K_0}{\sqrt{2\pi} \sigma} \left[ \sigma \sqrt{2\pi} \left( \exp\{P_0\} - K_2 \exp\{K_1(u_0 + \phi \bar{u}) + Q_0\} \sum_{l=0}^{n} \frac{1}{l!} (K_1 \phi)^l (u - \bar{u})^l \right) + \exp\{Q_0\} \sqrt{2}\sigma \right] \quad (124)$$

$$\left( \sum_{k=0}^{n} b_k \sum_{i=0}^{k} \binom{k}{i} \sum_{l=0}^{n} \frac{1}{l!} w_{i,k}^{l}(u - \bar{u})^l \right) \sum_{j=0}^{i} \binom{i}{j} (\sigma^2 [1 - \gamma + K_1])^{i-j} (\sqrt{2}\sigma)^j \Gamma\left[ \frac{j}{2} \right] (1/2)(1 + (-1)^j)$$

$$= b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + ... + b_n(u - \bar{u})^n + O((u - \bar{u})^{n+1})$$

We can equate the coefficients to yield the system of linear equations in the $b_i$’s stated in the text. The solution of the system follows the same procedure as in the Mehra-Prescott case.
Error estimation for Abel case.

The logic in this section is as in the Mehra-Prescott case so we give a shorter version of it. The integral equation in the Abel case is:

\[
P \frac{D(u)}{P(u)} = \frac{K_0}{\sqrt{2\pi\sigma}} \frac{\exp\{K_1u\}}{1 - K_2 \exp\{K_1u\}} [I_1(u) + I_2(u)].
\]  

(125)

As in the previous case we have to account for the infinite support for the normal distribution of dividend growth by examining the property of the function:

\[
\chi(u) = \frac{1 - K_2 \exp\{K_1u\} P\left(\frac{u}{K_1}\right)}{\exp\{K_1u\} P\left(\frac{u}{K_1}\right)}. 
\]

(126)

We can now restate the second integral as:

\[
\exp\{K_1(u_0 + \phi u)\} \int_{-\infty}^{+\infty} \exp\{(1 - \gamma + K_1)x - (x^2/2\sigma^2)\}\chi(u_0 + \phi u + x)dx = 
\]

(127)

\[
\exp\{K_1(u_0 + \phi u)\}h(u),
\]

where \(h(u)\) is defined exactly the same as in the Mehra and Prescott case. We also define \(\omega = u_0 + \phi u + x\).

The integral equation, \(I_2(u)\), may now be written as:

\[
\chi(u) = K_0 \left( \exp\{P_0\} - K_2 \exp\{K_1(u_0 + \phi u) + Q_0\} + \frac{1}{\sqrt{2\pi\sigma}} \exp\{K_1(u_0 + \phi u)\}h(u) \right).
\]

(128)

This integral equation is the same as in the Mehra and Prescott case except for the additional term \(K_2 \exp\{K_1(u_0 + \phi u) + Q_0\}\) which is analytic. As a result the proof, that the price dividend function is analytic, will follow the same logic.

Recall that:

\[
\int_{-\infty}^{\infty} \exp\{[1 - \gamma + K_1]x - x^2/2\sigma^2\} dx = \sigma \sqrt{2\pi} \exp\{Q_0\}. 
\]

(129)
Assume:
\[ |\chi(\omega)| < M_0. \] (130)

Substituting (130) into (127) we obtain:

\[ h(u) < M_0 \sigma \sqrt{2\pi} \exp\{Q_0\}. \] (131)

Now substituting (131) into (128), the integral equation is bounded by:

\[ |\chi(\omega)| < K_0 \left( \exp\{P_0\} - K_2 \exp\{K_1(u_0 + \phi u) + Q_0\} + \exp\{K_1[u_0 + \phi u] + Q_0\}M_0 \right). \] (132)

Define:

\[ A_0 \equiv \exp\{P_0\} - K_2 \exp\{K_1(u_0 + \phi u) + Q_0\} \] (133)

and:

\[ B_0 \equiv \exp\{K_1[u_0 + \phi u] + Q_0\}. \] (134)

Then \( M_1 = K_0(A_0 + B_0M_0) \) is a new upper bound. Define \( M_{n+1} \equiv K_0(A_0 + B_0M_n) \). As a result a sequence of upper bounds may be expressed as:

\[ \nu_0 \equiv \lim_{n \to \infty} M_n = \frac{K_0A_0}{1 - K_0B_0} \] (135)

provided \( |K_0B_0| < 1 \). This condition for a bounded solution is identical to the Mehra and Prescott case, except that \( K_1 \) has a new definition.

The derivative \( h^{(n)}(u) \) is the same as in the Mehra-Prescott case so that we can look at the bound on the first derivative.

\[ \chi^{(1)} = \frac{K_0}{\sqrt{2\pi} \sigma} \left( -K_2 \exp\{K_1(u_0 + \phi u) + Q_0\} + \exp\{K_1[u_0 + \phi u]\}h^{(1)}(u) \right). \] (136)

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Now assume:

$$|\chi^{(1)}(\omega)| < M_0^1$$  \hspace{1cm} (137)

so that $h^{(1)} < M_0^1 \phi \sigma \sqrt{2\pi} \exp\{Q_0\}$ and $h(u) < \nu_0 \sigma \sqrt{2\pi} \exp\{Q_0\}$. The first derivative is bounded by

$$\chi^{(1)} < K_0 B_0 (- K_2 \phi K_1 + \phi K_1 \nu_0 + M_0^3 \phi).$$  \hspace{1cm} (138)

Define:

$$A_0^1 \equiv - K_2 B_0 K_1 \phi + B_0 K_1 \nu_0$$  \hspace{1cm} (139)

$$B_0^1 \equiv \phi B_0;$$  \hspace{1cm} (140)

For a bound we need:

$$|K_0 \phi B_0| < 1$$  \hspace{1cm} (141)

which is true given the bound on $|K_0 B_0| < 1$. As a result the bound $\nu_1$ on $\chi^{(1)}$ exist. The bound is given by:

$$\nu_1 \equiv \lim_{n \to \infty} M_n^1 = \frac{K_0 A_0^1}{1 - K_0 B_0^1}.$$  \hspace{1cm} (142)

For the $n^{th}$ derivative, we assume:

$$|\chi^{(n)}(\omega)| < M_0^n$$  \hspace{1cm} (143)

so that $h^{(n)}(u) < M_0^n \phi^n \sigma \sqrt{2\pi} \exp\{Q_0\}$ and $h^{(n-1)}(u) < \nu_{n-1} \phi^{n-1} \sigma \sqrt{2\pi} \exp\{Q_0\}$. Let:

$$f(u) \equiv \exp\{K_1(u_0 + \phi u)\} h(u)$$  \hspace{1cm} (144)

By Leibnitz’s rule:
\[ f^{(n)}(u) = \exp\{K_1(u_0 + \phi u)\} \sum_{k=0}^{n} \binom{n}{k} (K_1\phi)^{n-k} h^{(k)}(u) < \]

\[ \sigma\sqrt{2\pi B_0} \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \nu_k \phi^k + \sigma\sqrt{2\pi B_0} M_0^n \phi^n \]

so that:

\[ \chi^{(n)}(u) < K_0 \left( -K_2B_0(\phi K_1)^n + B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \nu_k \phi^k + B_0 M_0^n \phi^n \right) \]

Define:

\[ A_0^n = -K_2B_0(\phi K_1)^n + B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \nu_k \phi^k \]

\[ B_0^n = \phi^n B_0. \]

For a bound we need:

\[ |K_0\phi^n B_0| < 1 \]

which is true for all \( n \). Thus there is a bound to all the derivatives of \( \chi \) as long as limit as \( n \to \infty \), then \( |A_n| \) is bounded. The bound is given by:

\[ \nu_n = \frac{-K_2K_0B_0(\phi K_1)^n + K_0B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \nu_k \phi^k}{1 - K_0\phi^n B_0}. \]

provides a bound for \( \chi^{(n)}(u) \) for all \( n \).

Remember that:

\[ \chi(u) = \frac{1 - K_2\exp\{K_1 u\}}{\exp\{K_1 u\}} \frac{P}{D}(u) \]

The Taylor expansion of \( \chi(u) \) around \( u = \bar{u} \) is:

\[ \chi(u) = b_0 + b_1(u - \bar{u}) + b_2(u - \bar{u})^2 + \cdots + b_n(u - \bar{u})^n + R_n \]

where \( |R_n| \leq \frac{\nu_{n+1}(u_{\text{max}} - \bar{u})^{n+1}}{(n+1)!} \). Thus we have a bound on the error term as long as \( |K_0B_0| < 1 \).
The price-dividend function is analytic in the Abel case.

The proof, that the price dividend function is analytic, follows the same logic as in the previous case. First for $n = 0$:

$$|\chi^{(0)}(u)| \leq M$$

which we know is true for $M = \nu_0$.

Now assume:

$$|\chi^{(k)}(u)| \leq k!C^k M \forall \ u \in [a,b]$$

for $k = 0, 1, \cdots, n - 1$. From the previous section we bound:

$$|\chi^{(n)}(u)| < K_0B_0 \left( - K_2(\phi K_1)^n + \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \phi^k C^k M + \phi^n \frac{K_0B_0 \sum_{k=0}^{n-1} \binom{n}{k} (K_1\phi)^{n-k} \phi^k k!C^k M}{1 - K_0\phi^n B_0} \right)$$

$$= -K_0B_0 K_2(\phi K_1)^n (n-1)!M + \frac{K_0B_0}{1 - K_0\phi^n B_0} \left[ [K_1\phi + \phi C]^n - [\phi C]^n \right] (n-1)!M < n!MC^n.$$

As a result we need to prove there exist a $C$ such that:

$$-K_0B_0 K_2(\phi K_1)^n (n-1)!M + \frac{K_0B_0}{1 - K_0\phi^n B_0} \left[ [K_1\phi + \phi C]^n - [\phi C]^n \right] < nC^n. \quad (156)$$

For $n=1$ we have:

$$-K_0B_0 K_2(\phi K_1)^1 (n-1)!M + \frac{K_0B_0}{1 - K_0\phi^n B_0} K_1\phi < C \quad (157)$$

So we can make $C$ big enough to make this true. It is also clear that the first term in (156) $|K_0B_0 \frac{K_2(\phi K_1)^n}{(n-1)!M}|$ converges monotonically toward zero. As a result we can focus only on finding a $C$ such that:

$$K_0B_0 [K_1\phi + \phi C]^n < [n - (n-1)K_0\phi^n B_0] C^n. \quad (158)$$
As a result we have to choose $C$ such that:

\[
\frac{[K_0B_0]^\frac{1}{n} K_1\phi}{[n(1-K_0\phi^nB_0) + K_0\phi^nB_0]^\frac{1}{n} - \phi [K_0B_0]^\frac{1}{n}} < C.
\] (159)

Now note that the denominator is positive as long as $n [1 - K_0\phi^nB_0] > 0$, which is true for $u$ within the range of convergence. As in the Mehra-Prescott case, $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. As a result, choose $C > \left|\frac{K_1\phi}{1-\phi}\right| + \left|K_0B_0\frac{K_2(\phi K_1)}{M}\right|$ to assure the Cauchy estimate is satisfied. Thus, the $\frac{P}{D}(u)$ function is analytic for $u \in [a, b]$. 

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The procedure we use is the Gaussian quadrature procedure that one can read about in detail in Judd [1998]. We will give an overview of the methodology. The problem is we want to solve a functional equation of the following form:

$$f(u) = \int_{-\infty}^{+\infty} f(u_0 + \phi u + x) \exp(-x^2)dx$$

(160)

where $x$ is a standard normal random variable. We want to solve for the function $f$ so we first set $f$ equal to a sum of Chebyshev polynomial multiplied by some coefficients that we want to solve for:

$$f(u) = \sum_{i=0}^{n} a_i T_i(u)$$

(161)

where $a_i$ is a coefficient we will solve for numerically and $T_i(u)$ is the $i^{th}$ Chebyshev polynomial. We then substitute (161) into (160) to obtain:

$$\sum_{i=0}^{n} a_i T_i(u) = \int_{-\infty}^{+\infty} \sum_{i=0}^{n} a_i T_i(u_0 + \phi u + x) \exp(-x^2)dx$$

(162)

We now need to evaluate the integral on the right hand side of equation (162). To do this we use the Gaussian quadrature weights, see Judd [1998], pp 259-263. The problem is to evaluate an integral $\int_{-\infty}^{+\infty} f(x) \exp(-x^2)dx$ using $n$ endpoints. Since we wish to bound the support of the distribution, we will be evaluating the integral $\int_{a}^{b} f(x) \exp(-x^2)dx$ as an approximation to the integral over the infinite support. The Gaussian quadrature weights and nodes are chosen by the method, $x_i$ and $\omega_i$, $i = 1, \ldots, n$ so that we can use the approximation:

$$\int_{a}^{b} f(x) \exp(-x^2)dx \approx \sum_{i=1}^{n} \omega_i f(x_i)$$

(163)

This is the point where we bound the support of the distribution since one cannot choose $\infty$ as a possibility in the approximation in certain asset pricing models. Note that if we choose $a$ and $b$ to be negative infinity and infinity, we could use the Gauss-Hermite quadrature procedure.

Gaussian quadrature
though since, as for example in the habit persistence model, we cannot use the entire support of the distribution, we want to solve this integral for some finite support. One can choose any limits of integration in this approximation, we choose three, four and five standard deviations above and below the mean.\textsuperscript{40} We then substitute this into (162). Now we need to solve for the coefficients $a_i$. To do so we use a collocation method which is if we use $n$ coefficients, then we evaluate equation (162) at $n$ points and then solve for the $n$ coefficients, that is we have a system of $n$ equations and $n$ unknowns. To choose the $n$ points to evaluate (162) we solve for the roots of the $n^{th}$ Chebyshev polynomial which are the optimal points to use, see Judd [1998], and then substitute them into (162). Once we have those coefficients, we substitute those coefficients into (161) and we have our estimate of the solution. Note that one should use enough coefficients until the solution is good according to some measure. The measure used is the residual from the Euler equation divided by the policy function:

$$\text{Error} = \left( f(u) - \int_{-\infty}^{+\infty} f(u_0 + \phi u + x) \exp(-x^2)dx \right) / f(u).$$\hspace{1cm} (164)

One can read this as the error in the Euler equation as a percent of the function of interest. If this is small, machine zero, it tells us that you will not improve much on the solution to the policy function by adding more coefficients.

\textsuperscript{40}Note that both integrals will give the same approximate solution if we choose $|a|$ and $|b|$ large enough with $a < 0$ and $b > 0$.\textsuperscript{58}


need reference for Judd and Goo (2002).


