Abstract: A new method is provided for finding the optimal consumption and investment decision of an investor in the presence of time varying expected return on stocks using continuous time stochastic control. Although our method works for any state variable, such as dividend-price ratio, cay, or productive shocks, we demonstrate the method using the model of Campbell et. al. (2004). Expanding on our earlier work, the lifetime utility of the investor is an analytic function within a region large enough to cover all relevant values of the state variable. Subsequently, a computer algorithm for calculating a high order polynomial approximation is developed. Based on analyticity of the lifetime utility, an uniform bound is found on the error in the numerical procedure. As a result, we can quickly find a solution to the investor’s problem to any degree of accuracy desired by the investor. The true optimal investment decision is radically different from that found in Campbell et. al. which demonstrates the benefit of finding an accurate solution to the investor’s problem.

Journal of Economic Literature Classification Numbers: G12, G13, C63, D51

Key Words: Portfolio Decisions
1 Introduction

Over 35 years ago Merton (1969, 1971) developed the theory of optimal consumption and portfolio rules in the presence of time varying investment alternatives using continuous time stochastic control. He was able to completely characterize the solution to this problem for time additively separable utility within the HARA class of utility functions. Unfortunately, these preferences are inconsistent with a reasonable equity premium and return predictability. As a result, researchers have had to compromise their study of optimal consumption and investment decisions in order to meet the limitations of known mathematical results. In this paper we develop a procedure for accurately solving Merton’s problem in the face of return predictability with recursive utility preferences.

Over the eighties and nineties evidence and theory have been developed to explain the time variation and long term predictability of stock returns. Starting with Campbell (1987), Campbell and Shiller (1988a, 1988b), and Fama and French (1988, 1989) evidence suggest that the dividend price ratio tends to predict long horizon stock returns. The recent article by Cochrane (2007) outlines the arguments against this result and develops a defense of the long run predictability of stock returns. Furthermore, Fama and French’s (1996) multi-factor models, used to explain the cross section of stock returns, are also based on return predictability. On the theoretical front various arguments have been put forward to explain return predictability. As pointed out by Mehra and Prescott (1985, 2003) a representative investor with constant relative risk averse utility, a subset of HARA utility, cannot explain the equity premium. In addition, time additively separable utility cannot deliver time varying risk premium necessary to explain predictable stock returns. As a result, alternative preference structures have been introduced to explain return predictability including habit models, such as Abel (1990,1999), Constantinides (1990), Campbell and Cochrane (1999), Menzly, Santos and Veronesi (2004), and Wachter (2006); Durable consumption including Constantinides and Ferson (1991), Yogo (2006); Recursive utility models including Epstein and Zin (1988,1989, 1991), Duffie and Epstein (1991,1992), and Bansal and Yaron (2004); and behavioral models such as Jegdeesh and Titman (1993, 2001), and Barberis, Huang, and Sanots (2001). Rather than changing preferences other researchers explain return predictability based on persistence in some other variable which
helps explain stock returns. Included in this work are consumption relative to assets (cay), Lettau and Ludvigson (2001a, 2001b, 2004, 2005); distorted beliefs, Cecchetti, Lam and Mark (2000); labor income, Santos and Veronesi (2005); Housing, Piazzesi, Schneider, and Tuzel (2007); and Investment Opportunities, Balvers and Huang (2007).

Given the substantial work on return predictability, there has been renewed interest in Merton’s problem. Despite this interest, there are still only a few special cases in which analytic solutions exist.\(^1\) Two of these cases were solved by Merton in which returns are **not** predictable. The more recent problem solved deals with the case in which the investor only cares about terminal wealth, the utility over wealth is HARA, and returns can be predictable.\(^2\) In the last decade progress has been made on more realistic versions of Merton’s Problem in both discrete and continuous time by using various approximation methods.\(^3\) However, the range of accuracy of these procedures have not been established. Rather the answers are compared to the known solutions to Merton’s problem.

In this paper we use the analytic methods, developed by Calin, Chen, Cosimano, and Himonas (2005) and Chen, Cosimano, and Himonas (2006, 2007a, 2007b), to accurately solve Merton’s problem as formulated by Campbell, Chacko, Rodriguez and Viceira (2004). We solve this case since Campbell et al use the preferences of Duffie and Epstein (1992a, 1992b) with the specific functional form of Kreps and Porteus (1978). These preferences are the continuous time version of Epstein and Zin (1989, 1990, 1991), which separates the investor’s aversion to risk from a willingness to substitute consumption across time. Bansal and Yaron (2004) demonstrate that these preferences are consistent with the long horizon predictability of stock returns. In addition, Campbell *et. al.* use an empirical model of stock returns in which a single state variable, the dividend price ratio, predicts the expected return on stocks.\(^4\) If one wished to use an alternative state variable, such

\(^1\)See Cochrane (2007) for a summary.

\(^2\)See Wachter (2002) who also shows that these results generalize to the circumstances in which the investor consumes each period when markets are complete. Sangvinatsos and Wachter (2005) extend this work to the problem of bond return predictability.


\(^4\)While this model is *ad hoc* it is consistent with the Campbell and Cochrane’s (1999) model of asset returns. The preferences of Campbell and Cochrane are not used to represent the investor’s behavior since Epstein and Zin preferences have the investor assess the future while Campbell and Cochrane’s preferences are backward looking. The PDE for an investor subject to Campbell and Cochrane’s preferences is no more complicated than the one solved here.
as cay or productivity shocks, to predict the expected return on stocks, then our method would apply as well. Thus, our solution for Merton’s problem allows for an accurate evaluation of investor behavior over more realistic circumstances.

In solving the Campbell et al. model we overcome two technical difficulties relative to Chen, Cosimano, and Himonas’s (2007) method for solving continuous time asset pricing models. Chen et. al. use the Cauchy-Kovalevsky Theorem to show that the solution to the ordinary differential equation subject to two initial values, which solves the Campbell and Cochrane asset pricing problem, is an analytic function within a reasonable radius of convergence. In the first stage of our solution to Merton’s problem we represent the investor’s problem subject to two initial conditions which are functions of the initial optimal consumption and investment decision. This differential equation is non-linear, rather than the linear differential equation found in asset pricing problems. Campbell et al. overcome this problem by using a first order Taylor polynomial expansion for the non-linear term. They can then guess the functional form of the solution based on previous work by Merton. We do not have to make this approximation in our procedure. The Cauchy-Kovalevsky Theorem applies to this case as well so that the solution to this differential equation is an analytic function within a given radius of convergence. This radius of convergence is large enough so that any realistic variation in the dividend-price ratio is accounted for. Thus, we can represent the solution to the investor’s problem as an \( n^{th} \) order Taylor polynomial expansion. In addition, we are able to establish a bound on the error associated with the Taylor expansion, as well as its first derivative. Consequently, we can add sufficient order to the Taylor polynomial so that the solution is as accurate as desired by the investor.

The second problem to overcome in solving the Campbell et al. model is that the differential equation is a boundary value problem rather than the initial value problem stated in the Cauchy-Kovalevsky Theorem. As a result, in the first step we had to assume an initial value for both the consumption-wealth ratio and the percentage of wealth invested in stocks. This step yields an accurate solution to the investor’s differential equation assuming the investor starts at a particular point. However, we do not know which initial level of consumption and allocation to stock would satisfy the transversality condition. This transversality condition assures that the lifetime utility of
the investor converges to zero when the investor has an infinite horizon or to some finite value under a finite horizon. To address this problem, we use the logic of the shooting method of Judd (1998, 350-362) and Stoer and Bulirsch (2002, 539-557) to find a consumption and investment allocation which satisfy the transversality condition. Using Ito’s lemma we are able to derive the transversality condition as a function of the initial conditions for consumption and investment in the initial value problem and the distribution of the expected return on stocks. We then use the Gauss Hermite Quadrature procedure following Judd (1998, pp.261-263) and Stoer and Bulirsch (2002, 171-181) to evaluate the expectation in the transversality condition by integrating over the distribution of the expected return on stocks. Finally, we choose the level of consumption and investment so that the boundary value problem is satisfied. Once this condition is shown to hold we use our first procedure to represent the optimal behavior of the investor in the face of predictable moment of the expected return on stocks.

The optimal solution is dramatically different from that found in the approximation by Campbell et. al. First, the optimal percentage of wealth invested in stocks is significantly less. For example with a coefficient of risk aversion of 2 and an intertemporal elasticity of substitution of 1.33, the investor finds it optimal to invest 96.89% of her wealth in stocks which is less than half that found in Campbell et. al.. It is shown that this result follows from their approximating the consumption wealth ratio around its value under certainty. It turns out that the optimal consumption to wealth ratio is 10.7% higher, when the true solution to the investor’s problem is solved. In addition, Campbell et. al.’s approximation of the solution to the investor’s problem yields a linear hedging demand as the expected return on stocks changes. However, the true hedging demand has a significant non-linearity. The hedging demand is a modest 3.46% at the stationary expected return on stocks. Yet, there is substantial shorting of stocks at both high (-46% of wealth) and low (-20% of wealth) expected return on stocks, which is contrary to the positive relation between hedging demand for stocks and its expected return. This investment behavior is very effective at stabilizing the lifetime utility of the investor as the expected return on stocks varies. Thus, the optimal behavior of the investor in the face of time varying expected return on stocks is substantially different from Campbell et. al.
2 The Investor’s Problem

The investor takes the empirical characteristics of stock returns as given. Following Campbell et al. stock returns has a time varying expected return on stocks, so that

\[ \frac{dS(t)}{S(t)} = \mu(t)dt + \sigma_S d\omega_S,t, \]  

with

\[ d\mu(t) = \kappa(\theta - \mu(t))dt + \sigma_{\mu} d\omega_{\mu,t}. \]  

The risk free interest rate, \( r \) is assumed to be constant.\(^5\)

Campbell et al. use the expected impact of the dividend-price ratio on stock returns for the state variable \( \mu \). Using quarterly data from 1947.1 to 1995.4 Campbell et al. estimate a VAR for stock returns and the dividend-price ratio. Given the quarterly estimates, they approximate the continuous time stochastic processes (2.1) and (2.2).\(^6\) Their parameter values are given in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>0.00081</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.01398</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0.043875</td>
</tr>
<tr>
<td>( \sigma_S )</td>
<td>0.078959</td>
</tr>
<tr>
<td>( \sigma_{\mu} )</td>
<td>0.005738</td>
</tr>
<tr>
<td>( \rho )</td>
<td>-0.9626</td>
</tr>
</tbody>
</table>

Table 1: Parameter Values 1947.1 to 1995.4

Following the procedure in Campbell and Viceira (1999) the parameters are updated to include the run up and collapse of the price-dividend ratio over the last decade. These new parameters are given in Table 2. While the impact of the dividend-price ratio is weaker, when the last decade is included, the net impact on the long run expected return on stocks, \( \theta \), is less than one basis point. The standard deviation of stock returns \( \sigma_S \) increases while the standard deviation of the dividend-price ratio falls. Finally, the persistence of the dividend-price ratio increases, which yields a decline in \( \kappa \).

\(^5\)Campbell and Viceria (2001, 2002), Brennan and Xia (2002), Wachter (2003), Campbell, Chen and Viciera (2003) and Sangvinatsos and Wachter (2005) look at uncertainty in the risk free rate and bond returns. Currently we can look at only one state variable since more state variables adds to the order of the differential equation which has to be solved. Cochrane (2007b) and Viceira (2001) discuss the impact of labor income on optimal portfolio problems.

\(^6\)The alternative would be to estimate a continuous time process given that the econometrician only observes the data at discrete intervals. See Ait-Sahalia (1999, 2002) for details. We follow Campbell et al. so that any difference here follow from the solution method for the investor’s problem.
The investor assumes that stock returns follow the stochastic processes $(2.1)$ and $(2.2)$ and chooses consumption, $C(t)$ and percentage of wealth, $W(t)$, invested in stocks, $\alpha(t)$ each period so as to maximize

$$J(W(t), \mu(t)) = \max_{\alpha(t), C(t)} E_t \left[ \int_t^T f(C(\tau), J(W(\tau), \mu(\tau)))d\tau \right] + e^{-\beta} E_t \left[ B(W(T), \mu(T)) \right]$$

subject to the stochastic process for wealth

$$dW(t) = rW(t)dt + \alpha(t)W(t)[(\mu(t) - r)dt + \sigma_S d\omega_{S,t}] - C(t)dt.$$  

This specification of the investor’s problem is the general form specified by Merton (1971) in which the expected return on stocks may move over time. Following Campbell et. al. we use the recursive utility of Duffie and Epstein (1992a, 1992b) with the particular functional form of Kreps and Porteus (1978).

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}}(1 - \gamma)J \left[ \left( \frac{C}{((1 - \gamma)J)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right].$$

The investor bases her utility on a comparison between current consumption and the expected lifetime utility from following the optimal strategy, $J(W(t), \mu(t))$. As a result, the investor is forward looking rather than backward looking as in the habit model of Campbell and Cochrane (1999) and the prospect theory which was formulated by Barberis, Huang and Yaron (2004). In addition, the pricing kernel and risk premium are time varying since the marginal utility of consumption is dependent on the lifetime utility of the investor, which varies as the state variables, wealth and expected return on stocks, changes. For the recursive utility the investor’s response to risk can be separated from her willingness to substitute consumption across time. While both stocks and bonds are vehicles for substituting consumption across time, only stocks are subject to uncertainty. Thus,

<table>
<thead>
<tr>
<th>$r$</th>
<th>$0.00250$</th>
<th>$\sigma_S = 0.080204$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$0.01452$</td>
<td>$\kappa = 0.015346$</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td>$0.002161$</td>
<td>$\rho = -0.9583$</td>
</tr>
</tbody>
</table>

Table 2: Parameter Values 1947.1 to 2006.4

Schroder and and Skiadas (1999, 2002) use Duffie and Epstein preferences and examine the relation with habit models.
the recursive utility representation of investor behavior allows for a reasonable equity premium as
demonstrated by Epstein and Zin (1990, 1991), and Bansal and Yaron (2004) in discrete time models
of asset returns. The Kreps and Porteus (1978) functional form allows the two parameters $\gamma$ and $\psi$
to independently control the investor’s aversion to risk, $\gamma$, and her rate of substitution of consumption
across time, $\psi$. Thus, we can separately examine how risk aversion and intertemporal substitution
of consumption impacts the optimal decisions of the investor.

To complete the investor’s problem a bequest function, $B(W(T), \mu(T))$, must be specified. We
choose the form

$$B(W(T), x(T)) = e^{-\beta T} J(W(T), x(T)). \quad (2.6)$$

The parameter $\beta$ measures how much weight the investor places on the welfare of some future person
for example the investor’s children. If $\beta = 0$, then the weight is the same as if it is herself. If $\beta \to \infty$,
then no weight is put on this future person. In this work we let $\beta$ be the same as the investor’s
subjective rate of discount.\footnote{Note it is conceivable that the parameters in $J(W(T), \mu(T))$ are different for her children. $\beta$ is taken to be 6\% on an annual basis following Campbell et. al.}

In this work we follow Campbell et. al. and consider an infinite horizon so that the investor’s
transversality condition is

$$\lim_{T \to \infty} e^{-\beta T} E_t [J(W(T), x(T))] = 0. \quad (2.7)$$

We also follow Campbell et. al. in that the solution to the investor’s lifetime utility can be
separated into separate functions of the two state variables, wealth and the expected return on
stocks.

$$J(W(t), \mu(t)) = H(\mu(t))^{-\frac{1}{1-\gamma}} \frac{W(t)^{1-\gamma}}{1-\gamma}. \quad (2.8)$$

As a result, the optimal consumption decision is proportional to wealth

$$C(t) = W(t) \frac{(\beta)^{\psi}}{H(\mu(t))}. \quad (2.9)$$

The optimal percentage invested in stocks is given by

$$\alpha(t) = \frac{1}{\gamma} \frac{\mu(t) - r}{\sigma^2} - \frac{\rho \sigma_{\mu}}{\gamma \sigma S} \frac{1 - \gamma}{1-\psi} \frac{H'}{H}. \quad (2.10)$$
The first term is the myopic portfolio choice originally identified by Markowitz (1952). The second term is the intertemporal hedging effect developed by Merton (1971). This hedging effect is dependent on the elasticity of lifetime utility with respect to the expected return on stocks, \(-\mu(t)\frac{1-\gamma}{1-\psi} H'(\mu(t))\). It is also affected by the regression of expected returns on actual returns \(\frac{\rho \sigma_{\mu} \sigma_{S}}{\sigma_{S}^2}\). Thus, it seems straightforward to formulate an optimal investment strategy.

Unfortunately, we do not know the form of \(H(\mu(t))\), since it is a solution to the following ordinary differential equation:

\[
0 = -\frac{\beta^\psi}{H} + \beta \psi + (1 - \psi) r + \frac{(1 - \psi)}{2\gamma} \left( \frac{\mu(t) - r}{\sigma_S} \right)^2 - \frac{\sigma_{\mu}^2 H''}{2 H} - \left[ \kappa (\theta - \mu(t)) + \frac{1 - \gamma}{\gamma} \rho \sigma_{\mu} \left( \frac{\mu(t) - r}{\sigma_S} \right) \right] \frac{H'}{H} + \frac{\sigma_{\mu}^2}{2} \left[ \left( 1 + \frac{1 - \gamma}{(1 - \psi)} \right) + \rho^2 \frac{(1 - \gamma)^2}{\gamma(1 - \psi)} \right] \left( \frac{H'}{H} \right)^2.
\]

Campbell et. al. approximate the solution to this differential equation by replacing the first term with a log-linear approximation of (2.9) around the value \(\frac{C(0)}{W(0)} = \beta = 0.015\), which is given by

\[
-\frac{\beta^\psi}{H(\mu(t))} \approx a_0 + a_1 \ln H(\mu(t))
\]

based on (2.9), which implicitly sets the first initial condition imposed on the differential equation (2.11). In this case, they guess and verify the solution

\[
H(\mu(t)) = \exp \left\{ A_1 + B_1 \mu(t) + \frac{1}{2} C_1 \mu(t)^2 \right\}.
\]

Consequently, their hedging demand for stocks is a linear function of the expected return on stocks. In addition, the steady state expected return on stocks \(\theta\) automatically determines the second initial condition imposed on the differential equation (2.11) such that the percentage of wealth invested in stocks is implicitly defined. We demonstrate below that this implicit condition establishes a level for the hedging demand for stocks which is associated with too low a level of consumption to wealth ratio. Consequently, the investor’s wealth is expected to grow too fast relative to consumption, since the expected return on the investor’s portfolio is higher and her consumption is lower. In addition, a 500th order Taylor polynomial approximation to the solution of the ordinary differential equation (2.11) must be used for the solution to be accurate given the possible fluctuations in the expected return on stocks. Thus, the actual hedging demand for stocks is dramatically different from that found in Campbell et. al.
3 Solution Method for Investor’s Problem

In this section, we use the analytic methods discussed in Chen, Cosimano and Himonas (2007a) to solve the ordinary differential equation (2.11). The ordinary differential equation (2.11) subject to the optimal conditions (2.9), (2.10), and the transversality condition (2.7) is a boundary value problem. Our solution method has five steps: First, treat the differential equation as an initial value problem in which the solution is dependent on two given initial values. For this problem we can use the Cauchy-Kovalevsky Theorem to show that the solution to this initial value problem is an analytic function with a given radius of convergence. This radius of convergence includes any reasonable value for expected return on stocks. Second, we represent the solution to this initial value problem as an $n^{th}$ order Taylor polynomial. Here the order of the polynomial is chosen to make the solution and its first derivative as accurate as desired by the investor. Third, we relate the two initial conditions in the initial value problem to the optimal choice of consumption and the percentage of wealth invested in stock following (2.9) and (2.10). Fourth, we use the logic in the shooting method, discussed in Judd (1998, 350-362) and Stoer and Bulirsch (2002, 539-557), to find the consumption and investment which satisfies the transversality condition (2.7). In this step we use Ito’s lemma to write the future lifetime utility of the investor in terms of the solution to the initial value problem and the distribution for the expected return on stocks. We then integrate the future lifetime utility over the distribution of expected stock returns following the Gauss Hermite Quadrature procedure. This yields a function which is dependent on the initial values of consumption and investment used in the initial value problem. Finally, the optimal consumption and investment is found by satisfying the transversality condition. Thus, we can construct a lifetime utility function which accurately represents the optimal behavior of the investor.

3.1 Initial Value Problem

We seek a solution $H(\mu)$ to the ordinary differential equation (2.11) subject to two initial conditions at the unconditional mean, $\mu = \theta$ of the stochastic process (2.2) for the expected return on stocks.

$$H(\theta) = h_0 \text{ and } H'(\theta) = h_1.$$  

(3.14)
The Cauchy-Kovalevsky Theorem says that the solution to this initial value problem is a analytic function near $\mu = \theta$ with radius of convergence $r_c$. Thus, the solution may be represented by

$$H(\mu; h_0, h_1) = \sum_{n=0}^{\infty} \frac{h_n}{(c\sigma_{\mu})^n}(\mu - \theta)^n$$

for $|\mu - \theta| < r_c$. Here $\epsilon$ is a parameter chosen to minimize the division by small numbers in the computer program.

For this result to be useful in the subsequent calculation of an accurate solution we must identify a radius of convergence, which includes reasonable values for the expected return on stocks, $\mu$. This problem is harder for the differential equation (2.11) relative to the linear ordinary differential equation found in asset pricing models. If the non-linear term $\left(\frac{\mu'}{\mu}\right)^2$ did not exists, then the Theorem in Chen, Cosimano and Himonas (2007a) would apply without change. To get the solution to the actual problem, we tailor the argument in the Cauchy-Kovalevsky Theorem to the particular differential equation (2.11). The result is a subroutine to find the radius of convergence. Thus, we can accurately represent the solution to the initial value problem by a $n^{th}$ order Taylor polynomial for the expected return on stocks within a reasonable interval.

The first step is to develop a recursive rule for the coefficients in the power series (3.15). Starting with the initial conditions (3.14) the remaining coefficients follow the recursive rule.

$$(n + 1)(n + 2)h_0 h_{n+2} = A_0 \sum_{k=0}^{n} (k + 1)(n - k + 1)h_{k+1} h_{n-k+1}$$

$$+ B_1 \sum_{k=0}^{n-1} (k + 1)h_{k+1} h_{n-k-1} + B_0 \sum_{k=0}^{n} (k + 1)h_{k+1} h_{n-k}$$

$$+ C_2 \sum_{k=0}^{n-2} h_k h_{n-k-2} + C_1 \sum_{k=0}^{n-1} h_k h_{n-k-1} + C_0 \sum_{k=0}^{n} h_k h_{n-k}$$

$$+ D_0 h_n - \sum_{k=0}^{n-1} (k + 1)(k + 2)h_{k+2} h_{n-k} \text{ for } n \geq 2.$$

where the parameters are given by

1. $A_0 = 1 + \frac{1-\gamma}{1-\psi} + \frac{\rho^2(1-\gamma)^2}{\gamma(1-\psi)}$,

2. $B_0 = 2\epsilon^2 \left[ \kappa \left( x_0 - \frac{\theta}{\epsilon \sigma_{\mu}} \right) - \frac{\rho \sigma_{\mu} (1-\gamma)}{\gamma \sigma_S} \left( x_0 - \frac{r}{\epsilon \sigma_{\mu}} \right) \right]$,
3. \( B_1 = 2e^2 \left[ \kappa - \frac{r \sigma_{\mu}(1 - \gamma)}{\gamma \sigma_{\gamma}} \right] \),

4. \( C_0 = \frac{\epsilon_4^2 \sigma_{\mu}^2 (1 - \psi)}{\gamma \sigma_{\gamma}^2} \left( x_0 - \frac{r}{\epsilon_{\mu}} \right)^2 + 2e^2 r(1 - \psi) + 2e^2 \beta \psi \),

5. \( C_1 = \frac{2 \epsilon_4^2 \sigma_{\mu}^2 (1 - \psi)}{\gamma \sigma_{\gamma}^2} \left( x_0 - \frac{r}{\epsilon_{\mu}} \right) \),

6. \( C_2 = \frac{\epsilon_4^2 \sigma_{\mu}^2 (1 - \psi)}{\gamma \sigma_{\gamma}^2} \),

7. \( D_0 = -2e^2 \beta \psi \).

Now we must find conditions under which the power series based on this recursive rule converges. The strategy is to bound the original rule with another recursive rule which does converge. First, the recursive rules are defined in terms of the variable, \( \tilde{h}_n = \frac{n^2 L(h_n/h_0)}{h_0} \) for all \( n \geq 1 \), where \( L \) is a positive number. The second step is to show there exist some real number \( L > 0 \) and integer \( N \geq 4 \) such that for all \( n \geq N \), we have

\[
\begin{align*}
&\frac{n + 2}{n + 1} \left[ |A_0| \sum_{k=0}^{n} \frac{1}{(k + 1)(n - k + 1)} + \frac{|B_1|}{L} \sum_{k=0}^{n-2} \frac{1}{(k + 1)(n - k - 1)} + \frac{|B_0|}{L} \sum_{k=0}^{n-1} \frac{1}{(k + 1)(n - k)^2} \\
&+ \frac{|C_2|}{L} \sum_{k=1}^{n-3} \frac{1}{k^2(n - k - 2)^2} + \frac{|C_1|}{L} \sum_{k=1}^{n-2} \frac{1}{k^2(n - k - 1)^2} + \frac{|C_0|}{L} \sum_{k=1}^{n-1} \frac{1}{k^2(n - k)^2} + \frac{1}{L} \cdot \frac{\pi^2}{6} \\
&+ \frac{|B_1|}{n} + \frac{|B_0|}{n + 1} + \frac{2|C_2|}{(n - 2)^2} + \frac{2|C_1|}{(n - 1)^2} + \frac{2|C_0|}{n^2} + \frac{|D_0/h_0|}{n^2} \right] < 1. \\
&\text{(3.17)}
\end{align*}
\]

The final step is to develop a power series which dominates (3.15). Consequently, we prove in the appendix the following theorem.

**Theorem 3.1.** Choose a real number \( L > 0 \) and an integer \( N \geq 4 \) such that the inequality (3.17) holds and set

\[
M = \max \left\{ 1, \sqrt[n^2 L|h_n/h_0|}{1 \leq n \leq N + 1} \right\} \quad \text{and} \quad r_c = \epsilon \sigma_{\mu}/M.
\]

\[
\text{The power series solution (3.15) for the initial value problem (2.11) and (3.14) has coefficients, } h_n, \text{ which are determined by the recurrence relation (3.16). This power series solution is convergent in the interval } \theta - r_c \leq \mu \leq \theta + r_c.
\]
The numerical representation of the power series solution is a \( n^{th} \) order Taylor Polynomial for \( H(\mu) \) around \( \mu = \theta \) which is given by

\[
T_n(\mu; h_0, h_1) = \sum_{k=0}^{n} \frac{h_k}{(\epsilon \sigma_\mu)^k} (\mu - \mu_0)^k,
\]

where the coefficients, \( h_k \), are determined by (3.16).

The main benefit of Theorem 3.1 is that bounds on the errors can be found for both the solution to the initial value problem (2.11) and (3.14), as well as its derivatives.

**Corollary 3.2.** For any positive real number \( \nu < 1 \), we have

\[
\max_{|\mu - \mu_0| \leq \nu r_c} |H(\mu) - T_n(\mu)| \leq \frac{|h_0|}{L} \sum_{k=0}^{n} \frac{\nu^k}{k^2} \leq \frac{|h_0|}{(n+1)^2L} \cdot \frac{\nu^{n+1}}{1-\nu} \tag{3.20}
\]

and

\[
\max_{|\mu - \mu_0| \leq \nu r_c} |H'(\mu) - T'_n(\mu)| \leq \frac{M|h_0|}{\epsilon \sigma_\mu L} \sum_{k=0}^{n} \frac{\nu^k}{k+1} \leq \frac{M|h_0|}{(n+1)\epsilon \sigma_\mu L} \cdot \frac{\nu^n}{1-\nu}. \tag{3.21}
\]

Thus, we can choose the order of the Taylor polynomial (3.19) such that the solution to the initial value problem and its derivative are as close to the actual values as desired by the investor, as long as the expected return on stocks stays within the interval \( \theta - \nu r_c \leq \mu \leq \theta + \nu r_c \). We want to know the accuracy of \( H'(\mu) \), since the optimal investment decision (2.10) is dependent on the elasticity of the lifetime utility with respect to the expected return on stocks, \(-\mu(t) \frac{1-\gamma}{1-\psi} H'(\mu(t)) \).

In a typical case, discussed below, the radius of convergence is 6.4962 \( \times \sigma_\mu \), so that reasonable values of the expected return on stocks are included in this interval. For a 500\( ^{th} \) order Taylor polynomial (3.19) and \( \nu = 0.8 \) the errors in (3.20) and (3.21) are 5.6994 \( \times 10^{-29} \) and 2.2600 \( \times 10^{-24} \), respectively. Thus, we have an accurate solution to the initial value problem (2.11) and (3.14), for \( \theta - 5.2\sigma_\mu \leq \mu \leq \theta + 5.2\sigma_\mu \). If the investor wants to know the solution for a larger range of the expected return on stocks, then we can use the accuracy of \( H(\mu) \) and \( H'(\mu) \) given by Corollary 3.2.

At the point \( \theta + 5.2\sigma_\mu \) we can take \( h_0 = T_n(\theta + 5.2\sigma_\mu) \) and \( h_1 = T'_n(\theta + 3.6\sigma_\mu) \) as new initial points and calculate the new power series solution (3.15) based on the recursive relations (3.16). We have found that the radius of convergence falls each time the interval is extended to the right or left. By following this procedure we can extend the solution seven times to the right and only two times to
the left. These extensions makes the solution accurate for expected return on stocks in the interval \( \theta - 8.6\sigma_{\mu} = -0.0185 \leq \mu \leq 0.074 = \theta + 32.5\sigma_{\mu} \). However, any equilibrium model of stock returns would not yield an expected return on stocks below the risk free return \( r = 0.0025 \) since this would violate the no arbitrage condition. Thus, the solution to the initial value problem (2.11) and (3.14) can be made as accurate as desired by the investor in a reasonable range for the expected return on stocks.

3.2 Initial Values

We now consider how to choose the initial conditions to solve the investor’s problem. We want these initial conditions to satisfy the conditions for optimal consumption (2.9) and investment (2.10) as well as the transversality condition (2.7). As a result, the initial value problem (2.11) and (3.14) becomes a boundary value problem.

The optimal consumption condition (2.9) implies that

\[
H(\mu(t)) = \beta^\psi \frac{W(t)}{C(t)}. \tag{3.22}
\]

Consequently, at the initial point we want

\[
h_0 = H(\mu(0)) = \beta^\psi \frac{W(0)}{C(0)}. \tag{3.23}
\]

Using the optimal percentage of wealth to invest in stock (2.10) we have

\[
H'(\mu(t)) = \frac{(1 - \psi)}{\rho\sigma_{\mu}(1 - \gamma)} \left[ \frac{\mu(t) - r}{\sigma_S} - \sigma_S\gamma\alpha(t) \right] H(\mu(t)). \tag{3.24}
\]

As a result, we can set the second initial condition so that

\[
h_1 = H'(\mu(0)) = \frac{(1 - \psi)}{\rho\sigma_{\mu}(1 - \gamma)} \left[ \frac{\mu(0) - r}{\sigma_S} - \sigma_S\gamma\alpha(0) \right] h_0. \tag{3.25}
\]

Thus, our solution of the initial value problem (2.11) and (3.14) problem becomes

\[
H \left( \mu, \frac{W(0)}{C(0)}, \alpha(0) \right) = \sum_{n=0}^{\infty} \frac{h_n}{(\epsilon \sigma_{\mu})^n}(\mu - \theta)^n, \tag{3.26}
\]

which we approximate with

\[
T_n \left( \mu, \frac{W(0)}{C(0)}, \alpha(0) \right) = \sum_{k=0}^{n} \frac{h_k}{(\epsilon \sigma_{\mu})^k}(\mu - \mu_0)^k. \tag{3.27}
\]
At this stage we do not know the optimal consumption and investment decision at the initial time \( t = 0 \). To find these values we must treat the differential equation (2.11) as a boundary value problem subject to the terminal condition (2.7) which we write as

\[
F \left( \frac{W(0)}{C(0)}, \alpha(0); \mu(0), W(0) \right) = \lim_{T \to \infty} e^{-\beta T} E_0 \left[ \left( H \left( \frac{\mu(T)}{C(0)}, \alpha(0) \right) \right) - \frac{1-\gamma}{1-\psi} W(T)^{1-\gamma} \right] = 0, \tag{3.28}
\]

where the initial value of expected return \( \mu(0) \) and the investor’s wealth \( W(0) \) is known by the investor. Consequently, we want the investor to choose consumption \( C(0) \) and investment \( \alpha(0) \) so that (3.28) is true.

We cannot directly evaluate (3.28) since we do not know the future value of wealth. In the next steps we use Ito’s lemma to find a differential equation for the lifetime utility of the investor. We then find a solution to this differential equation given the stochastic process for the expected return on stocks (2.1).

The stochastic process for wealth (2.4) is subject to the two optimal conditions for consumption (2.9) and investment (2.10) so that

\[
\frac{dW(t)}{W(t)} = \left[ r - \frac{\beta \psi}{H \left( \frac{\mu(t)}{C(0)}, \alpha(0) \right)} + \frac{1}{\gamma} \left( \frac{\mu(t) - r}{\sigma^2_S} - \frac{\rho \sigma \mu}{\sigma^2_S} \frac{1-\gamma}{1-\psi} H' \left( \frac{\mu(t)}{C(0)}, \alpha(0) \right) \right) (\mu(t) - r) \right] dt
\]

\[
\quad + \frac{\sigma_S}{\gamma} \left( \mu(t) - r \right) \frac{1}{\sigma_S} \frac{1-\gamma}{1-\psi} H' \left( \frac{\mu(t)}{C(0)}, \alpha(0) \right) \right] d\omega_{S,t}.
\tag{3.29}
\]

Thus, the optimal choice of consumption and investment determines the path for the investor’s wealth.

Since we know the stochastic process for wealth and the expected return on stocks we can find the stochastic process for the lifetime utility using Ito’s Lemma. We write this differential equation as

\[
\frac{de^{-\beta t} J(\mu(t))}{e^{-\beta t} J(\mu(t))} = A \left( \frac{\mu(t)}{C(0)}, \alpha(0) \right) dt + B \left( \frac{\mu(t)}{C(0)}, \alpha(0) \right) d\omega_{S,t}
\]

\[
- C \left( \frac{\mu(t)}{C(0)}, \alpha(0) \right) d\omega_{\mu,t}. \tag{3.30}
\]

14
The functions $A(\cdot), B(\cdot)$ and $C(\cdot)$ are defined in the appendix.

This stochastic differential equation has the solution

$$e^{-\beta T} J(\mu(T)) = \left( H \left( \mu(0); \frac{W(0)}{C(0)}, \alpha(0) \right) \right)^{-\frac{1-\gamma}{\gamma}} \frac{W(0)^{1-\gamma}}{1-\gamma} \exp \left\{ \int_0^T A \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) \, dt \right\} \left( \frac{W(0)}{C(0)}, \alpha(0) \right)$$

$$+ \int_0^T B \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) \, d\omega_{S,t} - \int_0^T C \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) \, d\omega_{\mu,t} , \quad (3.31)$$

To capture the concept of an adaptive process (See Shreve 2003 p.97), we assume that the contemporary random shocks $d\omega_{S,t}$ and $d\omega_{\mu,t}$ are independent of the long term distribution of $\mu(t)$. Following Arnold (1974), the solution to the stochastic differential equation for the expected return on stocks (2.2) is

$$\mu(t) = \theta + \exp \left[ -\kappa t \right] \left[ \mu(0) - \theta \right] + \sigma_\mu \int_0^t \exp \left[ -\kappa(t - s) \right] \, d\omega_{\mu,t} . \quad (3.32)$$

Using the result from Shreve (2003), we have

$$\sigma_\mu \int_0^t \exp \left[ -\kappa(t - s) \right] \, d\omega_{\mu,t} \sim N \left( 0, \frac{\sigma_\mu^2}{2\kappa} \left[ 1 - \exp \left[ -2\kappa t \right] \right] \right) , \quad (3.33)$$

so that the expected return on stocks, $\mu(t)$, is normally distributed.

For given $\mu(t)$ the stochastic integrals in (3.31) have a normal distribution with mean zero and variance

$$\int_0^T B \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right)^2 \, dt + \int_0^T C \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right)^2 \, dt \quad (3.34)$$

$$- 2\rho \int_0^T B \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) C \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) \, dt$$

following Shreve (2003, p. 149) and the formula for the variance of the sum of random variables Mood, Graybill, and Boes (1974, p. 178). As a result,

$$E \left[ e^{-\beta T} J(\mu(T)) \mid \mu(t) \right] = J(\mu(0), W(0)) \exp \left\{ \int_0^T A \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) \, dt \right\}$$

$$- \rho \int_0^T B \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) C \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right) \, dt$$

$$+ \frac{1}{2} \int_0^T B \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right)^2 \, dt + \frac{1}{2} \int_0^T C \left( \mu(t); \frac{W(0)}{C(0)}, \alpha(0) \right)^2 \, dt \right\} , \quad (3.35)$$

15
To obtain the transversality condition (3.28) we integrate over $\mu(t)$ using the Gauss-Hermite Quadrature procedure following Judd (1998, pp. 261-263), and Stoer and Bulirsch (2002, 171-181). As a result, we have

$$F\left(\frac{W(0)}{C(0)}, \alpha(0); \mu(0), W(0)\right) = \lim_{T \to \infty} \int_{-\infty}^{\infty} E\left[ e^{-\beta T J(\mu(T))}\right] f(\mu) d\mu,$$

(3.36)

where $f(\mu)$ is the probability distribution (3.33).

The investor is given the current value of wealth $W(0)$ and expected return on stocks $\mu(0)$. They maximize their lifetime utility (3.31) by choosing consumption $C(0)$ and percentage of wealth allocated to stock $\alpha(0)$ such that (3.36) is zero. If the investor has a shorter time horizon, then consumption and investment would be chosen so that (3.36) is equal to the objective of the investor at the finite terminal time $T$.

In simulating (3.36) we found that the integral does not make sense for negative values of the expected return on stocks. Upon reflection it is clear that (2.2) violates the no arbitrage condition for equilibrium stock prices. The reason is that investors concerned about uncertainty would not purchase stock when the expected return on stocks is less than the riskfree rate. We will fix this in the future by switching (3.36) to a log normal stochastic process. In addition, (2.1) would be expressed in terms of the equity premium rather than the return on stocks. Actually, these changes are consistent with the original specification of Campbell et. al.

For now we limit the number of nodes in the Gauss-Hermite quadrature procedure to six so that the possible values of the expected return on stocks is $\mu \in [-2.3 \sigma_{\mu}, 2.3 \sigma_{\mu}]$. We also integrate the function over the time interval of one quarter $T = 1$ with 20 subintervals used in the trapozoid rule.\(^9\) We take the initial wealth to be 1, and the initial expected return to be $\mu(0) = \theta$. The other parameters are reported in Table 2. For the transversality condition (3.36) to converge to zero we want the exponent on the exponential function to be less than zero or this exponential function to be less than one. In Figure 1 we plot this exponential function for various values of the wealth to consumption ratio $\frac{W(0)}{C(0)} \in [50, 65]$, while the percentage invested in stocks is $\alpha(0) \in [0.75, 2]$.\(^{10}\)

\(^9\)We increase $T$ to 300 quarters or 75 years. In this case, the exponential function very slowly converges to zero.

\(^{10}\)This plot took 6 minutes on a standard PC using the software Matlab.
In Campbell et al., it is typical for them to find a percentage of wealth invested in stocks of about 2. From Figure 1, we can see why they get this result. They undertook their approximation (2.12) around the point \( \frac{C(0)}{W(0)} = \beta = 0.15 \), since this is the standard result under certainty. From Figure 1 we see that they implicitly approximation the solution around the point \([W(0)/C(0), \alpha(0)] = [65, 2]\). This point does satisfy a transversality condition for the approximated problem, but it is not a global optima, since one can increase consumption out of wealth \( \frac{C(0)}{W(0)} \) or reduce \( \frac{W(0)}{C(0)} \) and reduce the percentage invested in stocks and still have the transversality condition satisfied. Thus, the investor has more consumption and less risk when he is on the upward slope of this plane, i.e., the left hand side of the graph closer to the front.

We can simplify the search for the solution in (3.36) by making the assumption that consumption and wealth grow at the same rate in the steady state. This assumption is consistent with the empirical evidence of Lettau and Ludvigson (2001a, 2001b, 2004, 2005), who demonstrate that consumption, financial assets and labor wealth are cointegrated. It is also consistent with the linear homogeniety of the optimal consumption decision (2.9) in the investor’s wealth, since \( \mu \) and subsequently \( H(\mu) \) converge to a stationary distribution. Apply Ito’s lemma to the optimal consumption to wealth ratio.
(2.9) to find

$$\frac{dC}{CW} = -\left[\frac{H'(\mu(t))}{H(\mu(t))}\kappa(\theta - \mu(t)) + \frac{H''(\mu(t))H(\mu(t))^2 - 2(H'(\mu(t)))^2}{H(\mu(t))^3}\sigma_\mu^2\right] dt - \frac{H'(\mu(t))}{H(\mu(t))}\sigma_\mu d\omega_{\mu,t}$$

For the consumption wealth ratio to be stationary in the long term, when $\mu(t) \to \theta$, we need

$$H''(\theta)H(\theta)^2 = 2(H'(\theta))^2$$

or $h_2 = h_1^2$.

Using this result and the rule for $h_2$ following the recursive rule (3.16) we see that

$$h_1 = \frac{-2B_0 h_0 \pm \sqrt{B_0^2 h_0^2 - 4(A_0 - 2) [C_0 h_0 + D_0] h_0}}{2(A_0 - 2)}.$$ (3.37)

As a result, we can reduce the choice in (3.36) to just the optimal consumption, $C(0)$. Thus, we can simplify the search for the solution to the investor’s problem.

**Notes on further work.** There are still several open questions:

- Given the argument on page 12, can we find the global property of the power series solution (3.15)?

- Currently, it is assumed that $d\omega_{S,t}$, and $d\omega_{\mu,t}$ are independent of the long term distribution of $\mu(t)$. Can we justify this or at least place a bound on how much error is introduced by this assumption?

- In (3.36) we approximate the integral with Gauss-Hermite. This has to change so that the no arbitrage condition is not violated.

### 4 Preliminary Simulation Results

We now consider the optimal behavior for various values of the intertemporal elasticity of substitution $\psi$ and the risk aversion of the investor $\gamma$. We first consider the solutions given the assumption of stationarity of the consumption to wealth ratio (3.37). With this simplification we can demonstrate the properties of the solution in two dimensional graphs, since we only have to vary consumption to find the optimum decision for the investor. We use the parameters in Table 2 which are based on stock return and the dividend price ratio from 1947.1 to 2006.4.
4.1 The intertemporal elasticity of consumption greater than one

We first consider the case in which the intertemporal elasticity of substitution is greater than one, \( \psi = 1 \frac{1}{3} \). A higher value is advocated by Bansal and Yaron (2004) since it leads to a positive relation between consumption and the expected rate of return on stock. The aversion to risk is kept constant in this subsection at \( \gamma = 2 \). The subjective discount factor is always \( \beta = 0.015 \) which is the value used by Campbell et al. The graphs are plotted for the expected return on stock in the interval \( \theta - \frac{\sigma_\mu}{\sqrt{2k}} \leq \mu \leq \theta + \frac{\sigma_\mu}{\sqrt{2k}} \) where \( \frac{\sigma_\mu}{\sqrt{2k}} = 0.012335 \) is the long run standard deviation of the expected return on stocks from (3.33) or 5.7080\( \sigma_\mu \). Also \( \theta - \frac{\sigma_\mu}{\sqrt{2k}} \leq r \), so that the no arbitrage condition is not violated. Consequently, all the graphs are an accurate representation of the decisions for the investor.

In Figure 2 a graph is provided for the exponential function in (3.36) for a time horizon of one quarter, \( T = 1 \). Again we use only six nodes in the Gauss Hermite quadrature procedure. For a time horizon of one quarter we find that the optimal consumption to wealth ratio is 1.75% or \( \frac{W(0)}{C(0)} = 57 \). With an expected return on stocks of \( \theta = 1.452\% \) and the percentage invested in stocks 96.89% the investor is spending more than the return on her portfolio, so that wealth is expected to decrease slowly over time.

![Figure 2](image-url)

Figure 3 uses the same parameters with \( T = 200 \) so that the investor’s time horizon is now 50 years. Again we want the exponential function to be less than one, so that it converges to zero over
time. In this case the optimal wealth to consumption ratio increases to \( \frac{W(0)}{C(0)} = 62.8 \), so that the investor now spends 1.59% of her wealth each quarter. Consequently, an investor with a longer time horizon chooses to set aside more to assure that the wealth declines toward zero at a slower rate. The amount invested in stocks decreases by about 3% to 93.44%. Thus, the longer time horizon makes the investor more cautious.

![Figure 3](image)

To understand how the optimal comes about, we consider three possible levels of the initial consumption to wealth ratio in the graphs so that the choice of optimal consumption to wealth ratio and investment may be illustrated. In Figure 4 we graph the solution for \( \psi = 1 \ 1/3 \). The optimal consumption to wealth (2.9) is \( \frac{C(0)}{W(0)} = 1.754\% \) at \( \theta = 1.452\% \). In addition, the optimal percentage of wealth invested in stocks (2.10) is \( \alpha(0) = 96.89\% \). As a result, the investor spends more that the return on her portfolio so that her wealth slowly declines by \(-0.3399\%\) over time at \( \theta = 1.452\% \).

To see how this result is found in each graph of Figure 4, we plot the current lifetime utility for \( \frac{W(0)}{C(0)} \) of 47, 57, and 67 which are represented by the dash (blue), dash dot (green), and solid (red) curves, respectively. The middle value corresponds to the optimal decision. The first graph gives the current lifetime utility which is positively related to the wealth to consumption ratio. The investor
does not go to the higher lifetime utility, dash (blue) curve, since the expected change in lifetime utility is positive. The change in the expected lifetime utility is all the integrands in (3.35). As a result, the area under this graph gives a sufficient condition for the convergence of the integral in (3.36), since we are integrating over the possible values of the expected return on stocks, $\mu$. Thus, the wealth consumption ratio of 57 satisfies the transversality condition. The individual does not want to go to the higher wealth to consumption level, since it lowers her lifetime utility, solid (red) curve. Thus, the middle curves represent the optimal decision for the investor.

The optimal consumption to wealth ratio is positively related to the expected return on stocks, as suspected with an intertemporal elasticity of consumption above 1. However, there is little variation in the optimal consumption to wealth ratio which ranges from 1.737% to 1.83% or 10 basis points. This explains why there is little variation in the lifetime utility of the investor in response to the expected return on stocks. This stability is accomplished with little variation in optimal investment relative to the myopic investment, which we represent by the long dash (black) line in the graph of percentage of wealth invested in stock. In particular, the hedging demand starts at $-20.18\%$ at the lowest expected return, rises to 3.46% at the stationary expected return $\theta = 1.452\%$, and declines to $-46.04\%$. Compared to Campbell et. al. the level of hedging is significantly lower under the optimal solution. In addition, the hedging demand is not linear as is implied by the approximation (2.13) of Campbell et. al.. Thus, the optimal decision of the investor is significantly different from that reported by Campbell et. al..

The optimal behavior is also significantly different from that found in known solutions to the problem in which investors only care about the terminal wealth of their portfolio. The typical investment strategy is not optimal and can be represented by the solid (red) curves in Figure 4. In this case the hedging demand is positively related to the expected return on stocks. For example, the investor would go long in stocks for high expected return, i.e., a high dividend-price ratio, so that wealth grows. However, we see that the level of consumption falls since the investor is allocating more resources to stocks at a time when the actual return on stocks is low. The optimal policy minimizes

\footnote{Strictly speaking in the solution to (3.36) we take a weighted average based on the normal probability distribution (3.33) of $\mu$, so that the values of the expected return on stocks near $\theta = 1.452\%$ are more important.}
this variation in consumption, while at the same time increases the level of the consumption to wealth ratio at all levels of the expected return on stocks.

4.1.1 Increasing aversion to risk

We can also change the coefficient of risk aversion $\gamma$. In Figure 5 we look at the exponential function in (3.36) to find the optimal level of consumption when the coefficient of risk aversion doubles to $\gamma = 4$. All the other parameters are the same as the first example considered in this section. In particular the time horizon is taken to be one quarter $T = 1$. In this case the transversality condition is satisfied when consumption is 1.46% of wealth. Thus, the investor consumes a smaller percentage of her wealth, when the coefficient of risk aversion doubles. In Figure 6 the optimal consumption to wealth ratio is plotted against the expected return on stocks. For lower levels of the expected return on stocks the consumption to wealth ratio is positively related to the expected return on stocks, but it tends to decline for expected stock returns above $\mu(0) = \theta = 1.452\%$. 

Figure 4
Figure 5

Figure 6

The investor now places 49.81% of her wealth in stocks, when $\mu(0) = \theta = 1.452\%$. The myopic percentage invested in stocks is 47.03%. As a result, the hedging demand is reduced to 2.78%. Consequently, the hedging demand at the stationary expected return on stocks is reduced by 0.68% when the coefficient of risk aversion doubles to four. In Figure 6 we plot the optimal investment (* line), hedging demand (dash line), and myopic investment (□ line). The investor does not do much hedging when the expected return goes above its steady state value. The highest value of hedging demand is 22.8%. On the other hand there is rather substantial short selling when the expected return on stocks is below its steady state value. For example the hedging demand is $-100\%$ for an
expected return on stocks of $\mu(0) = 0.7\%$.

4.2 The intertemporal elasticity of consumption less than one

In Figure 8 we consider the optimal decision of the investor when the intertemporal elasticity of substitution goes below one to $\psi = 2/3$, while keeping the coefficient of risk aversion at $\gamma = 2$. Again the middle curves corresponds to the optimal decision of the investor at $\frac{W(0)}{C(0)} = 83$ at $\mu = \theta$, while the dash (blue) and solid (red) curves corresponds to subtracting and adding 10 to the optimal wealth to consumption ratio, respectively. The order of the graphs have reversed in that the higher wealth to consumption ratio is associated with the larger lifetime utility. In this case the behavior of the lifetime utility in the extreme cases is important for determining the optimal policy for the investor. For example the lower wealth to consumption ratio $\frac{W(0)}{C(0)} = 73$ is inferior, since the lower tail of the distribution for the expected return on stocks is associated with a higher expected change in the lifetime utility function. In fact we had to increase the range to $\theta - 2\frac{\sigma_\mu}{\sqrt{2}} \leq \mu \leq \theta + 2\frac{\sigma_\mu}{\sqrt{2}}$ for the graph of the expected change of the lifetime utility function to identify the difference in the transversality condition (3.36).

In this case the optimal at $\mu = \theta$ consists of a consumption to wealth ratio of $\frac{C(0)}{W(0)} = 1.205\%$ and a percentage invested in stocks of $\alpha(0) = 94.02\%$ so that the investor’s wealth is expected to grow at the rate $0.1752\%$. Thus, a lower intertemporal elasticity of consumption tends to generate
positive growth in wealth, while wealth falls when the elasticity is greater than one. In addition, the optimal consumption to wealth ratio falls as the expected return on stocks increases, so that investors tend to save more in response to a higher expected return on stocks. This behavior contributes to larger growth of wealth and a higher lifetime utility as the expected return on stocks increases. The hedging demand for stocks follows a similar pattern as the intertemporal elasticity of consumption changes. For an expected return on stocks of \( \theta - \frac{\sigma}{\sqrt{2\kappa}} \), the hedging demand is \(-14.08\%\). There is a slight positive demand of 0.61% at the stationary value of the expected return on stocks. For the higher expected return on stocks \( \theta + \frac{\sigma}{\sqrt{2\kappa}} \), the hedging demand for stocks falls to \(-52.93\%\). Thus, the hedging demand for stocks is a non-linear function of the expected return on stocks.

![Graphs showing various economic indicators](image.png)

Figure 8

5 Appendix

Proof of Theorem 3.1 and Corollary 3.2.
The Power Series Solution. Notice that the standard deviation $\sigma_\mu$ in Eq.(2.11) is small. Apply the change of variable:

$$\mu = \epsilon \sigma_\mu x \quad \text{for some } \epsilon \geq 1.$$ (5.1)

By the chain rule, we obtain

$$\frac{dH}{dx} = \epsilon \sigma_\mu \frac{dH}{d\mu} \quad \text{and} \quad \frac{d^2H}{dx^2} = \epsilon^2 \sigma_\mu^2 \frac{d^2H}{d\mu^2}.$$ (5.2)

In terms of the new variable $x$, the initial value problems (2.11) and (3.14) can be rewritten as

$$H'' = -2\epsilon^2 \beta \psi \frac{H'}{H} + \frac{\epsilon^4 \sigma_\mu^2 (1 - \psi)}{\gamma \sigma_S^2} \left( x - \frac{r}{\epsilon \sigma_\mu} \right)^2 + 2\epsilon^2 r (1 - \psi) + 2\epsilon^2 \beta \psi$$

$$+ 2\epsilon^2 \left[ \kappa \left( x - \frac{\theta}{\epsilon \sigma_\mu} \right) - \rho \sigma_\mu (1 - \gamma) \left( x - \frac{r}{\epsilon \sigma_\mu} \right) \right] \left( \frac{H'}{H} \right)^2$$

$$+ \left[ 1 + \frac{1 - \gamma}{\gamma (1 - \psi)} + \frac{\rho^2 (1 - \gamma)^2}{\gamma (1 - \psi)} \right] \left( \frac{H'}{H} \right)^2$$

and

$$H(x_0) = \tilde{h}_0 \quad \text{and} \quad H'(x_0) = \epsilon \sigma_\mu \tilde{h}_1,$$ (5.4)

where $x_0 = \mu_0 / (\epsilon \sigma_\mu)$.

Eq.(5.3) is also equivalent to

$$HH'' = A_0 (H')^2 + [B_1 (x - x_0) + B_0] H H' + \left[ C_2 (x - x_0)^2 + C_1 (x - x_0) + C_0 \right] H^2 + D_0 H,$$ (5.5)

where

1. $A_0 = 1 + \frac{1 - \gamma}{1 - \psi} + \frac{\rho^2 (1 - \gamma)^2}{\gamma (1 - \psi)}$,
2. $B_0 = 2\epsilon^2 \left[ \kappa \left( x_0 - \frac{\theta}{\epsilon \sigma_\mu} \right) - \rho \sigma_\mu (1 - \gamma) \left( x_0 - \frac{r}{\epsilon \sigma_\mu} \right) \right]$,
3. $B_1 = 2\epsilon^2 \left[ \kappa - \rho \sigma_\mu (1 - \gamma) \right]$,
4. $C_0 = \frac{\epsilon^4 \sigma_\mu^2 (1 - \psi)}{\gamma \sigma_S^2} \left( x_0 - \frac{r}{\epsilon \sigma_\mu} \right)^2 + 2\epsilon^2 r (1 - \psi) + 2\epsilon^2 \beta \psi$,
5. $C_1 = \frac{2\epsilon^4 \sigma_\mu^2 (1 - \psi)}{\gamma \sigma_S^2} \left( x_0 - \frac{r}{\epsilon \sigma_\mu} \right)$,
6. $C_2 = \frac{\epsilon^4 \sigma_\mu^2 (1 - \psi)}{\gamma \sigma_S^2}$,
7. $D_0 = -2\epsilon^2 \beta \psi$. 

26
Suppose that Eq.(5.5) has a power series solution of the form:

\[ H(x) = \sum_{n=0}^{\infty} h_n(x - x_0)^n \quad \text{for} \quad x_0 - r_x < x < x_0 + r_x, \tag{5.6} \]

where \( h_0 = \tilde{h}_0 \) and \( h_1 = \sigma_\mu \tilde{h}_1 \) are determined by the initial conditions in (5.4). Then

\[ H'(x) = \sum_{n=0}^{\infty} (n + 1)h_{n+1}(x - x_0)^n \quad \text{and} \quad H''(x) = \sum_{n=0}^{\infty} (n + 1)(n + 2)h_{n+2}(x - x_0)^n. \tag{5.7} \]

We can also calculate

\[
H(x)H''(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (k + 1)(k + 2)h_{k+2}h_{n-k} \right) (x - x_0)^n, \tag{5.8}
\]

\[
[H'(x)]^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (k + 1)(n - k + 1)h_{k+1}h_{n-k+1} \right) (x - x_0)^n, \tag{5.9}
\]

\[
H(x)H'(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (k + 1)h_{k+1}h_{n-k} \right) (x - x_0)^n, \tag{5.10}
\]

\[
(H(x))^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \tilde{h}_k h_{n-k} \right) (x - x_0)^n. \tag{5.11}
\]

By Eqs.(5.10) and (5.11), we get

\[
[B_1(x - x_0) + B_0] H(x)H'(x) = \sum_{n=0}^{\infty} \left[ B_1 \sum_{k=0}^{n-1} (k + 1)h_{k+1}h_{n-k-1} + B_0 \sum_{k=0}^{n} (k + 1)h_{k+1}h_{n-k} \right] (x - x_0)^n \tag{5.12}
\]

and

\[
[C_2(x - x_0)^2 + C_1(x - x_0) + C_0] [H(x)]^2 = \sum_{n=0}^{\infty} \left[ C_2 \sum_{k=0}^{n-2} \tilde{h}_k h_{n-k-2} + C_1 \sum_{k=0}^{n-1} \tilde{h}_k h_{n-k-1} + C_0 \sum_{k=0}^{n} \tilde{h}_k h_{n-k} \right] (x - x_0)^n. \tag{5.13}
\]

Substitute Eqs.(5.6) through (5.13) into Eq.(5.5) and then equate the coefficients of the same degree in the power series.

\[
\sum_{k=0}^{n} (k + 1)(k + 2)h_{k+2}h_{n-k} = A_0 \sum_{k=0}^{n} (k + 1)(n - k + 1)h_{k+1}h_{n-k+1}
\]

\[+ B_1 \sum_{k=0}^{n-1} (k + 1)h_{k+1}h_{n-k-1} + B_0 \sum_{k=0}^{n} (k + 1)h_{k+1}h_{n-k} \]

\[+ C_2 \sum_{k=0}^{n-2} \tilde{h}_k h_{n-k-2} + C_1 \sum_{k=0}^{n-1} \tilde{h}_k h_{n-k-1} + C_0 \sum_{k=0}^{n} \tilde{h}_k h_{n-k} + D_0 h_n \tag{5.14}
\]
Solve Eq. (5.14) for the term on the left hand side that contains $h_{n+2}$:

$$(n + 1)(n + 2)h_0 h_{n+2} = A_0 \sum_{k=0}^{n} (k + 1)(n - k + 1)h_{k+1}h_{n-k+1}$$

$$+ B_1 \sum_{k=0}^{n-1} (k + 1)h_{k+1}h_{n-k-1} + B_0 \sum_{k=0}^{n} (k + 1)h_{k+1}h_{n-k}$$

$$+ C_2 \sum_{k=0}^{n-2} h_k h_{n-k-2} + C_1 \sum_{k=0}^{n-1} h_k h_{n-k-1} + C_0 \sum_{k=0}^{n} h_k h_{n-k}$$

$$+ D_0 h_n - \sum_{k=0}^{n-1} (k + 1)(k + 2)h_{k+2}h_{n-k}$$

(5.15)

The recurrence relation (5.15) determines all coefficients in the power series (5.6).

**The Convergence of Power Series Solution.** Rewrite Eq. (5.15) in the following equivalent forms. Dividing Eq. (5.15) by $h_0^2$ gives

$$(n + 1)(n + 2)h_{n+2}/h_0 = A_0 \sum_{k=0}^{n} (k + 1)(n - k + 1)(h_{k+1}/h_0)(h_{n-k+1}/h_0)$$

$$+ B_1 \sum_{k=0}^{n-1} (k + 1)(h_{k+1}/h_0)(h_{n-k-1}/h_0) + B_0 \sum_{k=0}^{n} (k + 1)(h_{k+1}/h_0)(h_{n-k}/h_0)$$

$$+ C_2 \sum_{k=0}^{n-2} (h_k/h_0)(h_{n-k-2}/h_0) + C_1 \sum_{k=0}^{n-1} (h_k/h_0)(h_{n-k-1}/h_0) + C_0 \sum_{k=0}^{n} (h_k/h_0)(h_{n-k}/h_0)$$

$$+ (D_0/h_0)(h_n/h_0) - \sum_{k=0}^{n-1} (k + 1)(k + 2)(h_{k+2}/h_0)(h_{n-k}/h_0).$$

(5.16)

For $n \geq 4$, Eq. (5.16) is equivalent to

$$(n + 1)(n + 2)(h_{n+2}/h_0) = A_0 \sum_{k=0}^{n} (k + 1)(n - k + 1)(h_{k+1}/h_0)(h_{n-k+1}/h_0)$$

$$+ B_1 \sum_{k=0}^{n-2} (k + 1)(h_{k+1}/h_0)(h_{n-k-1}/h_0) + B_0 \sum_{k=0}^{n-1} (k + 1)(h_{k+1}/h_0)(h_{n-k}/h_0)$$

$$+ C_2 \sum_{k=1}^{n-3} (h_k/h_0)(h_{n-k-2}/h_0) + C_1 \sum_{k=1}^{n-2} (h_k/h_0)(h_{n-k-1}/h_0) + C_0 \sum_{k=1}^{n-1} (h_k/h_0)(h_{n-k}/h_0)$$

$$- \sum_{k=0}^{n-1} (k + 1)(k + 2)(h_{k+2}/h_0)(h_{n-k}/h_0) + nB_1(h_n/h_0) + (n + 1)B_0(h_{n+1}/h_0)$$

$$+ 2C_2(h_{n-2}/h_0) + 2C_1(h_{n-1}/h_0) + 2C_0(h_n/h_0) + (D_0/h_0)(h_n/h_0).$$

(5.17)
Define \( \hat{h}_n = n^2L(h_n/h_0) \) for all \( n \geq 1 \), where \( L \) is a positive number to be determined later. For \( n \geq 4 \), rewrite Eq. (5.17) in terms of \( \hat{h}_n \) as

\[
\hat{h}_{n+2} = \frac{n+2}{n+1} \left[ A_0 \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} \hat{h}_{k+1} \hat{h}_{n-k+1} \right. \\
+ \frac{B_1}{L} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)^2} \hat{h}_{k+1} \hat{h}_{n-k-1} + \frac{B_0}{L} \sum_{k=0}^{n-1} \frac{1}{(k+1)(n-k)^2} \hat{h}_{k+1} \hat{h}_{n-k} \\
+ \frac{C_2}{L} \sum_{k=1}^{n-3} \frac{1}{k^2(n-k-2)^2} \hat{h}_k \hat{h}_{n-k-2} + \frac{C_1}{L} \sum_{k=1}^{n-2} \frac{1}{k^2(n-k-1)^2} \hat{h}_k \hat{h}_{n-k-1} \\
+ \frac{C_0}{L} \sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} \hat{h}_k \hat{h}_{n-k} - \frac{1}{L} \sum_{k=0}^{n-1} \frac{k+1}{(k+2)(n-k)^2} \hat{h}_{k+2} \hat{h}_{n-k} \\
+ \frac{B_1}{n} \hat{h}_n + \frac{B_0}{n+1} \hat{h}_{n+1} + \frac{2C_2}{(n-2)^2} \hat{h}_{n-2} + \frac{2C_1}{(n-1)^2} \hat{h}_{n-1} + \frac{2C_0}{n^2} \hat{h}_n + \frac{D_0/h_0}{n^2} \hat{h}_n \right].
\]  

(5.18)

**Lemma 5.1.** If \( a \) and \( b \) are two nonnegative integers, then

\[
\lim_{n \to \infty} \frac{1}{n+a} \sum_{k=1}^{n-b} \frac{1}{k} = 0.
\]  

(5.19)

**Proof.** Write \( c \) for Euler’s constant. We have

\[
\lim_{n \to \infty} \left[ \sum_{k=1}^{n-b} \frac{1}{k} - \ln(n-b) \right] = c \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n+a} \left[ \sum_{k=1}^{n-b} \frac{1}{k} - \ln(n-b) \right] = 0.
\]

Then Eq. (5.19) follows immediately from the fact that

\[
\lim_{n \to \infty} \frac{\ln(n-b)}{n+a} = 0. \quad \blacksquare
\]

**Lemma 5.2.** If \( a, b, c, d \) are integers such that \( a \geq 0, \ b \geq 0, \ a + c > 0, \) and \( b + d > 0 \), then

\[
\lim_{n \to \infty} \sum_{k=a}^{n-b} \frac{1}{(k+c)(n-k+d)} = 0.
\]  

(5.20)

**Proof.** By Lemma 5.1, we obtain

\[
\lim_{n \to \infty} \sum_{k=a}^{n-b} \frac{1}{(k+c)(n-k+d)} = \lim_{n \to \infty} \frac{1}{n+c+d} \sum_{k=a}^{n-b} \left( \frac{1}{k+c} + \frac{1}{n-k+d} \right) \\
= \lim_{n \to \infty} \frac{1}{n+c+d} \left( \sum_{k=a+c}^{n-b+c} \frac{1}{k} + \sum_{k=b+d}^{n-a+d} \frac{1}{k} \right) = 0. \quad \blacksquare
\]
Note that $\sum_{k=1}^{\infty}(1/k^2) = \pi^2/6$. There exist some real number $L > 0$ and integer $N \geq 4$ such that for all $n \geq N$, we have

$$
\frac{n+2}{n+1} \left[ \frac{|A_0|}{L} \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} + \frac{|B_1|}{L} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)^2} + \frac{|B_0|}{L} \sum_{k=0}^{n-1} \frac{1}{(k+1)(n-k)^2} + \frac{|C_2|}{L} \sum_{k=1}^{n-3} \frac{1}{k^2(n-k-2)^2} + \frac{|C_1|}{L} \sum_{k=1}^{n-2} \frac{1}{k^2(n-k-1)^2} + \frac{|C_0|}{L} \sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} \right] < 1. 
$$

(5.21)

Apply the following algorithm to construct a sequence $\{H_n\}$ of nonnegative numbers.

1. Use the recurrence relation (5.15) and the given values of $h_0$ and $h_1$ to calculate $h_n$, where $2 \leq n \leq N + 1$.

2. Calculate $H_n = n^2L|h_n/h_0|$ for $1 \leq n \leq N + 1$.

3. Calculate the remaining terms $H_{n+2}$ with $n \geq N$ by the recurrence relation:

$$
H_{n+2} = \frac{n+2}{n+1} \left[ \frac{|A_0|}{L} \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} H_{k+1}H_{n-k+1} + \frac{|B_1|}{L} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)^2} H_{k+1}H_{n-k-1} + \frac{|B_0|}{L} \sum_{k=0}^{n-1} \frac{1}{(k+1)(n-k)^2} H_{k+1}H_{n-k} + \frac{|C_2|}{L} \sum_{k=1}^{n-3} \frac{1}{k^2(n-k-2)^2} H_kH_{n-k-2} + \frac{|C_1|}{L} \sum_{k=1}^{n-2} \frac{1}{k^2(n-k-1)^2} H_kH_{n-k-1} + \frac{|C_0|}{L} \sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} H_kH_{n-k} + \frac{1}{L} \sum_{k=0}^{n-1} \frac{k+1}{(k+2)(n-k)^2} H_{k+2}H_{n-k} \right] + \frac{|B_1|}{n} H_n + \frac{|B_0|}{n+1} H_{n+1} + \frac{2|C_2|}{(n-2)^2} H_{n-2} + \frac{2|C_1|}{(n-1)^2} H_{n-1} + \frac{2|C_0|}{n^2} H_n + \frac{|D_0/h_0|}{n^2} |H_n|.
$$

(5.22)

Choose a real number $M \geq 1$ such that $H_n \leq M^n$ for $1 \leq n \leq N + 1$. By induction on $n$, we can show that

$$
n^2L|h_n/h_0| \leq H_n \leq M^n \quad \text{or} \quad |h_n| \leq \frac{|h_0|}{L} \frac{M^n}{n^2} \quad \text{for all } n \geq 1.
$$

(5.23)

Theorem 3.1 follows easily from the second inequality in (5.23) and the relation $\mu = \sigma_\mu x$ that
Note that
\[
H'(\mu) = \sum_{n=0}^{\infty} \frac{(n+1)h_{n+1}(\mu - \mu_0)^{n+1}}{(\epsilon \sigma \mu)^n}.
\] (5.24)
so that Corollary 3.2 follows from Theorem 3.1.

**Stochastic differential equation for lifetime utility** (3.30). Applying Ito’s lemma to (2.8) using the stochastic differential equations for the expected return on stocks (2.2) and (3.29) yields the following stochastic differential equation which is represented by (3.30).

\[
\begin{align*}
\frac{de^{-\beta t}J(\mu(t), W(t))}{e^{-\beta t}J(\mu(t), W(t))} &= \left[ -\beta - \frac{1 - \gamma}{1 - \psi} \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \kappa(\theta - \mu(t)) + \right. \\
&\left. (1 - \gamma) \left[ r - \frac{\beta \psi}{H(\mu(t); W(0) C(0), \alpha(0))} + \right. \\
&\frac{1}{\gamma} \left\{ \frac{\mu(t) - r}{\sigma_S^2} - K_0 \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \right\} (\mu(t) - r) \\
&\left. + \frac{1}{2} \left\{ \frac{1 - \gamma}{1 - \psi} \left[ 1 - \gamma \right] \left( \frac{1}{1 - \psi} + 1 \right) \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \right. \\
&\left. - \frac{1 - \gamma}{1 - \psi} \left[ 1 - \gamma \right] \frac{H''(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \right\} \sigma^2_{\mu} + \\
&\left. - \frac{1 - \gamma}{1 - \psi} (1 - \gamma) \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \frac{\sigma_S \sigma_{\mu \rho}}{\gamma} \left\{ \frac{\theta - r}{\sigma_S^2} - K_0 \right\} \\
&\left. - \frac{1}{2} \frac{(1 - \gamma)\sigma_S^2}{\gamma} \left\{ \frac{\mu(t) - r}{\sigma_S^2} - K_0 \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \right\}^2 dt \\
&\left. + \frac{(1 - \gamma) \sigma_S}{\gamma} \left\{ \frac{\mu(t) - r}{\sigma_S^2} - K_0 \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \right\} d\omega_{S,t} \\
&\left. - \frac{1 - \gamma}{1 - \psi} \frac{H'(\mu(t); W(0) C(0), \alpha(0))}{H(\mu(t); W(0) C(0), \alpha(0))} \sigma_{\mu} d\omega_{\mu,t}, \right. \end{align*}
\]
where

$$K_0 = \frac{\rho \sigma \mu}{\sigma_s} \frac{1 - \gamma}{1 - \psi}.$$ 

Gauss-Hermite Quadrature Procedure applied to \( (5.25) \). Let \( \omega_i \) and \( x_i \) for \( i = 1 \cdots n \) be the Gauss-Hermite quadrature weights following Judd (1998, pp. 261-263) and Stoer and Bullirsch (2002, pp. 171-181). The expectation of a function is approximated by

$$E[f(x)] = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i f(\sqrt{2} \sigma x_i + \mu). \tag{5.26}$$

In this case we want to calculate the expectation in \( (3.28) \), where we take

$$\mu(t) = \theta + \exp[-\kappa t] (\mu(0) - \theta), \quad \text{and} \quad \sigma(t)^2 = \frac{\sigma_x^2}{2\kappa} \left[ 1 - \exp[-2\kappa t] \right].$$

We also need to have finite values of time \( t \) where we keep the exponential function constant so that the deterministic integral from 0 to \( T \) is calculated.

$$= \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i \exp \left\{ \sum_{t=0}^{N} \left[ -\beta - \frac{1 - \gamma}{1 - \psi} H \left( \frac{\sigma(t)x_i + \mu(t)}{\sigma(0)}; \frac{W(0)}{C(0)} \right) \kappa (\theta - \sigma(t)x_i - \mu(t)) + \frac{\beta^2}{H(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))} \right] \right\}$$

$$+ \frac{1}{2} \left\{ \frac{1 - \gamma}{1 - \psi} \right\} \left[ \frac{H'(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))}{H(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))} \right]^2 - \frac{1 - \gamma}{1 - \psi} \frac{H''(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))}{H(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))} \right\} \left\{ \frac{\theta - r}{\sigma_x^2} - K_0 \right\}$$

$$- \frac{1 - \gamma}{1 - \psi} \left( 1 - \gamma \right) \frac{H'(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))}{H(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))} \frac{\sigma_x \sigma \mu}{\gamma} \left\{ \frac{\theta - r}{\sigma_x^2} - K_0 \right\}$$

$$- \frac{1}{2} \frac{1 - \gamma}{\gamma} \left( \frac{1}{1 - \psi} \right) \left\{ \frac{\sigma(t)x_i + \mu(t) - r}{\sigma_x^2} - K_0 \right\} H' \left( \frac{\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0)}{H(\sigma(t)x_i + \mu(t); \frac{W(0)}{C(0)}, \alpha(0))} \right)^2$$

32
\begin{align*}
+ \frac{1}{2} \left[ (1 - \gamma) \frac{\sigma_S}{\gamma} \left\{ \frac{\sigma(t)x_i + \mu(t) - r}{\sigma_S^2} - K_0 \frac{H' \left( \frac{\sigma(t)x_i + \mu(t); W(0, C(0))}{\alpha(0)} \right)}{H \left( \frac{\sigma(t)x_i + \mu(t); W(0, C(0))}{\alpha(0)} \right)} \right\} \right]^2 \\
- \rho \left[ (1 - \gamma) \frac{\sigma_S}{\gamma} \left\{ \frac{\sigma(t)x_i + \mu(t) - r}{\sigma_S^2} - K_0 \frac{H' \left( \frac{\sigma(t)x_i + \mu(t); W(0, C(0))}{\alpha(0)} \right)}{H \left( \frac{\sigma(t)x_i + \mu(t); W(0, C(0))}{\alpha(0)} \right)} \right\} \right]^2 \\
\times \left[ 1 - \frac{\gamma}{\alpha} \frac{H' \left( \frac{\sigma(t)x_i + \mu(t); W(0, C(0))}{\alpha(0)} \right)}{H \left( \frac{\sigma(t)x_i + \mu(t); W(0, C(0))}{\alpha(0)} \right)} \sigma_\mu \right]
\end{align*}

Here \( \Delta = \frac{T}{N} \) with \( N \) significantly bigger than \( T \).

**Parameter Estimation for (2.1) and (2.2).** We estimate the vector autoregressive system (VAR) of Campbell and Viceira (1999) using data from 1947.1 to 2006.4.

\[
\begin{pmatrix}
  r_{1,t+1} - r_f \\
  d_{t+1} - p_{t+1}
\end{pmatrix}
= \begin{pmatrix}
  \theta_0 \\
  \theta_1
\end{pmatrix}
+ \begin{pmatrix}
  \beta_0 \\
  \beta_1
\end{pmatrix}
\begin{pmatrix}
  d_t - p_t \\
  \epsilon_{1,t+1}
\end{pmatrix}
+ \begin{pmatrix}
  \epsilon_{2,t+1}
\end{pmatrix}.
\]  

The variance-covariance is

\[
\begin{pmatrix}
  \Omega_{11} & \Omega_{12} \\
  \Omega_{21} & \Omega_{22}
\end{pmatrix}
= E_t \begin{pmatrix}
  \epsilon_{1,t+1} & \epsilon_{2,t+1}
\end{pmatrix}
\begin{pmatrix}
  \epsilon_{1,t+1} \\
  \epsilon_{2,t+1}
\end{pmatrix}.
\]

Here \( r_{1,t+1} \) is the logarithm of the stock return at time \( t + 1 \), \( r_f \) is the logarithm of the return on a one quarter government bond. \( d_t - p_t \) is the logarithm of the dividend-price ratio at time \( t \). \( E_t(\cdot) \) refers to the expectation conditional on information at time \( t \).

In Table A1 the estimates are updated for the time period 1947 quarter 1 to 2006 quarter 4. The data is quarterly returns from CRSP for this time period. The CRSP value-weighted index for NYSE, AMEX and NASDAQ stock is used for the stock return \( r_{1,t+1} \). The return with and without dividends from CRSP are used to calculate the stock price and dividend series. The three-month Treasury bill yield is from the Risk Free File of the CRSP bond tape is used for the risk free rate \( r_f = 0.0025 \) per quarter.

Following Assumption (A2) of CV (1999) we want to represent the equity premium as

\[
E_t \left[ r_{1,t+1} - r_f \right] = x_t,
\]

33
where
\[ x_{t+1} = \mu + \phi (x_t - \mu) + \eta_{t+1}. \] (5.30)

Based on the VAR (5.27) the state variable is
\[ x_t = \theta_0 + \theta_1 (d_t - p_t). \] (5.31)

The second equation in the VAR may be written as
\[ d_{t+1} - p_{t+1} = \beta_0 + \beta_1 (d_t - p_t) + \epsilon_{2,t+1}. \]

Let the lag operator \( L \) be defined so that
\[ L (d_{t+1} - p_{t+1}) = (d_t - p_t). \]

As a result, we have
\[ d_{t+1} - p_{t+1} = \frac{\beta_0}{1 - \beta_1 L} + \frac{\epsilon_{2,t+1}}{1 - \beta_1 L}. \]

Now substitute this result into (5.31) to get
\[ x_{t+1} = \theta_0 + \theta_1 \left( \frac{\beta_0}{1 - \beta_1 L} + \frac{\epsilon_{2,t+1}}{1 - \beta_1 L} \right). \] (5.32)

As a result we have
\[ x_{t+1} = \theta_0 + \theta_1 \frac{\beta_0}{1 - \beta_1} + \beta_1 \left( x_t - \theta_0 - \theta_1 \frac{\beta_0}{1 - \beta_1} \right) + \theta_1 \epsilon_{2,t+1}. \] (5.33)

Introduce the definitions
\[ \mu = \theta_0 + \theta_1 \frac{\beta_0}{1 - \beta_1}, \quad \phi = \beta_1, \]
and
\[ \eta_{t+1} = \theta_1 \epsilon_{2,t+1}, \]
so that (5.33) becomes
\[ x_{t+1} = \mu + \phi (x_t - \mu) + \eta_{t+1}. \] (5.34)

This corresponds to equation 3 of CV.
Use the parameters in Table A1 we get

\[ \mu = 0.008802789, \quad \phi = 0.984771. \]

For the standard error terms we get

\[ \sigma_\epsilon = \sqrt{\Omega_{11}} = 0.079176, \sigma_\eta = \theta_1 \sqrt{\Omega_{22}} = 0.00212805, \text{ and } \sigma_{\epsilon \eta} = \theta_1 \Omega_{12} = -0.000161287. \]

To set the parameters of the model we use equations (16) to (21) of CCRV with \( \Delta t = 1. \) By (16) and (18) we have

\[ r = r_f = 0.0025 \text{ and } \kappa = -\ln(\phi) = 0.015346. \]

Equation (19) of CCRV yeilds

\[ \sigma_\mu = \sigma_\eta \sqrt{\frac{2\kappa^3}{(1-e^{-\kappa})^2(1-e^{-2\kappa})}} = 0.002161. \]

From Equation (20) we have

\[ \sigma_{\mu S} = \frac{\kappa^2}{(1-e^{-\kappa})^2} \left[ \sigma_\epsilon + \frac{\sigma_\mu^2}{2\kappa^3} (1-e^{-2\kappa})(1-e^{-\kappa}) - \frac{\sigma_\mu^2}{\kappa^3} (1-e^{-\kappa})^2 \right] = 0.00016609475. \]

From equation (21) of CCRV we have

\[ \sigma_S = \sqrt{\sigma_\epsilon^2 + \frac{2\sigma_{\mu S}^2}{\kappa^2} (1-e^{-\kappa}) - \frac{2\sigma_{\mu S}^2}{\kappa^2} - \frac{2\sigma_\mu^2}{\kappa^3} (1-e^{-\kappa}) - \frac{\sigma_\mu^2}{\kappa^3} (1-e^{-2\kappa})} = 0.08020384093. \]

We also have

\[ \rho = \frac{\sigma_{\mu S}}{\sigma_S \sigma_\mu} = -.9583564720. \]

Finally, equation (17) yields

\[ \theta = \mu + \frac{\sigma_S^2}{2} + r = 0.01451911726. \]

<table>
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Notes: These estimates reproduces Campbell and Viciera (1999) over the longer time period from first quarter of 1947 to the last quarter of 2006. The standard errors are in parenthesis. The ML estimates are for the vector autoregressive system

$$\begin{pmatrix} r_{1,t+1} - r_f \\ d_{t+1} - p_{t+1} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \beta_1 \end{pmatrix} \begin{pmatrix} d_t - p_t \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{pmatrix}$$

The variance-covariance is

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = E_t \begin{pmatrix} \epsilon_{1,t+1} & \epsilon_{1,t+1} \\ \epsilon_{2,t+1} & \epsilon_{2,t+1} \end{pmatrix}$$

Quarterly returns from CRSP for this time period is used. The CRSP value-weighted index for NYSE, AMEX and NASDAQ stock is used for the stock return $r_{1,t+1}$. The return with and without dividends from CRSP are used to calculate the stock price and dividend series. The three-month Treasury bill yield is from the Risk Free File of the CRSP bond tape is used for the risk free rate $r_f = 0.0025$ per quarter.
References


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