DEFINABLE TYPES AND F-GENERICS IN PRESBURGER ARITHMETIC

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The goal of this note is to characterize certain definable types and f-generics in Presburger arithmetic (i.e., the complete first-order theory of the ordered group of integers $(\mathbb{Z}, +, <, 0)$). Specifically, we consider f-generic types in $S_n(G)$, where G is a saturated model of Presburger arithmetic (i.e. f-generic types with respect to the definable group $\mathbb{Z}^n$). Let G^* be sufficiently saturated monster model in which we can realize global types over G. Given $p \in S_n(G)$, and a realization $\bar{a} \models p$, we let $G(\bar{a})$ denote the divisible hull of the subgroup of G^* generated by $G\bar{a}$ (i.e. $G(\bar{a})$ is the definable closure of $G\bar{a}$ in G^*). Our main characterization is: given $p \in S_n(G)$ and $\bar{a} \models p$ in G^*, p is f-generic if and only if

(i) $\bar{a}$ is algebraically independent over G, and
(ii) $G(\bar{a})$ is an end extension of G (i.e. if $x \in G(\bar{a})\backslash G$ then $x > G$ or $x < G$).

Along the way, we give a similar characterization of definable types in $S_n(G)$, and we show that all f-generics are definable over $\emptyset$.

0.1. **Definable groups and f-generics.** Let T be a complete theory, with a monster model M. We also work with a larger monster model M^* in which we can take realizations of global types over M.

Suppose $G = G(M)$ is a definable group in T. We let $S_G(M)$ denote the space of global types containing the formula defining G. Given $p \in S_G(M)$ and $g \in G$, we let gp denote the translate $\{\varphi(g^{-1}x) : \varphi(x) \in p\}$ of p.

Definition 0.1. Let $p \in S_G(M)$ be a global G-type.

(a) p is **definable (over G)** if, for any formula $\varphi(\bar{x}, \bar{y})$, there is a formula $d_p[\varphi](\bar{y})$ over G such that, for any $\bar{b} \in G$, $\varphi(\bar{x}, \bar{b}) \in p$ if and only if $G \models d_p[\varphi](\bar{b})$.
(b) p is a **(strong) f-generic** if there is a small model M_0 such that no translate gp of p forks over M_0.
(c) p is a **definable f-generic** if there is a small model M_0 such that every translate gp is definable over M_0.

Note that by saturation of M, a definable f-generic is also a (strong) f-generic. In current literature, the terminology *f-generic* is used to mean that, for every formula $\varphi(x)$ in p, there is a small model M_0 such that no translate of $\varphi(x)$ forks over M_0. In the subsequent results on Presburger arithmetic, we will not distinguish between the two notions. This is justified by the following result.

Fact 0.2 (Levi, Kaplan, Simon [2]). *If G is definably amenable and dp-minimal, then any f-generic type in $S_G(M)$ is strongly f-generic.*

As $(\mathbb{Z}, +, 0)$ is abelian, it is amenable, and so the expansion $(\mathbb{Z}, +, <, 0)$ is definably amenable. $(\mathbb{Z}, +, <, 0)$ is also dp-minimal (see [1]).

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0.2. **End extensions of discrete orders.** Let T, M, and M^* be as in the previous subsection. Assume $\mathcal{L}$ contains a symbol $<$ for an ordering and T extends the theory of linear orders. We say that the theory T is *definably complete* if any nonempty definable subset of M, with an upper bound in M, has a least upper bound in M, and similarly for lower bounds. Note that this does not depend on the model M. If T is definably complete, and we further assume that M is discretely ordered by $<$, then it follows that definable subsets of M contain their least upper bound and greatest lower bound (whenever they exist). We will say T is *discretely ordered* to indicate that the ordering $<$ on M is discrete.

Recall that, in a totally ordered structure, algebraic closure and definable closure coincide. Given a tuple $\bar{a} \in (M^*)^n$, we let $M(\bar{a}) = \operatorname{dcl}(M\bar{a})$.

Definition 0.3. Given subsets $A \subseteq B$ of M^*, we say B is an **end extension of** A if, for all $b \in B \backslash A$, either $b < a$ for all $a \in A$ or $b > a$ for all $a \in A$.

Lemma 0.4. *Suppose T is discretely ordered and definably complete. Fix a non-isolated type $p \in S_n(M)$ and a realization $\bar{a}$ in M^*. If $M(\bar{a})$ is not an end extension of M then:*

(a) *p is not definable,*
(b) *p has at least two distinct coheirs to M^*.*

Proof. Since $M(\bar{a})$ is not an end extension of M, we may fix an M-definable function $f : (M^*)^n \longrightarrow M^*$, and $m_1, m_2 \in M$ such that $f(\bar{a}) \notin M$ and $m_1 < f(\bar{a}) < m_2$. Define the upwards closed set

$$X = \{m \in M : p \models f(\bar{x}) < m\}.$$

Then m_1 and m_2 witness that X is nonempty and not all of M. If X has a minimal element m_0, and m_0^- is the immediate predecessor of m_0 in M, then we must have $m_0^- \leq f(\bar{a}) < m_0$, and so $f(\bar{a}) = m_0^- \in M$, which is a contradiction. So X has no minimal element, and therefore cannot be M-definable. This proves part (a).

Now define

$$C = \{c \in M^* : m < c < m' \text{ for all } m \in M\backslash X \text{ and } m' \in X\}.$$

Then $f(\bar{a}) \in C$, and so $C \neq \emptyset$. We define the following partial types over M^*:

$$q_1 = p \cup \{m < f(\bar{x}) < c : m \in M\backslash X,\ c \in C\}$$
$$q_2 = p \cup \{c < f(\bar{x}) < m : c \in C,\ m \in X\}.$$

Note that q_1 and q_2 are distinct since $C \neq \emptyset$. If we can show that they are each finitely satisfiable in M, then they will extend to distinct coheirs of p, which proves part (b). So we show q_1 is finitely satisfiable in M (the proof for q_2 is similar).

Fix a formula $\varphi(\bar{x}) \in p$ and some $m \in G\backslash X$ (which exists since X is not all of M). Set

$$A = \{m' \in f(\varphi(M^n)) : m < m'\}.$$

Then A is an M-definable subset of M, which is nonempty since $b \in A(M^*)$. Since A is bounded below by m, we may fix a minimal element $m_0 \in A$. By elementarity, m_0 is the minimal element of $A(M^*)$. In particular, $m_0 < f(\bar{a})$, and so $m_0 \in M\backslash X$. By definition of A, $m_0 = f(\bar{a}')$ for some $\bar{a}' \in M^n$ such that $M \models \varphi(\bar{a}')$. Altogether, we have $M \models \varphi(\bar{a}')$, and $m < f(\bar{a}') < c$ for any $c \in C$. $\square$

Suppose T is discretely ordered and definably complete. If, moreover, $\operatorname{dcl}(\emptyset)$ is nonempty, then T has definable Skolem functions by picking out either the maximal element of a definable set or the least element greater than some $\emptyset$-definable constant. It follows that $M(\bar{a})$ is the unique prime model over $M\bar{a}$.

0.3. **Presburger arithmetic.** Let $T = \operatorname{Th}(\mathbb{Z}, +, <, 0)$. Let G denote a sufficiently saturated model of T, and let G^* denote a larger elementary extension of G, which is sufficiently saturated with respect to G. We treat types over G as *global types*, but use G^* as an even larger monster model in which we can realize such types.

Note that T satisfies the properties discussed in the previous section: it is discretely ordered and definably complete, with $\operatorname{dcl}(\emptyset)$ nonempty. Therefore, for $\bar{a} \in G^*$, $G(\bar{a})$ is the prime model over $G\bar{a}$. Recall that T has quantifier elimination in the expanded language $\mathcal{L}^* = \{+, <, 0, 1, (D_n)_{n<\omega}\}$ where D_n is a unary predicate interpreted as $n\mathbb{Z}$. Consequently, given $\bar{a} \in G^*$, $G(\bar{a})$ is the divisible hull of the subgroup of G^* generated by $G\bar{a}$.

Given $a \in G^*$ and $n > 0$, let $[a]_n \in \{0, 1, \ldots, n-1\}$ be the unique remainder of a modulo n. Given $\bar{k} \in \mathbb{Z}^n$, we let $s_{\bar{k}}(\bar{x})$ denote the definable function $\bar{x} \mapsto k_1x_1 + \ldots + k_nx_n$.

Proposition 0.5.

(*a*) *Let $G_0 \prec G$ be a small model, and fix $a, b \in G$.*

  (*i*) *If $G_0 < a < b$ then there is some $c \in G$ such that $b < c$ and $a \equiv_{G_0} c$.*

  (*ii*) *If $a < b < G_0$ then there is some $c \in G$ such that $c < a$ and $b \equiv_{G_0} c$.*

(*b*) *For any $p \in S_n(G)$ and $\bar{a} \models p$, if $G(\bar{a})$ is not an end extension of G then there are $h_1, h_2 \in G$ and $\bar{k} \in \mathbb{Z}^n$ such that $h_1 < s_{\bar{k}}(\bar{a}) < h_2$ and $s_{\bar{k}}(\bar{a}) \notin G$.*

Proof. Part (*a*). We prove (*i*); the proof of (*ii*) is similar. By quantifier elimination and saturation of G it is enough to fix an integer $N > 0$, and find $c \in G$ such that $b < c$ and $[c]_n = [a]_n$ for all $0 < n \leq N$. To find such an element, simply note that $\bigcap_{0<n\leq N} nG + [a]_n$ is nonempty (as it contains a), and is therefore a single coset $mG + r$ for some $m, r \in \mathbb{Z}$. So we may choose $c = b - [b]_m + m + r$.

Part (*b*). By assumption, there is $b \in \operatorname{dcl}(G\bar{a})\backslash G$ and $h_1', h_2' \in G$ such that $h_1' < b < h_2'$. By the description of definable closure in Presburger arithmetic, there are integers $r \in \mathbb{Z}^+$, $\bar{k} \in \mathbb{Z}^n$ and some $h_0 \in G$ such that $rb = s_{\bar{k}}(\bar{a}) + h_0$. Now let $h_i = rh_i' - h_0$. □

0.4. **Definable types in Presburger arithmetic.** In the context of Definition 0.1, we consider the situation where G is the monster model M, and the definable group is $G^n = \mathbb{Z}^n(G)$, for a fixed $n > 0$, under coordinate addition. In particular, where we have the notation $S_G(M)$ in Definition 0.1, here we just have $S_n(G)$.

Definition 0.6. A type $p \in S_n(G)$ is **algebraically independent** if for all (some) $\bar{a} \models p$, $a_i \notin G(\bar{a}_{\neq i})$ for all $1 \leq i \leq n$.

Lemma 0.7. *Suppose $p \in S_n(G)$ is algebraically independent and for all (some) $\bar{a} \models p$, $G(\bar{a})$ is an end extension of G. Then p is definable over $\emptyset$.*

Proof. Let $\mathbb{Z}^n_*$ denote $\mathbb{Z}^n\backslash\{\bar{0}\}$. By quantifier elimination, it suffices to give definitions for atomic formulas of the following forms:

- $\varphi_1(\bar{x}, \bar{y}) := \big(s_{\bar{k}}(\bar{x}) = t(\bar{y})\big)$, where $\bar{k} \in \mathbb{Z}^n_*$ and $t(\bar{y})$ is a term in variables $\bar{y}$,
- $\varphi_2(\bar{x}, \bar{y}) := \big(s_{\bar{k}}(\bar{x}) > t(\bar{y})\big)$, where $\bar{k} \in \mathbb{Z}^n_*$ and $t(\bar{y})$ is a term in variables $\bar{y}$,

- $\varphi_3(\bar{x},\bar{y}) := \big([s_{\bar{k}}(\bar{x}) + t(\bar{y})]_m = 0\big)$, where $\bar{k} \in Z^n_*$, $m \in \mathbb{Z}^+$, and $t(\bar{y})$ is a term in variables $\bar{y}$.

Fix $\bar{a} \models p$ and fix $\bar{k} \in \mathbb{Z}^n_*$. Since p is algebraically independent, it follows that $s_{\bar{k}}(\bar{a}) \notin G$. Since $G(\bar{a})$ is an end extension of G, we may partition $\mathbb{Z}^n_* = S^+ \cup S^-$ where

$$S^+ = \{\bar{k} : s_{\bar{k}}(\bar{a}) > G\} \quad \text{and} \quad S^- = \{\bar{k} : s_{\bar{k}}(\bar{a}) < G\}.$$

Note that S^+ and S^- depend only on p, and not choice of realization $\bar{a}$. Moreover, for any $\bar{k} \in \mathbb{Z}^n$ and $m > 0$, the integer $[s_{\bar{k}}(\bar{a})]_m \in \{0, \ldots, m-1\}$ depends only on p. We now give the following definitions for p (note that they are formulas over $\emptyset$):

$$d_p[\varphi_1](\bar{y}) := \big(y_1 \neq y_1\big),$$

$$d_p[\varphi_2](\bar{y}) := \begin{cases} y_1 = y_1 & \text{if } \bar{k} \in S^+ \\ y_1 \neq y_1 & \text{if } \bar{k} \in S^-, \end{cases}$$

$$d_p[\varphi_3](\bar{y}) := \big([t(\bar{y}) + [s_{\bar{k}}(\bar{a})]_m]_m = 0\big).$$

□

Theorem 0.8. *Given $p \in S_n(G)$, the following are equivalent.*

(i) p is definable (over G).
(ii) p has a unique coheir to G^*.
(iii) For any (some) $\bar{a} \models p$, $G(\bar{a})$ is an end extension of G.

Proof. $(i) \Rightarrow (ii)$. This is true in any NIP theory (see [5, Exercise 2.74]), and has nothing to do with G being a group.

$(ii) \Rightarrow (iii)$. By Lemma 0.4(b).

$(iii) \Rightarrow (i)$. We may assume p is non-isolated. We proceed by induction on n. If $n = 1$ then p is algebraically independent since it is non-isolated, and so we apply Lemma 0.7. Assume the result for $n' < n$ and fix $p \in S_n(G)$. If p is algebraically independent then we apply Lemma 0.7. So assume, without loss of generality, that we have $\bar{a} \models p$ with $a_n \in G(\bar{a}_{<n})$. Let $q = \mathrm{tp}(\bar{a}_{<n}/G) \in S_{n-1}(G)$. By assumption, $G(\bar{a}_{<n}) = G(\bar{a})$ is an end extension of G, and so q is definable by induction. Fix a G-definable function $f : (G^*)^{n-1} \longrightarrow G^*$ such that $f(\bar{a}_{<n}) = a_n$. Fix a formula $\varphi(\bar{x},\bar{y})$ and define

$$\psi(\bar{x}_{<n},\bar{y}) := \varphi(\bar{x}_{<n}, f(\bar{x}_{<n}), \bar{y}).$$

Let $d_q[\psi](\bar{y})$ be an $\mathcal{L}_G$-formula such that, for any $\bar{b}$ in G, $\psi(\bar{x}_{<n},\bar{b}) \in q$ if and only if $G \models d_q[\psi](\bar{b})$. Then, for any $\bar{b}$ in G, we have

$$\varphi(\bar{x},\bar{b}) \in p \;\Leftrightarrow\; G^* \models \varphi(\bar{a},\bar{b}) \;\Leftrightarrow\; G^* \models \psi(\bar{a}_{<n},\bar{b}) \;\Leftrightarrow\; G \models d_q[\psi](\bar{b}).$$

This shows that p is definable. □

Remark 0.9.

(1) Neither Lemma 0.7 nor Theorem 0.8 require saturation of G.
(2) This result is similar to the Marker-Steinhorn theorem for o-minimal theories [3], which says that a type p over a model $\mathcal{M}$ of an o-minimal theory is definable if and only if $\mathcal{M}$ is *Dedekind closed* in $\mathcal{M}(\bar{a})$ (i.e. no Dedekind cut in M is realized in $\mathcal{M}(\bar{a})$), where $\mathcal{M}(\bar{a})$ is again the unique prime model over $M\bar{a}$.
(3) Lemma 0.4(a) gives a direct (and short) proof of $(i) \Rightarrow (iii)$, which does not appeal to NIP.
(4) Some relationship between definable 1-types and end extensions in Peano arithmetic is discussed in Section 11.4 of [4].

0.5. **f-generics in Presburger arithmetic.** We continue to work with the notations and conventions from the previous section.

Proposition 0.10. *Any f-generic $p \in S_n(G)$ is algebraically independent.*

Proof. Suppose p is not algebraically independent. Fix a small model $G_0 \prec G$. We find a translate of p which forks over G_0. Without loss of generality, fix $\bar{a} \models p$ with $a_n \in G(\bar{a}_{<n})$. Then there are $r, k_1, \ldots, k_{n-1} \in \mathbb{Z}$ and $b \in G$ such that $ra_n = b + k_1a_1 + \ldots + k_{n-1}a_{n-1}$. Pick $c \in rG$ such that $b + c \notin G_0$ and set $g = \frac{c}{r}$. Let $\bar{g} = (0, \ldots, 0, g)$. Then $b + c = r(a_n + g) - (k_1a + 1 + \ldots + k_{n-1}a_{n-1})$, and so $b + c \in \operatorname{acl}(\bar{g} + \bar{a})$. Moreover, $b + c \not\perp_{G_0} G$, and so $\bar{g}p = \operatorname{tp}((\bar{g} + \bar{a})/G)$ forks over G_0. Therefore p is not f-generic. □

Theorem 0.11. *For algebraically independent $p \in S_n(G)$, the following are equivalent.*

(*i*) *p is an f-generic.*
(*ii*) *p is a definable f-generic.*
(*iii*) *p is definable (over G).*
(*iv*) *p is definable over $\emptyset$.*
(*v*) *For any (some) $\bar{a} \models p$, $G(\bar{a})$ is an end extension of G.*

Proof. $(iii) \Rightarrow (v)$: By Theorem 0.8.

$(v) \Rightarrow (iv)$: By Lemma 0.7.

$(iv) \Rightarrow (iii)$: Trivial.

$(i) \Rightarrow (v)$: Suppose $G(\bar{a})$ is not an end extension of G, and fix $\bar{k} \in \mathbb{Z}^n$ and $h_1, h_2 \in G$ such that, setting $b = s_{\bar{k}}(\bar{a})$, we have $b \notin G$ and $h_1 < b < h_2$. Fix a small model $G_0 \prec G$. We find a translate gp of p which forks over G_0. Without loss of generality, assume $b > 0$ and also $h_1 > 0$. Let k_i be a nonzero element of the tuple $\bar{k}$. By saturation of G, we may find $g \in G$ such that $k_ig > c$ for all $c \in G_0$. Let $\bar{g} \in G^n$ be such that $g_j = 0$ for all $j \neq i$ and $g_i = g$. For $t \in \{1, 2\}$, set $c_t = h_t + k_ig \in G$. Let $d = b + k_ig = k_1(g_1 + a_1) + \ldots + k_n(g_n + a_n)$. Then $G_0 < c_1 < d < c_2$. Using Proposition 0.5(a), it is easy to show $c_1 < x < c_2$ forks over G_0, and so $\operatorname{tp}(d/G)$ forks over G_0. Since $d \in \operatorname{dcl}(\bar{g} + \bar{a})$, it follows that $\bar{g}p = \operatorname{tp}((\bar{g} + \bar{a})/G)$ forks over G_0. Altogether, p is not f-generic.

$(v) \Rightarrow (ii)$: Suppose $G(\bar{a})$ is an end extension of G. For any $\bar{g} \in G^n$, we have $G(\bar{a}) = G(\bar{g} + \bar{a})$, and $\bar{g}p$ is still algebraically independent. Therefore, for any $\bar{g} \in G^n$, we use Lemma 0.7 to conclude that $\bar{g}p$ is definable over $\emptyset$.

$(ii) \Rightarrow (i)$: Trivial. □

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