DIVIDING LINES IN UNSTABLE THEORIES

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The aim of this paper is to define various properties of formulas in first order theories, and prove the appropriate implications between these properties. Most definitions are taken from [3], but the definitions themselves and many of the proofs are due to Shelah (see [4, II]). We give citations at the beginning of proofs taken from other sources.

Recall that a theory is stable if no formula has the so-called "order property", and a theory is simple if no formula has the "tree property". We first define these properties, along with a few more complicated properties of the same type. We fix some theory T and a sufficiently saturated $\mathbb{M} \models T$. If φ is a sentence with parameters from $\mathbb{M}$, we write $\models \varphi$ if $\mathbb{M} \models \varphi$.

1. A Chain of Properties

Definition 1.1. *A formula $\varphi(x, y)$ has the **order property**, OP, if there are tuples $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$.*

*For $n \geq 3$, a formula $\varphi(x, y)$, with $l(x) = l(y)$, has the n-**strong order property**, SOP$_n$, if*

$$\models \neg\exists x_1, \ldots, x_n(\varphi(x_1, x_2) \wedge \varphi(x_2, x_3) \wedge \ldots \wedge \varphi(x_n, x_1)),$$

and there are tuples $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_j)$ for all $i < j < \omega$.

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and there are tuples $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_j)$ for all $i < j < \omega$.

*A formula $\varphi(x, y)$ has the **strict order property**, sOP, if there are tuples $(a_i)_{i<\omega}$ such that*

$$\models \exists x(\neg\varphi(x, a_i) \wedge \varphi(x, a_j)) \quad \Leftrightarrow \quad i < j.$$

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Consider the definition of SOP_n and its natural extension to $n = 1$ or $n = 2$. For $n = 2$ we have the order property. Moreover, any theory with an infinite model would satisfy the definition with $n = 1$ via the formula $x \neq y$. Therefore we will redefine SOP_2 and SOP_1 in the same vein as the next class of properties, which are defined using trees as index sets.

Before defining these properties, we specify some notation concerning trees.

Definition 1.2. *Let A be a set and define*

$$A^{<\omega} = \bigcup_{n\in\omega} A^n.$$

If $(a_0, \ldots, a_n), (b_0, \ldots, b_m) \in A^{<\omega}$, define

$$(a_0, \ldots, a_n)\hat{\ }(b_0, \ldots, b_m) := (a_0, \ldots, a_n, b_0, \ldots, b_m) \in A^{<\omega}.$$

If $\mu, \eta \in A^{<\omega}$, we say $\mu \prec \eta$ if there is some $\gamma \in A^{<\omega}$ such that $\eta = \mu\hat{\ }\gamma$. For $a \in A$ we identify a and $(a) \in A^{<\omega}$. If $n \in \omega$, we also define $(a)^n = (\underbrace{a, a, \ldots, a}_{n\ times}) \in A^{<\omega}$. Two elements $\mu, \eta \in A^{<\omega}$ are ***incomparable*** *if $\mu \not\prec \eta$ and $\eta \not\prec \mu$.*

The next class of properties on formulas are defined using tuples indexed by trees.

Definition 1.3. *A formula $\varphi(x, y)$ has the* ***tree property****, TP, if there are tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ and some $k \geq 2$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all $\eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta\hat{\ }n}) : n < \omega\}$ is k-inconsistent.*

A formula $\varphi(x, y)$ has the ***tree property 1****, TP_1, if there are tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all incomparable $\mu, \eta \in \omega^{<\omega}$, $\{\varphi(x, a_\mu), \varphi(x, a_\eta)\}$ is inconsistent.*

A formula $\varphi(x, y)$ has SOP_1 if there are tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all $\mu, \eta \in 2^{<\omega}$, if $\mu\hat{\ }0 \prec \eta$ then $\{\varphi(x, a_{\mu\hat{\ }1}), \varphi(x, a_\eta)\}$ is inconsistent.

A formula $\varphi(x, y)$ has SOP_2 if there are tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all incomparable $\mu, \eta \in 2^{<\omega}$, $\{\varphi(x, a_\mu), \varphi(x, a_\eta)\}$ is inconsistent.

The goal of this section is to prove the following chain of implications (when Q $\Rightarrow$ R is written with no other information, we read this as "if T has Q then T has R").

Theorem 1.4.

$$\mathrm{sOP} \Rightarrow \mathrm{SOP} \Rightarrow \ldots \Rightarrow \mathrm{SOP}_{n+1} \Rightarrow \mathrm{SOP}_n \Rightarrow \ldots \Rightarrow \mathrm{SOP}_3 \Rightarrow (\mathrm{TP}_1 \Leftrightarrow \mathrm{SOP}_2) \Rightarrow \mathrm{SOP}_1 \Rightarrow \mathrm{TP} \Rightarrow \mathrm{OP}.$$

Proposition 1.5. sOP $\Rightarrow$ SOP.

Proof. Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses sOP. Let $l(x_1) = l(x_2) = l(y)$ and define

$$\psi(x_1, x_2) := \forall x(\varphi(x, x_1) \to \varphi(x, x_2)) \wedge \exists x(\varphi(x, x_2) \wedge \neg\varphi(x, x_1)).$$

By assumption, $\models \psi(a_i, a_j)$ for all $i < j$. Suppose, towards a contradiction, that we have $n \geq 3$ and $b_1, \dots, b_n$ such that

$$\models \psi(b_1, b_2) \wedge \dots \wedge \psi(b_{n-1}, b_n) \wedge \psi(b_n, b_1).$$

If $B_i = \psi(\mathbb{M}, b_i)$ for $1 \leq i \leq n$, then we have $B_1 \subsetneq B_2 \subsetneq \dots \subsetneq B_n \subsetneq B_1$, which is a contradiction. Therefore $\psi(x_1, x_2)$, with $(a_i)_{i<\omega}$, witnesses SOP. $\square$

Proposition 1.6. SOP $\Rightarrow$ SOP$_n$ *for all* $n \geq 3$.

Proof. Follows by definition. $\square$

Proposition 1.7. *For* $n \geq 3$*,* SOP$_{n+1}$ $\Rightarrow$ SOP$_n$*.*

Proof. Suppose T has SOP$_{n+1}$, witnessed by $\varphi(x, y)$ and $(a_i)_{i<\omega}$. Define

$$\psi(x_1, x_2, y_1, y_2) := \varphi(x_1, x_2) \ \wedge \varphi(x_2, y_1) \ \wedge \ \varphi(x_2, y_2) \ \wedge \ \varphi(y_1, y_2).$$

If $i < j$ then $\models \psi(a_{2i}, a_{2i+1}, a_{2j}, a_{2j+1})$. Suppose, towards a contradiction, that $(b_{1,0}, b_{1,1}), \dots, (b_{n,0}, b_{n,1})$ are such that

$$\mathbb{M} \models \psi(b_{1,0}, b_{1,1}, b_{2,0}, b_{2,1}) \wedge \dots \wedge \psi(b_{n-1,0}, b_{n-1,1}, b_{n,0}, b_{n,1}) \wedge \psi(b_{n,0}, b_{n,1}, b_{1,0}, b_{1,1}).$$

Then we have

$$\mathbb{M} \models \varphi(b_{1,0}, b_{1,1}) \ \wedge \ \varphi(b_{1,1}, b_{2,1}) \wedge \dots \wedge \varphi(b_{n-1,1}, b_{n,1}) \wedge \varphi(b_{n,1}, b_{1,0}),$$

contradicting that $\varphi(x, y)$ SOP$_{n+1}$. Therefore $\psi(x_1, x_2, y_1, y_2)$, with $(a_{2i}, a_{2i+1})_{i<\omega}$, witnesses SOP$_n$. $\square$

Proposition 1.8. SOP$_3$ $\Rightarrow$ SOP$_2$.

Proof. [2] Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses SOP$_3$. We have $\models \varphi(a_i, a_j)$ for all $i < j$. By compactness, we can obtain $(b_q)_{q\in\mathbb{Q}}$ such that $\models \varphi(b_q, b_r)$ for all $q < r$. Set $z = (y_1, y_2)$ and define

$$\psi(x, z) := \varphi(y_1, x) \wedge \varphi(x, y_2).$$

We define $(c_\eta)_{\eta\in 2^{<\omega}}$ inductively by $c_\emptyset = (b_0, b_1)$, and if $c_\eta = (b_q, b_r)$, with $q < r$, then

$$c_{\eta\frown i} = \begin{cases} (b_q, b_{\frac{1}{3}(r-q)}) & i = 0 \\ (b_{\frac{2}{3}(r-q)}, b_r) & i = 1 \end{cases}$$

We claim that $\psi(x,z)$, with $(c_\eta)_{\eta\in 2^{<\omega}}$, witnesses SOP_2. To this end, suppose $\sigma\in 2^\omega$ and $n<\omega$. There are $0=q_0<\ldots<q_n<r_n<r_{n-1}<\ldots<r_0=1$ such that for $0\le i\le n$,

$$c_{\sigma|_i}=(b_{q_i},b_{r_i}).$$

If $q_n<q<r_n$ then $\models\varphi(b_{q_i},b_q)\wedge\varphi(b_q,b_{r_i})$ for all $0\le i\le n$. Thus $\{\psi(x,c_{\sigma|_i}):0\le i\le n\}$ is satisfiable, and so $\{\psi(x,c_{\sigma|_n}):n<\omega\}$ is consistent by compactness.

Now suppose $\mu,\eta\in 2^{<\omega}$ are incomparable. Then, without loss of generality, we have $q<r<s<t$ such that

$$c_\mu=(b_q,b_r)\ \text{ and }\ c_\eta=(b_s,b_t).$$

If d satisfies $\{\psi(x,c_\mu),\psi(x,c_\eta)\}$ then we have

$$\varphi(d,b_r)\wedge\varphi(b_r,b_s)\wedge\varphi(b_s,d),$$

contradicting that $\varphi(x,y)$ witnesses SOP_3. Therefore $\{\psi(x,c_\mu),\psi(x,c_\eta)\}$ is inconsistent. □

Proposition 1.9. $\mathrm{SOP}_2\Leftrightarrow\mathrm{TP}_1$.

Proof. [3] Suppose $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses SOP_2. Define $h:\omega^{<\omega}\longrightarrow 2^{<\omega}$ inductively by $h(\emptyset)=\emptyset$ and for $i<\omega$,

$$h(\eta\hat{\ }i)=h(\eta)\hat{\ }(1)^i\hat{\ }0.$$

If $\eta\prec\mu$, say $\mu=\eta\hat{\ }(n_1,\ldots,n_k)$ with $n_i\in\omega$, then $h(\mu)=h(\eta)\hat{\ }(1)^{\sum n_i}\hat{\ }0$ so $h(\eta)\prec h(\mu)$. Thus if $\sigma\in\omega^\omega$, we may define $h(\sigma):=\bigcup_{n<\omega}h(\sigma|_n)\in 2^\omega$.

By assumption, $\{\varphi(x,a_{h(\sigma)|_n}):n<\omega\}$ is consistent. If $\eta,\mu\in\omega^{<\omega}$ are incomparable then, without loss of generality, there are $\gamma,\eta_0,\mu_0\in\omega^{<\omega}$ and $i<j$ such that $\eta=\gamma\hat{\ }i\hat{\ }\eta_0$ and $\mu=\gamma\hat{\ }j\hat{\ }\mu_0$. It follows that there are $\eta_1,\mu_1\in 2^{<\omega}$ such that $h(\eta)=h(\gamma)\hat{\ }(1)^i\hat{\ }0\hat{\ }\eta_1$ and $h(\mu)=h(\gamma)\hat{\ }(1)^j\hat{\ }0\hat{\ }\mu_1$. Therefore h($\eta$) and $h(\mu)$ are incomparable, and so $\{\varphi(x,a_{h(\eta)}),\varphi(x,a_{h(\mu)})\}$ is inconsistent. In conclusion $\varphi(x,y)$ with $(a_{h(\eta)})_{\eta\in\omega^{<\omega}}$, witnesses TP_1.

Conversely, if $\varphi(x,y)$, with $(a_\eta)_{\eta\in\omega^{<\omega}}$, witnesses TP_1, then clearly $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses SOP_2. □

Proposition 1.10. $\mathrm{SOP}_2\Rightarrow\mathrm{SOP}_1$.

Proof. Suppose $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses SOP_2. For all $\mu,\eta\in 2^{<\omega}$, if $\mu\hat{\ }0\prec\eta$ then $\mu\hat{\ }1$ and η are incomparable, and so $\{\varphi(x,a_{\mu\hat{\ }1}),\varphi(x,a_\eta)\}$ is inconsistent. Thus $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses SOP_1. □

Proposition 1.11. $\mathrm{SOP}_1\Rightarrow\mathrm{TP}$.

Proof. [2] Suppose $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses SOP_1. Define $h:\omega^{<\omega}\longrightarrow 2^{<\omega}$ inductively such that $h(\emptyset)=\emptyset$ and for $i<\omega$,

$$h(\eta\hat{\ }i)=h(\eta)\hat{\ }(0)^i\hat{\ }1.$$

For $\eta\in\omega^{<\omega}$, set $b_\eta=a_{h(\eta)}$. As in the proof of Proposition 1.9, $\mu\prec\eta$ implies $h(\mu)\prec h(\eta)$. For $\sigma\in\omega^\omega$, define $h(\sigma)=\bigcup_{n<\omega}h(\sigma|_n)$. Then $\{\varphi(x,b_{\sigma|_n}):n<\omega\}\subseteq\{\varphi(x,a_{h(\sigma)|_n}):n<\omega\}$, so $\{\varphi(x,b_{\sigma|_n}):n<\omega\}$ is consistent.

Now fix $\eta\in\omega^{<\omega}$ and suppose $i<j$. Then $h(\eta)\hat{\ }(0)^i\prec h(\eta\hat{\ }j)$ and $h(\eta\hat{\ }i)=h(\eta)\hat{\ }(0)^i\hat{\ }1$, so

$$\{\varphi(x,a_{h(\eta\hat{\ }i)}),\varphi(x,a_{h(\eta\hat{\ }j)})\}$$

is inconsistent by assumption. Therefore $\{\varphi(x,b_{\eta\hat{\ }i}),\varphi(x,b_{\eta\hat{\ }j})\}$ is inconsistent, and so $\{\varphi(x,b_{\eta\hat{\ }n}):n<\omega\}$ is 2-inconsistent. Thus $\varphi(x,y)$, with $(b_\eta)_{\eta\in\omega^{<\omega}}$, witnesses TP. □

The only remaining implication in the statement of Theorem 1.4 is TP $\Rightarrow$ OP. This argument is a bit more technical than the previous one, and we break it into two steps, the proofs of which are taken from [4].

Lemma 1.12. *Suppose $\varphi(x,y)$ witnesses* TP *with respect to $k\geq 2$. Then there is an infinite set A such that $|S_\varphi(A)|>|A|$.*

Proof. [4, II] Let κ be an infinite cardinal such that $\kappa^\omega>\max\{2^\omega,\kappa\}$. By compactness we may assume that we have $(a_\eta)_{\eta\in\kappa^{<\omega}}$ such that for all $\sigma\in\kappa^\omega$,

$$\pi_\sigma=\{\varphi(x,a_{\sigma|_n}):n<\omega\}$$

is consistent; and for all $\eta\in\kappa^{<\omega}$, $\{\varphi(x,a_{\eta\hat{\ }i}):i<\kappa\}$ is k-inconsistent. Given $\sigma\in\kappa^\omega$, construct $F_\sigma\subseteq\kappa^\omega$ such that

(*i*) $\sigma\in F_\sigma$;

(*ii*) $\bigcup_{\tau\in F_\sigma}\pi_\tau$ is consistent.

(*iii*) for all $\rho\in\kappa^\omega\backslash F_\sigma$, $\pi_\rho\cup\bigcup_{\tau\in F_\sigma}\pi_\tau$ is inconsistent.

Let $T_\sigma=\{\tau|_n:n<\omega,\ \tau\in F_\sigma\}$. Then T_σ is a tree. Suppose, towards a contradiction, that there is $\eta\in T_\sigma$ and distinct $i_1,\ldots,i_k\in\kappa$ such that $\eta\hat{\ }i_j\in T_\sigma$ for all j. Then there are $\tau_1,\ldots,\tau_k\in F_\sigma$ such that $\eta\hat{\ }i_j\prec\tau_j$, which is a contradiction since $\{\varphi(x,a|_{\eta\hat{\ }i_j}):1\leq j\leq k\}$ is inconsistent. It follows that T_σ can be embedded into k^ω. In particular, $|F_\sigma|\leq 2^\omega$. Since $\kappa^\omega>2^\omega$, there is $F\subseteq\kappa^\omega$ such that $|F|=\kappa^\omega$ and $F_\sigma\neq F_\tau$ for all distinct $\sigma,\tau\in F$.

Let $A=(a_\eta)_{\eta\in\kappa^{<\omega}}$ and, for $\sigma\in F$, let $p_\sigma\in S_\varphi(A)$ be a complete φ-type containing $\bigcup_{\tau\in F_\sigma}\pi_\tau$. If $\sigma,\tau\in F$ are distinct then, without loss of generality, there is some $\rho\in F_\sigma\backslash F_\tau$. Then $\pi_\rho\subseteq p_\sigma$ and $p_\tau\cup\pi_\rho$ is inconsistent. Therefore $p_\sigma\neq p_\tau$, and so $|S_\varphi(A)|\geq\kappa^\omega>\kappa=|A|$. □

Definition 1.13. *Given formulas $\varphi(x,y)$, $\psi(y,x)$, a type p (ψ,φ)-**splits** over a set B if there are $a,b \in \mathrm{dom}(p)$ such that $\mathrm{tp}_\psi(a/B) = \mathrm{tp}_\psi(b/B)$, but $\varphi(x,a), \neg\varphi(x,b) \in p$.*

Proposition 1.14. TP $\Rightarrow$ OP.

Proof. [4, II] Suppose $\varphi(x,y)$ witnesses TP. By Lemma 1.12, there is some infinite cardinal κ, and a set A of size κ, such that $|S_\varphi(A)| > \kappa$. Let $(c_i)_{i<\kappa^+}$ be realizations of κ^+-many distinct φ-types in $S_\varphi(A)$. Set $\psi(y,x) = \varphi(x,y)$. Let $A_0 = A$ and, given A_n of size κ, define

$$A_{n+1} = A_n \cup \{a : a \models p,\ p \in S_\varphi(B) \cup S_\psi(B),\ B \subseteq A_n \text{ is finite}\}.$$

There are countably many finite subsets of A_n, and if B is finite then $S_\varphi(B) \cup S_\psi(B)$ is finite, so A_{n+1} still has size κ.

Claim: There is some $i < \kappa^+$ such that for all $n < \omega$ and for all $B \subseteq A_n$ finite, $\mathrm{tp}_\varphi(c_i/A_{n+1})$ (ψ,φ)-splits over B.

Proof: Suppose not. Then for all $i < \kappa^+$ there is a pair (n,B) such that $B \subseteq A_n$ is finite and $\mathrm{tp}_\varphi(c_i/A_{n+1})$ does not (ψ,φ)-split over B. There are only countably many such pairs (n,B). Thus, without loss of generality, there is a pair (n,B) such that $B \subseteq A_n$ is finite and for all $i < \kappa^+$, $\mathrm{tp}(c_i/A_{n+1})$ does not (ψ,φ)-split over B. By definition, there is a finite set C such that $B \subseteq C \subseteq A_{n+1}$ and all types in $S_\varphi(B) \cup S_\psi(B)$ are realized in C. Again, $S_\varphi(C)$ is finite, so without loss of generality we may assume $\mathrm{tp}_\varphi(c_i/C) = \mathrm{tp}_\varphi(c_j/C)$ for all $i,j < \kappa^+$.

Consider c_0, c_1. By assumption, there is some $a \in A_0$ such that $\models \varphi(c_1,a) \leftrightarrow \neg\varphi(c_0,a)$. Let $a' \in C$ such that $\mathrm{tp}_\psi(a'/B) = \mathrm{tp}_\psi(a/B)$. For all $i < \kappa^+$, $\mathrm{tp}_\varphi(c_i/A_{n+1})$ does not (ψ,φ)-split over B, so it follows that $\mathrm{tp}_\varphi(c_i/C)$ does not (ψ,φ)-split over B. Since $\mathrm{tp}_\psi(a/B) = \mathrm{tp}_\psi(a'/B)$, we have $\varphi(x,a) \in \mathrm{tp}_\varphi(c_i/C)$ if and only if $\varphi(x,a') \in \mathrm{tp}_\varphi(c_i/C)$. In other words, $\models \varphi(c_i,a) \leftrightarrow \varphi(c_i,a')$, for all $i < \kappa^+$. Altogether, we have

$$\models \varphi(c_0,a) \leftrightarrow \varphi(c_0,a') \leftrightarrow \varphi(c_1,a') \leftrightarrow \varphi(c_1,a) \leftrightarrow \neg\varphi(c_0,a),$$

which is a contradiction.$_{/\!/}$

By the claim, we have $i < \kappa^+$ such that for all $n < \omega$ and for all $B \subseteq A_n$ finite, $\mathrm{tp}(c_i/A_{n+1})$ (ψ,φ)-splits over B. Set $c = c_i$. Then $\mathrm{tp}_\varphi(c/A_1)$ (ψ,φ)-splits over $\emptyset$, so there are $a_0, b_0 \in A_1$ such that $\mathrm{tp}_\psi(a_0) = \mathrm{tp}_\psi(b_0)$ with $\varphi(x,a_0), \neg\varphi(x,b_0) \in \mathrm{tp}(c/A_1)$. Now $\{a_0,b_0\} \subseteq A_1$ so there is some $d_0 \in A_2$ realizing $\mathrm{tp}_\varphi(c/a_0,b_0)$.

Suppose $n > 0$ and we are given $(a_i,b_i,d_i)_{i<n}$ such that for all $i < n$,

(*i*) $\mathrm{tp}_\psi(a_i/\{d_j : j < i\}) = \mathrm{tp}_\psi(b_i/\{d_j : j < i\})$;
(*ii*) $d_i \in A_{2i+2}$ realizes $\mathrm{tp}_\varphi(c/\{a_j,b_j : j \leq i\})$;
(*iii*) $\models \varphi(c,a_i) \wedge \neg\varphi(c,b_i)$.

Then $\mathrm{tp}_\varphi(c/A_{2n+1})$ (ψ,φ)-splits over $\{d_i : i < n\} \subseteq A_{2n}$ so there are $a_n, b_n \in A_{2n+1}$ such that $\mathrm{tp}_\psi(a_n/\{d_i : i < n\}) = \mathrm{tp}_\psi(b_n/\{d_i : i < n\})$ and $\varphi(x, a_n), \neg\varphi(x, b_n) \in \mathrm{tp}_\varphi(c/A_{2n+1})$. But $\mathrm{tp}_\varphi(c/\{a_i, b_i : i \leq n\})$ is realized by some $d_n \in A_{2n+2}$. This process generates $(a_n, b_n, d_n)_{n<\omega}$ such that for all $n < \omega$,

(*i*) $\mathrm{tp}_\psi(a_n/\{d_i : i < n\}) = \mathrm{tp}_\psi(b_n/\{d_i : i < n\})$;

(*ii*) $d_n \in A_{2n+2}$ realizes $\mathrm{tp}_\varphi(c/\{a_i, b_i : i \leq n\})$;

(*iii*) $\models \varphi(c, a_n) \wedge \neg\varphi(c, b_n)$.

Note first that for all $j \leq i$, we have $\models \varphi(d_i, a_j) \wedge \neg\varphi(d_i, b_j)$. Moreover, for all $i < j$,

$$\models \varphi(d_i, a_j) \leftrightarrow \psi(a_j, d_i) \leftrightarrow \psi(b_j, d_i) \leftrightarrow \varphi(d_i, b_j).$$

Therefore we have

$$\models \varphi(d_i, a_j) \leftrightarrow \varphi(d_i, b_j) \quad \Leftrightarrow \quad i < j.$$

Altogether, if $z = (y_1, y_2)$ and $\theta(x, z) := \varphi(x, y_1) \leftrightarrow \varphi(x, y_2)$, then $\theta(x, z)$, with $(d_i)_{i<\omega}$ and $(a_i, b_i)_{i<\omega}$, witnesses OP. $\square$

This completes the proof of Theorem 1.4.

2. Further Properties

We now define two more properties, which do not fit exactly into the chain in Theorem 1.4.

Definition 2.1. *A formula $\varphi(x, y)$ has the **independence property**,* IP, *if there are $(a_i)_{i<\omega}$ and $(c_\sigma)_{\sigma\in 2^\omega}$ such that $\models \varphi(a_i, c_\sigma)$ if and only if $\sigma(i) = 1$.*

*A formula $\varphi(x, y)$ has the **tree property 2**,* TP_2, *if there are $(a_{i,j})_{i,j<\omega}$ such that for any $\sigma \in \omega^\omega$, $\{\varphi(x, a_{n,\sigma(n)}) : n < \omega\}$ is consistent; but for all $j < k < \omega$, $\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k})\}$ is inconsistent.*

Proposition 2.2. IP $\Rightarrow$ OP.

Proof. Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$ and $(c_\sigma)_{\sigma\in 2^\omega}$, witnesses IP. Given $i < \omega$, let $\sigma_i : \omega \longrightarrow \omega$ such that $\sigma_i(j) = 0$ if and only if $i \leq j$. Then we have

$$\models \varphi(a_i, c_{\sigma_j}) \quad \Leftrightarrow \quad \sigma_j(i) = 1 \quad \Leftrightarrow \quad i < j.$$

So $\varphi(x, y)$, with $(a_i)_{i<\omega}$ and $(c_{\sigma_i})_{i<\omega}$, witnesses OP. $\square$

Proposition 2.3. $\mathrm{TP}_2 \Rightarrow$ TP.

Proof. [1] Suppose $\varphi(x,y)$, with $(a_{i,j})_{i,j<\omega}$, witnesses TP_2. Fix an injection $f:\omega\times\omega\longrightarrow\omega$. Set $b_\emptyset = a_{0,0}$, and for $i<\omega$, set $b_{(j)} = a_{1,j}$. Suppose $0<n<\omega$ and for all $\eta\in\omega^n$ we have $j<\omega$ such that $b_\eta = a_{n,j}$. Let $(b_{\eta_i})_{i<\omega}$ be an enumeration of ω^n and for $j<\omega$ define $b_{\eta_i\hat{\ }j} = a_{n+1,f(i,j)}$.

We claim that $\varphi(x,y)$, with $(b_\eta)_{\eta\in\omega^{<\omega}}$, witnesses TP with respect to 2. If $\sigma\in\omega^\omega$ then for all $n<\omega$, $b_{\sigma|_n} = a_{n,j}$ for some $j<\omega$. So if $\tau:\omega\longrightarrow\omega$ is such that $\tau(n)=j$, we have that

$$\{\varphi(x,b_{\sigma|_n}) : n<\omega\} = \{\varphi(x,a_{n,\tau(n)}) : n<\omega\}$$

is consistent. Furthermore suppose $\eta\in\omega^{<\omega}$ and $j<k<\omega$. If $|\eta|=n$, then $b_{\eta\hat{\ }j} = a_{n+1,f(i,j)}$ and $b_{\eta\hat{\ }k} = a_{n+1,f(i,k)}$, where $\eta=\eta_i$ in the enumeration of ω^n. Since f is injective, it follows that $f(i,j)\neq f(i,k)$, and so

$$\{\varphi(x,b_{\eta\hat{\ }j}),\varphi(x,b_{\eta\hat{\ }k})\} = \{\varphi(x,a_{n+1,f(i,j)}),\varphi(x,a_{n+1,f(i,k)})\}$$

is inconsistent by assumption. □

Proposition 2.4. $\mathrm{TP}_2 \Rightarrow \mathrm{IP}$.

Proof. [1] Suppose $\varphi(x,y)$, with $(a_{i,j})_{i,j<\omega}$, witnesses TP_2. Let $\sigma\in 2^\omega$. By assumption,

$$\{\varphi(x,a_{i,1}) : \sigma(i)=1\}\cup\{\varphi(x,a_{i,0}) : \sigma(i)=0\}$$

is consistent, say satisfied by some b_σ. Furthermore, $\{\varphi(x,a_{i,0}),\varphi(x,a_{i,1})\}$ is inconsistent for all $i<\omega$, and so it follows that b_σ satisfies

$$\{\varphi(x,a_{i,1}) : \sigma(i)=1\}\cup\{\neg\varphi(x,a_{i,1}) : \sigma(i)=0\}.$$

Therefore $\varphi(x,y)$, with $(a_{i,1})_{i<\omega}$ and $(b_\sigma)_{\sigma\in 2^\omega}$, witnesses IP. □

Altogether, we have shown the following:

Theorem 2.5.

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Uparrow \qquad \Uparrow \\ \mathrm{sOP}\Rightarrow\mathrm{SOP}\Rightarrow\ldots\Rightarrow\mathrm{SOP}_{n+1}\Rightarrow\mathrm{SOP}_n\Rightarrow\ldots\Rightarrow\mathrm{SOP}_3\Rightarrow(\mathrm{TP}_1\Leftrightarrow\mathrm{SOP}_2)\Rightarrow\mathrm{SOP}_1\ \mathrm{TP}\ \mathrm{TP}_2 \Rightarrow \mathrm{TP} \Rightarrow \mathrm{IP}\ \mathrm{OP} \end{array}$$

Remark 2.6. In [4], the following equivalences are proved,

$$\mathrm{OP}\Leftrightarrow(\mathrm{IP}\ \text{ or }\ \mathrm{sOP})\quad\text{and}\quad\mathrm{TP}\Leftrightarrow(\mathrm{TP}_1\ \text{ or }\ \mathrm{TP}_2).$$

We detail these proofs in the last section.

Recall again that a theory T is stable if and only if T does not have OP; and T is simple if and only if T does not have TP.

3. Alternate Definitions

In the literature, it is easy to find sources with slightly different definitions of the properties discussed above. While this can sometimes make a nominal difference when considering the property with respect to the formula, it usually does not make any difference when considering the property with respect to a theory.

Theorem 3.1. *Let $n \geq 3$. Then T has SOP_n if and only if there is a formula $\varphi(x, y)$, with $l(x) = l(y)$, such that for all $k \leq n$,*

$$\models \neg\exists x_1, \ldots, x_k(\varphi(x_1, x_2) \wedge \ldots \wedge \varphi(x_{k-1}, x_k) \wedge \varphi(x_k, x_1)),$$

and there are $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$.

Proof. Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses SOP_n. Then for all $k < n$, there are $\varphi_k(x, y)$ and $(a_i^k)_{k<\omega}$ witnessing SOP_k if $k \geq 3$ and OP (respectively an infinite model) if $k = 2$ (resp. $k = 1$). Define

$$\psi(x_1, \ldots, x_n, y_1, \ldots, y_n) := \varphi(x_n, y_n) \wedge \bigwedge_{k<n} \varphi_k(x_k, y_k).$$

Clearly, for all $k \leq n$, we have

$$\models \neg\exists \bar{x}_1, \ldots, \bar{x}_k(\psi(\bar{x}_1, \bar{x}_2) \wedge \ldots \wedge \psi(\bar{x}_{k-1}, \bar{x}_k) \wedge \psi(\bar{x}_k, \bar{x}_1)),$$

Moreover, if $\bar{a}_i = (a_i^1, \ldots, a_i^{n-1}, a_i^n)$, then $\models \psi(\bar{a}_i, \bar{a}_{i+1})$ for all $i < \omega$.

Conversely, suppose we have $\varphi(x, y)$, with $l(x) = l(y)$ and $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and for all $k \leq n$,

$$\models \neg\exists x_1, \ldots, x_k(\varphi(x_1, x_2) \wedge \ldots \wedge \varphi(x_{k-1}, x_k) \wedge \varphi(x_k, x_1)).$$

□

Theorem 3.2. *T has sOP if and only if there is a formula $\psi(x, y)$, with $l(x) = l(y)$, defining a partial order (reflexive, antisymmetric, transitive) with infinite chains.*

Proof. Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses that T has sOP. Define the formula,

$$\psi(y_1, y_2) := y_1 = y_2 \vee \Big(\forall x(\varphi(x, y_1) \to \varphi(x, y_2)) \wedge \exists x(\neg\varphi(x, y_1) \wedge \varphi(x, y_2))\Big).$$

In other words, for all $b, c \in \mathbb{M}$,

$$\models \psi(b, c) \quad \Leftrightarrow \quad b = c \text{ or } \varphi(\mathbb{M}, b) \subsetneq \varphi(\mathbb{M}, c).$$

Therefore $\psi(y_1, y_2)$ defines a partial order. By assumption we have $\varphi(\mathbb{M}, a_i) \subsetneq \varphi(\mathbb{M}, a_j)$ for all $i < j$, so $(a_i)_{i<\omega}$ is an infinite chain with respect to $\psi(y_1, y_2)$.

Conversely, suppose we have $\psi(x, y)$ defining a partial order with infinite chains. Let $(a_i)_{i<\omega}$ be an infinite chain, i.e., $\models \psi(a_i, a_j)$ and $a_i \neq a_j$ for all $i < j$. We claim that $\psi(x, y)$, with $(a_i)_{i<\omega}$ witnesses sOP. Indeed, if $i < j$ then we have $\models \neg\psi(a_j, a_i) \wedge \psi(a_j, a_j)$. On the other hand, if $c \in \mathbb{M}$ such that $\models \neg\psi(c, a_i) \wedge \psi(c, a_j)$ then $i < j$, since otherwise we would have $\models \psi(c, a_j) \wedge \psi(a_j, a_i)$, and so $\models \psi(c, a_i)$ by transitivity. $\square$

4. Equivalence Theorems

Definition 4.1. *A formula $\varphi(x, y)$ is **unstable** if there is some infinite set A such that $|S_\varphi(A)| > |A|$.*

Recall that T is stable if and only if no formula is unstable.

Lemma 4.2. *A formula $\varphi(x, y)$ is unstable if and only if it has* OP.

Proof. [4, II] Suppose $\varphi(x, y)$ is unstable. As in the proof of Proposition 1.14, there are $(a_i, b_i, d_i)_{i<\omega}$ such that

$$\models \varphi(d_i, a_j) \leftrightarrow \varphi(d_i, b_j) \text{ for all } i < j, \quad \text{and} \quad \models \varphi(d_i, a_j) \wedge \neg\varphi(d_i, b_j) \text{ for all } j \leq i.$$

Let $[\omega] = \{(i, j) : i < j < \omega\}$ and define $f : [\omega] \longrightarrow \{0, 1\}$ such that $f(i, j) = 0$ if and only if $\models \varphi(d_i, a_j)$. By Ramsey's Theorem, there is an infinite subset $I \subseteq \omega$ such that f is constant on $\{(i, j) \in I^2 : i < j\}$. By renaming, we may assume f is constant on $[\omega]$. If $f \equiv 0$ then we have $\models \varphi(d_i, b_j)$ if and only if $i < j$, so $\varphi(x, y)$ has OP. If $f \equiv 1$ then we have $\models \neg\varphi(d_i, a_j)$ if and only if $i < j$. Define

$$\Delta = T \cup \{\varphi(x_i, y_j) : i < j < \omega\} \cup \{\neg\varphi(x_i, y_j) : j \leq i < \omega\}.$$

If $\Delta_0 \subseteq \Delta$ is finite then let n be maximal such that x_n or y_n occurs as a variable in Δ_0. For $i \leq n$, interpret x_i as d_{n-i} and y_j as a_{n-j}, which satisfies Δ_0. Therefore Δ is satisfied by compactness and so $\varphi(x, y)$ has OP.

Suppose $\varphi(x, y)$ has OP. By compactness we may assume OP is witnessed by $(a_q)_{q\in\mathbb{Q}}$ and $(b_q)_{q\in\mathbb{Q}}$. Note that for all $q < r$ we have $\models \varphi(a_q, b_r) \wedge \neg\varphi(a_q, b_q)$, so if $A = \{b_q : q < \omega\}$ then A is countably infinite. Given $t \in \mathbb{R}\backslash\mathbb{Q}$, define the φ-type $p_t = \{\varphi(x, b_q) : q > t\} \cup \{\neg\varphi(x, b_q) : q < t\}$. By assumption and compactness, each p_t is consistent. If $s < t$ are irrational and $q \in \mathbb{Q}$ with $s < q < t$ then $\varphi(x, b_q) \in p_s$ and $\neg\varphi(x, b_q) \in p_t$. Therefore $|S_\varphi(A)| > |A|$ and so $\varphi(x, y)$ is unstable. $\square$

Theorem 4.3. *A formula $\varphi(x, y)$ is unstable if and only if $\theta(y, x) := \varphi(x, y)$ has* IP *or, for some $n < \omega$ and $\eta \in 2^n$*

$$\psi_\eta(x, y_0, \ldots, y_{n-1}) := \bigwedge_{\eta(i)=1} \varphi(x, y_i) \ \wedge \bigwedge_{\eta(i)=0} \neg\varphi(x, y_i)$$

has sOP*.*

Proof. [4, II] First, if $\varphi(x, y)$ has IP then it is unstable by Proposition 2.2 and Lemma 3.2. On the other hand suppose there is some $n < \omega$ and $\eta \in 2^n$ such that $\psi_\eta(x, \bar{y})$ has sOP, witnessed by $(a_i)_{i<\omega}$. If b_i is such that $\models \neg\psi_\eta(b_i, a_i) \wedge \psi_\eta(b_i, a_{i+1})$, then $\models \psi_\eta(b_i, a_j)$ if and only if $i < j$, so $\psi_\eta(x, y)$ is unstable by Lemma 3.2. Let A be infinite such that $|S_{\psi_\eta}(A)| > |A|$. Given $p \in S_{\psi_\eta}(A)$, let $a_p \models p$ and define

$$\hat{p} = \{\varphi(x, a) : a \in A, \ \models \varphi(a_p, a)\} \cup \{\neg\varphi(x, a) : a \in A, \ \models \neg\varphi(a_p, a)\}.$$

Clearly, each $\hat{p}$ is a consistent φ-type. Furthermore, if $p, q \in S_{\psi_n}(A)$ and $\hat{p} = \hat{q}$, then $p = q$. Therefore $|S_\varphi(A)| \geq |S_{\psi_n}(A)| > |A|$, and so $\varphi(x, y)$ is unstable.

Conversely, suppose $\varphi(x, y)$ is unstable. By Lemma 3.2, there are $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ witnessing that $\varphi(x, y)$ has OP. By replacing $(a_i, b_i)_{i<\omega}$ with a realization of $EM((a_i, b_i)_{i<\omega})$, we may assume $(a_i, b_i)_{i<\omega}$ is indiscernible. Suppose that for all $n < \omega$ and $\mu \in 2^n$ we have

$$\models \exists x \left(\bigwedge_{\mu(i)=1} \varphi(x, b_i) \ \wedge \bigwedge_{\mu(i)=0} \neg\varphi(x, b_i) \right).$$

Then for any $\sigma \in 2^\omega$, we have a solution c_σ to $\{\varphi(x, b_i) : \sigma(i) = 0\} \cup \{\neg\varphi(x, b_i) : \sigma(i) = 1\}$ by compactness. Setting $\theta(y, x) = \varphi(x, y)$, it follows that $\theta(y, x)$, with $(b_i)_{i<\omega}$ and $(c_\eta)_{\eta \in 2^n}$, witnesses IP. Therefore we may assume that there is some $n < \omega$ and $\mu \in 2^n$ such that

$$\models \neg\exists x \left(\bigwedge_{\mu(i)=1} \varphi(x, b_i) \ \wedge \bigwedge_{\mu(i)=0} \neg\varphi(x, b_i) \right).$$

Let $X_0 = \{i : \mu(i) = 1\}$ and set $m = |X_0|$. Note that $0 < m < n$. For some $N < \omega$, we construct sets $X_0, \ldots, X_N$ satisfying the following properties:

- (i) $X_N = \{n - m, n - m + 1, \ldots, n - 1\}$;
- (ii) for all $k \leq N$, $|X_k| = m$ and $X_k \subseteq \{0, \ldots, n - 1\}$;
- (iii) for all $k < N$ there is some $l \in X_k$ such that $X_{k+1} = (X_k \backslash \{l\}) \cup \{l + 1\}$ (note that altogether this implies $l \in X_k \backslash X_{k+1}$ and $l + 1 \in X_{k+1} \backslash X_k$).

This can be done in the following way. Let $X_0 = \{l_1, \ldots, l_m\}$ with $l_1 < \ldots < l_m$. Then $l_i \leq n-1+m-i$ for all i. The next set in the sequence is obtained from the current one by choosing i maximal with $l_i < n-1+m-i$ and replacing l_i with $l_i + 1$. Eventually we find $l_i = n - 1 + m - i$ for all i.

We have

$$\models \neg\exists x\left(\bigwedge_{i\in X_0}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_0,\ i<n}\neg\varphi(x,b_i)\right)\ \text{ and }\ \models \exists x\left(\bigwedge_{i\in X_N}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_N, i<n}\neg\varphi(x,b_i)\right),$$

where the second statement is witnessed with $x = a_{n-m-1}$. Therefore there is some $k < N$ such that

$$\models \neg\exists x\left(\bigwedge_{i\in X_k}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_k,\ i<n}\neg\varphi(x,b_i)\right)\ \text{ and }\ \models \exists x\left(\bigwedge_{i\in X_{k+1}}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_{k+1}, i<n}\neg\varphi(x,b_i)\right),$$

Let $l \in X_k$ be such that $X_{k+1} = (X_k\backslash\{l\}) \cup \{l+1\}$. Set

$$\psi(x,y,y_0,\dots,y_{l-1},y_{l+2},\dots,y_{n-1}) := \varphi(x,y)\ \wedge \bigwedge_{i\in X_k\backslash\{l\}}\varphi(x,y_i)\ \wedge \bigwedge_{i\notin X_{k+1}\cup\{l\},\ i<n}\neg\varphi(x,y_i).$$

For $r<\omega$, let $\bar{b}_r = (b_0,\dots,b_{l-1},b_{l+2+r},\dots,b_{n-1+r})$. Then we have $\models \exists x(\psi(x,b_{l+1},\bar{b}_0)\wedge\neg\varphi(x,b_l))$. Fixing $r<\omega$, for all $i,j<\omega$ with $l\leq i<j<l+2+r$, we have by indiscernibility

$$\models \exists x(\psi(x,b_j,\bar{b}_r)\wedge\neg\varphi(x,b_i)).$$

But $\models \neg\exists x(\psi(x,b_l,\bar{b}_0)\wedge\neg\varphi(x,b_{l+1}))$ so, similarly, for $r<\omega$ and $l\leq i<j<l+2+r$, we have

$$\models \neg\exists x(\psi(x,b_i,\bar{b}_r)\wedge\neg\varphi(x,b_j)).$$

It follows that for all $r<\omega$ and $l\leq i<j<l+2+r$,

$$\models \exists x(\psi(x,b_j,\bar{b}_r)\wedge\neg\psi(x,b_i,\bar{b}_r))\ \text{ and }\ \models\neg\exists x(\psi(x,b_i,\bar{b}_r)\wedge\neg\psi(x,b_j,\bar{b}_r)).$$

For $r<\omega$ and $i<r$, let $\bar{a}^r_i = (b_{l+i},\bar{b}_r)$. Then for all $r<\omega$ we have

$$\models \exists x(\neg(\psi(x,\bar{a}^r_i)\wedge\varphi(x,\bar{a}^r_j)\quad\Leftrightarrow\quad i<j.$$

By compactness, $\psi(x,y)$ has sOP. Clearly, ψ is of the desired form ψ_η, for some $\eta\in 2^{<\omega}$. □

Corollary 4.4. OP $\Leftrightarrow$ (IP *or* sOP). □

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