The aim of this paper is to define various properties of formulas in first order theories, and prove the appropriate implications between these properties. Most definitions are taken from [3], but the definitions themselves and many of the proofs are due to Shelah (see [4, II]). We give citations at the beginning of proofs taken from other sources.

Recall that a theory is stable if no formula has the so-called “order property”, and a theory is simple if no formula has the “tree property”. We first define these properties, along with a few more complicated properties of the same type. We fix some theory $T$ and a sufficiently saturated $M \models T$. If $\phi$ is a sentence with parameters from $M$, we write $M \models \phi$ if $M \models \phi$.

1. A Chain of Properties

Definition 1.1. A formula $\phi(x, y)$ has the **order property**, OP, if there are tuples $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ such that $M \models \phi(a_i, b_j)$ if and only if $i < j$.

For $n \geq 3$, a formula $\phi(x, y)$, with $l(x) = l(y)$, has the **$n$-strong order property**, SOP$_n$, if

$$M \models \neg \exists x_1, \ldots, x_n (\phi(x_1, x_2) \land \phi(x_2, x_3) \land \ldots \land \phi(x_n, x_1)),$$

and there are tuples $(a_i)_{i<\omega}$ such that $M \models \phi(a_i, a_j)$ for all $i < j < \omega$.

A formula $\phi(x, y)$, with $l(x) = l(y)$, has the **strong order property**, SOP, if for all $n \geq 3$

$$M \models \neg \exists x_1, \ldots, x_n (\phi(x_1, x_2) \land \phi(x_2, x_3) \land \ldots \land \phi(x_n, x_1)),$$

and there are tuples $(a_i)_{i<\omega}$ such that $M \models \phi(a_i, a_j)$ for all $i < j < \omega$.

A formula $\phi(x, y)$ has the **strict order property**, sOP, if there are tuples $(a_i)_{i<\omega}$ such that

$$M \models \exists x (\neg \phi(x, a_i) \land \phi(x, a_j)) \iff i < j.$$
Consider the definition of $\text{SOP}_n$ and its natural extension to $n = 1$ or $n = 2$. For $n = 2$ we have the order property. Moreover, any theory with an infinite model would satisfy the definition with $n = 1$ via the formula $x \neq y$. Therefore we will redefine $\text{SOP}_2$ and $\text{SOP}_1$ in the same vein as the next class of properties, which are defined using trees as index sets.

Before defining these properties, we specify some notation concerning trees.

**Definition 1.2.** Let $A$ be a set and define

$$A^{<\omega} = \bigcup_{n \in \omega} A^n.$$  

If $(a_0, \ldots, a_n), (b_0, \ldots, b_m) \in A^{<\omega}$, define

$$(a_0, \ldots, a_n) \cdot (b_0, \ldots, b_m) := (a_0, \ldots, a_n, b_0, \ldots, b_m) \in A^{<\omega}.$$  

If $\mu, \eta \in A^{<\omega}$, we say $\mu \prec \eta$ if there is some $\gamma \in A^{<\omega}$ such that $\eta = \mu \cdot \gamma$. For $a \in A$ we identify $a$ and $(a) \in A^{<\omega}$. If $n \in \omega$, we also define $(a)^n = (a, a, \ldots, a) \in A^{<\omega}$. Two elements $\mu, \eta \in A^{<\omega}$ are **incomparable** if $\mu \not\prec \eta$ and $\eta \not\prec \mu$.

The next class of properties on formulas are defined using tuples indexed by trees.

**Definition 1.3.** A formula $\varphi(x, y)$ has the **tree property**, $\text{TP}$, if there are tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ and some $k \geq 2$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|\eta}) : n < \omega\}$ is consistent; but for all $\eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta^{-n}}) : n < \omega\}$ is $k$-inconsistent.

A formula $\varphi(x, y)$ has the **tree property 1**, $\text{TP}_1$, if there are tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|\eta}) : n < \omega\}$ is consistent; but for all incomparable $\mu, \eta \in \omega^{<\omega}$, $\{\varphi(x, a_\mu), \varphi(x, a_\eta)\}$ is inconsistent.

A formula $\varphi(x, y)$ has $\text{SOP}_1$ if there are tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x, a_{\sigma|\eta}) : n < \omega\}$ is consistent; but for all $\mu, \eta \in 2^{<\omega}$, if $\mu , 0 < \eta$ then $\{\varphi(x, a_{\mu^{-1}}), \varphi(x, a_\eta)\}$ is inconsistent.

A formula $\varphi(x, y)$ has $\text{SOP}_2$ if there are tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x, a_{\sigma|\eta}) : n < \omega\}$ is consistent; but for all incomparable $\mu, \eta \in 2^{<\omega}$, $\{\varphi(x, a_\mu), \varphi(x, a_\eta)\}$ is inconsistent.

The goal of this section is to prove the following chain of implications (when $Q \Rightarrow R$ is written with no other information, we read this as “if $T$ has Q then $T$ has R”).

**Theorem 1.4.**

$s\text{OP} \Rightarrow \text{SOP} \Rightarrow \ldots \Rightarrow \text{SOP}_{n+1} \Rightarrow \text{SOP}_n \Rightarrow \ldots \Rightarrow \text{SOP}_3 \Rightarrow (\text{TP}_1 \Leftrightarrow \text{SOP}_2) \Rightarrow \text{SOP}_1 \Rightarrow \text{TP} \Rightarrow \text{OP}$. 
Proposition 1.5. sOP $\Rightarrow$ SOP.

Proof. Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses sOP. Let $l(x_1) = l(x_2) = l(y)$ and define
\[
\psi(x_1, x_2) := \forall x (\varphi(x, x_1) \rightarrow \varphi(x, x_2)) \land \exists x (\varphi(x, x_2) \land \lnot \varphi(x, x_1)).
\]
By assumption, $\models \psi(a_i, a_j)$ for all $i < j$. Suppose, towards a contradiction, that we have $n \geq 3$ and $b_1, \ldots, b_n$ such that
\[
\models \psi(b_1, b_2) \land \ldots \land \psi(b_{n-1}, b_n) \land \psi(b_n, b_1).
\]
If $B_i = \psi(M, b_i)$ for $1 \leq i \leq n$, then we have $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n \subseteq B_1$, which is a contradiction. Therefore $\psi(x_1, x_2)$, with $(a_i)_{i<\omega}$, witnesses SOP. \hfill $\square$

Proposition 1.6. SOP $\Rightarrow$ SOP$_n$ for all $n \geq 3$.

Proof. Follows by definition. \hfill $\square$

Proposition 1.7. For $n \geq 3$, SOP$_{n+1} \Rightarrow$ SOP$_n$.

Proof. Suppose $T$ has SOP$_{n+1}$, witnessed by $\varphi(x, y)$ and $(a_i)_{i<\omega}$. Define
\[
\psi(x_1, x_2, y_1, y_2) := \varphi(x_1, x_2) \land \varphi(x_2, y_1) \land \varphi(x_2, y_2) \land \varphi(y_1, y_2).
\]
If $i < j$ then $\models \psi(a_{2i}, a_{2i+1}, a_{2j}, a_{2j+1})$. Suppose, towards a contradiction, that $(b_{1,0}, b_{1,1}), \ldots, (b_{n,0}, b_{n,1})$ are such that
\[
M \models \psi(b_{1,0}, b_{1,1}, b_{2,0}, b_{2,1}) \land \ldots \land \psi(b_{n-1,0}, b_{n-1,1}, b_{n,0}, b_{n,1}) \land \psi(b_{n,0}, b_{n,1}, b_{1,0}, b_{1,1}).
\]
Then we have
\[
M \models \varphi(b_{1,0}, b_{1,1}) \land \varphi(b_{1,1}, b_{2,1}) \land \ldots \land \varphi(b_{n-1,1}, b_{n,1}) \land \varphi(b_{n,1}, b_{1,0}),
\]
contradicting that $\varphi(x, y)$ SOP$_{n+1}$. Therefore $\psi(x_1, x_2, y_1, y_2)$, with $(a_{2i}, a_{2i+1})_{i<\omega}$, witnesses SOP$_n$. \hfill $\square$

Proposition 1.8. SOP$_3 \Rightarrow$ SOP$_2$.

Proof. [2] Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses SOP$_3$. We have $\models \varphi(a_i, a_j)$ for all $i < j$. By compactness, we can obtain $(b_q)_{q \in \mathbb{Q}}$ such that $\models \varphi(b_q, b_r)$ for all $q < r$. Set $z = (y_1, y_2)$ and define
\[
\psi(x, z) := \varphi(y_1, x) \land \varphi(x, y_2).
\]
We define $(c_\eta)_{\eta \in 2^{<\omega}}$ inductively by $c_0 = (b_0, b_1)$, and if $c_\eta = (b_q, b_r)$, with $q < r$, then
\[
c_\eta^i = \begin{cases} 
(b_q, b_{\frac{1}{2}q(r-q)}) & i = 0 \\
(b_{\frac{1}{2}(r-q)}, b_r) & i = 1
\end{cases}
\]
We claim that $\psi(x, z)$, with $(c_\eta)_{\eta \in 2^{<\omega}}$, witnesses SOP$_2$. To this end, suppose $\sigma \in 2^\omega$ and $n < \omega$. There are $0 = q_0 < \ldots < q_n < r_n < r_{n-1} < \ldots < r_0 = 1$ such that for $0 \leq i \leq n$,

$$c_{\sigma|_i} = (b_{q_i}, b_{r_i}).$$

If $q_n < q < r_n$ then $\models \varphi(b_{q_i}, b_q) \land \varphi(b_q, b_{r_n})$ for all $0 \leq i \leq n$. Thus $\{\psi(x, c_{\sigma|_i}) : 0 \leq i \leq n\}$ is satisfiable, and so $\{\psi(x, c_{\sigma|_i}) : n < \omega\}$ is consistent by compactness.

Now suppose $\mu, \eta \in 2^{<\omega}$ are incomparable. Then, without loss of generality, we have $q < r < s < t$ such that

$$c_\mu = (b_{q_i}, b_r) \quad \text{and} \quad c_\eta = (b_s, b_t).$$

If $d$ satisfies $\{\psi(x, c_\mu), \psi(x, c_\eta)\}$ then we have

$$\varphi(d, b_r) \land \varphi(b_r, b_s) \land \varphi(b_s, d),$$

contradicting that $\varphi(x, y)$ witnesses SOP$_3$. Therefore $\{\psi(x, c_\mu), \psi(x, c_\eta)\}$ is inconsistent. \hfill $\Box$

**Proposition 1.9.** SOP$_2 \iff$ TP$_1$.

*Proof.* [3] Suppose $\varphi(x, y)$, with $(a_\eta)_{\eta \in 2^{<\omega}}$, witnesses SOP$_2$. Define $h : \omega^{<\omega} \rightarrow 2^{<\omega}$ inductively by $h(\emptyset) = \emptyset$ and for $i < \omega$,

$$h(\eta \hat{i}) = h(\eta)^*(1)^i.0.$$ 

If $\eta < \mu$, say $\mu = \eta^*(n_1, \ldots, n_k)$ with $n_i \in \omega$, then $h(\mu) = h(\eta)^*(1)^{\sum n_i}0$ so $h(\eta) \prec h(\mu)$. Thus if $\sigma \in \omega^{\omega}$, we may define $h(\sigma) := \bigcup_{n<\omega} h(\sigma|_n) \in 2^\omega$.

By assumption, $\{\varphi(x, a_{h(\sigma)|_n}) : n < \omega\}$ is consistent. If $\eta, \mu \in \omega^{<\omega}$ are incomparable then, without loss of generality, there are $\gamma, \eta_0, \mu_0 \in \omega^{<\omega}$ and $i < j$ such that $\eta = \gamma \hat{i} \eta_0$ and $\mu = \gamma \hat{j} \mu_0$. It follows that there are $\eta_1, \mu_1 \in 2^{<\omega}$ such that $h(\eta) = h(\gamma)^*(1)^i0 \eta_1$ and $h(\mu) = h(\gamma)^*(1)^j0 \mu_1$. Therefore $h(\eta)$ and $h(\mu)$ are incomparable, and so $\{\varphi(x, a_{h(\eta)}), \varphi(x, a_{h(\mu)})\}$ is inconsistent. In conclusion $\varphi(x, y)$ with $(a_{h(\eta)})_{\eta \in \omega^{<\omega}}$, witnesses TP$_1$.

Conversely, if $\varphi(x, y)$, with $(a_\eta)_{\eta \in \omega^{<\omega}}$, witnesses TP$_1$, then clearly $\varphi(x, y)$, with $(a_\eta)_{\eta \in 2^{<\omega}}$, witnesses SOP$_2$. \hfill $\Box$

**Proposition 1.10.** SOP$_2 \Rightarrow$ SOP$_1$.

*Proof.* Suppose $\varphi(x, y)$, with $(a_\eta)_{\eta \in 2^{<\omega}}$, witnesses SOP$_2$. For all $\mu, \eta \in 2^{<\omega}$, if $\mu'0 \prec \eta$ then $\mu'1$ and $\eta$ are incomparable, and so $\{\varphi(x, a_{\mu'1}), \varphi(x, a_\eta)\}$ is inconsistent. Thus $\varphi(x, y)$, with $(a_\eta)_{\eta \in 2^{<\omega}}$, witnesses SOP$_1$. \hfill $\Box$

**Proposition 1.11.** SOP$_1 \Rightarrow$ TP.
Proof. [2] Suppose \( \varphi(x,y) \), with \( (a_\eta)_{\eta < 2^{<\omega}} \), witnesses SOP\(_1\). Define \( h : \omega^{<\omega} \rightarrow 2^{<\omega} \) inductively such that \( h(\emptyset) = \emptyset \) and for \( i < \omega \),

\[
\eta h^i = h(\eta)^i \cdot 1.
\]

For \( \eta \in \omega^{<\omega} \), set \( b_\eta = a_h(\eta) \). As in the proof of Proposition 1.9, \( \mu < \eta \) implies \( h(\mu) < h(\eta) \). For \( \sigma \in \omega^{\omega} \), define \( h(\sigma) = \bigcup_{n < \omega} h(\sigma|_n) \). Then \( \{\varphi(x, b_{\sigma|_n}) : n < \omega\} \subseteq \{\varphi(x, a_{h(\sigma|_n)}) : n < \omega\} \), so \( \{\varphi(x, b_{\sigma|_n}) : n < \omega\} \) is consistent.

Now fix \( \eta \in \omega^{<\omega} \) and suppose \( i < j \). Then \( h(\eta)^i(0) < h(\eta^j) \) and \( h(\eta^i) = h(\eta^j)^i(1) \), so

\[
\{\varphi(x, a_{h(\eta^i)}), \varphi(x, a_{h(\eta^j)})\}
\]

is inconsistent by assumption. Therefore \( \{\varphi(x, b_{\eta^i}), \varphi(x, b_{\eta^j})\} \) is inconsistent, and so \( \{\varphi(x, b_{\eta|n}) : n < \omega\} \) is 2-inconsistent. Thus \( \varphi(x,y) \), with \( (b_\eta)_{\eta \in \omega^{<\omega}} \), witnesses TP. \( \square \)

The only remaining implication in the statement of Theorem 1.4 is TP \( \Rightarrow \) OP. This argument is a bit more technical than the previous one, and we break it into two steps, the proofs of which are taken from [4].

Lemma 1.12. Suppose \( \varphi(x,y) \) witnesses TP with respect to \( k \geq 2 \). Then there is an infinite set \( A \) such that \( |S_\varphi(A)| > |A| \).

Proof. [4, II] Let \( \kappa \) be an infinite cardinal such that \( \kappa^\omega > \max\{2^\omega, \kappa\} \). By compactness we may assume that we have \( (a_\eta)_{\eta < \kappa^{<\omega}} \) such that for all \( \sigma \in \kappa^{\omega} \),

\[
\pi_\sigma = \{\varphi(x, a_{\sigma|_n}) : n < \omega\}
\]

is consistent; and for all \( \eta \in \kappa^{<\omega}, \{\varphi(x, a_{\eta|n}) : i < \kappa\} \) is \( \kappa \)-inconsistent. Given \( \sigma \in \kappa^{\omega} \), construct \( F_\sigma \subseteq \kappa^{\omega} \) such that

\[
\begin{align*}
(i) & \quad \sigma \in F_\sigma; \\
(ii) & \quad \bigcup_{\tau \in F_\sigma} \pi_\tau \text{ is consistent.} \\
(iii) & \quad \text{for all } \rho \in \kappa^{\omega} \setminus F_\sigma, \pi_\rho \cup \bigcup_{\tau \in F_\sigma} \pi_\tau \text{ is inconsistent.}
\end{align*}
\]

Let \( T_\sigma = \{\tau|_n : n < \omega, \tau \in F_\sigma\} \). Then \( T_\sigma \) is a tree. Suppose, towards a contradiction, that there is \( \eta \in T_\sigma \) and distinct \( i_1, \ldots, i_k \in \kappa \) such that \( \eta^i j \in T_\sigma \) for all \( j \). Then there are \( \tau_1, \ldots, \tau_k \in F_\sigma \) such that \( \eta^i j \prec \tau_1 \), which is a contradiction since \( \{\varphi(x, a_{\eta|_j}) : 1 \leq j \leq k\} \) is inconsistent. It follows that \( T_\sigma \) can be embedded into \( k^{\omega} \). In particular, \( |F_\sigma| \leq 2^\omega \). Since \( \kappa^{\omega} > 2^\omega \), there is \( F \subseteq \kappa^{\omega} \) such that \( |F| = \kappa^{\omega} \) and \( F_\sigma \neq F_\tau \) for all distinct \( \sigma, \tau \in F \).

Let \( A = (a_\eta)_{\eta \in \kappa^{<\omega}} \) and, for \( \sigma \in F \), let \( p_\sigma \in S_\varphi(A) \) be a complete \( \varphi \)-type containing \( \bigcup_{\tau \in F_\sigma} \pi_\tau \). If \( \sigma, \tau \in F \) are distinct then, without loss of generality, there is some \( \rho \in F_\sigma \setminus F_\tau \). Then \( \pi_\rho \subseteq p_\sigma \) and \( p_\sigma \cup \pi_\rho \) is inconsistent. Therefore \( p_\sigma \neq p_\tau \), and so \( |S_\varphi(A)| \leq \kappa^{\omega} > \kappa = |A| \). \( \square \)
**Definition 1.13.** Given formulas \( \varphi(x,y), \psi(y,x) \), a type \( p(\psi, \varphi) \)-splits over a set \( B \) if there are \( a,b \in \text{dom}(p) \) such that \( \text{tp}_\psi(a/B) = \text{tp}_\psi(b/B) \), but \( \varphi(x,a), \neg\varphi(x,b) \in p \).

**Proposition 1.14.** TP \( \Rightarrow \) OP.

**Proof.** [4, II] Suppose \( \varphi(x,y) \) witnesses TP. By Lemma 1.12, there is some infinite cardinal \( \kappa \), and a set \( A \) of size \( \kappa \), such that \( |S_\varphi(A)| > \kappa \). Let \( (c_i)_{i<\kappa^+} \) be realizations of \( \kappa^+ \)-many distinct \( \varphi \)-types in \( S_\varphi(A) \). Set \( \psi(y,x) = \varphi(x,y) \). Let \( A_0 = A \) and, given \( A_n \) of size \( \kappa \), define

\[
A_{n+1} = A_n \cup \{ a : a \models p, p \in S_\varphi(B) \cup S_\psi(B), B \subseteq A_n \text{ is finite} \}.
\]

There are countably many finite subsets of \( A_n \), and if \( B \) is finite then \( S_\varphi(B) \cup S_\psi(B) \) is finite, so \( A_{n+1} \) still has size \( \kappa \).

**Claim:** There is some \( i < \kappa^+ \) such that for all \( n < \omega \) and for all \( B \subseteq A_n \), \( \text{tp}_\varphi(c_i/A_{n+1}) \) \( (\psi, \varphi) \)-splits over \( B \).

**Proof:** Suppose not. Then for all \( i < \kappa^+ \) there is a pair \((n, B)\) such that \( B \subseteq A_n \) is finite and \( \text{tp}_\varphi(c_i/A_{n+1}) \) does not \( (\psi, \varphi) \)-split over \( B \). There are only countably many such pairs \((n, B)\). Thus, without loss of generality, there is a pair \((n, B)\) such that \( B \subseteq A_n \) is finite and for all \( i < \kappa^+ \), \( \text{tp}(c_i/A_{n+1}) \) does not \( (\psi, \varphi) \)-split over \( B \). By definition, there is a finite set \( C \) such that \( B \subseteq C \subseteq A_{n+1} \) and all types in \( S_\varphi(B) \cup S_\psi(B) \) are realized in \( C \). Again, \( S_\varphi(C) \) is finite, so without loss of generality we may assume \( \text{tp}_\varphi(c_i/C) = \text{tp}_\varphi(c_j/C) \) for all \( i,j < \kappa^+ \).

Consider \( c_0, c_1 \). By assumption, there is some \( a \in A_0 \) such that \( \models \varphi(c_1, a) \leftrightarrow \neg\varphi(c_0, a) \). Let \( a' \in C \) such that \( \text{tp}_\varphi(a'/B) = \text{tp}_\varphi(a/B) \). For all \( i < \kappa^+ \), \( \text{tp}_\varphi(c_i/A_{n+1}) \) does not \( (\psi, \varphi) \)-split over \( B \), so it follows that \( \text{tp}_\varphi(c_i/C) \) does not \( (\psi, \varphi) \)-split over \( B \). Since \( \text{tp}_\varphi(a/B) = \text{tp}_\varphi(a'/B) \), we have \( \varphi(x, a) \in \text{tp}_\varphi(c_i/C) \) if and only if \( \varphi(x, a') \in \text{tp}_\varphi(c_i/C) \). In other words, \( \models \varphi(c_i, a) \leftrightarrow \varphi(c_i, a') \), for all \( i < \kappa^+ \). Altogether, we have

\[
\models \varphi(c_0, a) \leftrightarrow \varphi(c_0, a') \leftrightarrow \varphi(c_1, a') \leftrightarrow \varphi(c_1, a) \leftrightarrow \neg\varphi(c_0, a),
\]

which is a contradiction.

By the claim, we have \( i < \kappa^+ \) such that for all \( n < \omega \) and for all \( B \subseteq A_n \) finite, \( \text{tp}(c_i/A_{n+1}) \) \( (\psi, \varphi) \)-splits over \( B \). Set \( c = c_i \). Then \( \text{tp}_\varphi(c/A_1) \) \( (\psi, \varphi) \)-splits over \( B_0 \), so there are \( a_0, b_0 \in A_1 \) such that \( \text{tp}_\psi(a_0) = \text{tp}_\psi(b_0) \) with \( \varphi(x, a_0), \neg\varphi(x, b_0) \in \text{tp}(c/A_1) \). Now \( \{a_0, b_0\} \subseteq A_1 \) so there is some \( d_0 \in A_2 \) realizing \( \text{tp}_\varphi(c/a_0, b_0) \).

Suppose \( n > 0 \) and we are given \((a_i, b_i, d_i)_{i<n}\) such that for all \( i < n \),

\[
(i) \text{ tp}_\varphi(a_i/\{d_j : j < i\}) = \text{tp}_\psi(b_i/\{d_j : j < i\}); \\
(ii) d_i \in A_{2i+2} \text{ realizes } \text{tp}_\varphi(c/\{a_j, b_j : j \leq i\}); \\
(iii) \models \varphi(c, a_i) \land \neg\varphi(c, b_i).
\]
Then $tp\phi(c/A_{2n+1}) (\psi, \varphi)$-splits over $\{d_i : i < n\} \subseteq A_{2n}$ so there are $a_n, b_n \in A_{2n+1}$ such that $tp\phi(a_n/\{d_i : i < n\}) = tp\phi(b_n/\{d_i : i < n\})$ and $\varphi(x, a_n), \neg \varphi(x, b_n) \in tp\phi(c/A_{2n+1})$. But $tp\phi(c/\{a_i, b_i : i \leq n\})$ is realized by some $d_n \in A_{2n+2}$. This process generates $(a_n, b_n, d_n)_{n<\omega}$ such that for all $n < \omega$,

(i) $tp\phi(a_n/\{d_i : i < n\}) = tp\phi(b_n/\{d_i : i < n\})$;
(ii) $d_n \in A_{2n+2}$ realizes $tp\phi(c/\{a_i, b_i : i \leq n\})$;
(iii) $\models \varphi(c, a_n) \land \neg \varphi(c, b_n)$.

Note first that for all $j \leq i$, we have $\models \varphi(d_i, a_j) \land \neg \varphi(d_i, b_j)$. Moreover, for all $i < j$,

$\models \varphi(d_i, a_j) \leftrightarrow \psi(a_j, d_i) \leftrightarrow \psi(b_j, d_i) \leftrightarrow \varphi(d_i, b_j)$.

Therefore we have

$\models \varphi(d_i, a_j) \leftrightarrow \varphi(d_i, b_j) \iff i < j$.

Altogether, if $z = (y_1, y_2)$ and $\theta(x, z) := \varphi(x, y_1) \leftrightarrow \varphi(x, y_2)$, then $\theta(x, z)$, with $(d_i)_{i<\omega}$ and $(a_i, b_i)_{i<\omega}$, witnesses OP.

This completes the proof of Theorem 1.4.

2. Further Properties

We now define two more properties, which do not fit exactly into the chain in Theorem 1.4.

**Definition 2.1.** A formula $\varphi(x, y)$ has the **independence property**, IP, if there are $(a_i)_{i<\omega}$ and $(c_\sigma)_{\sigma \in 2^\omega}$ such that $\models \varphi(a_i, c_\sigma)$ if and only if $\sigma(i) = 1$.

A formula $\varphi(x, y)$ has the **tree property 2**, TP$_2$, if there are $(a_{i,j})_{i,j<\omega}$ such that for any $\sigma \in \omega^\omega$, $
\{\varphi(x, a_n, \sigma(n)) : n < \omega\}$ is consistent; but for all $j < k < \omega$, $\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k})\}$ is inconsistent.

**Proposition 2.2.** IP $\Rightarrow$ OP.

*Proof.* Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$ and $(c_\sigma)_{\sigma \in 2^\omega}$, witnesses IP. Given $i < \omega$, let $\sigma_i : \omega \rightarrow \omega$ such that $\sigma_i(j) = 0$ if and only if $i \leq j$. Then we have

$\models \varphi(a_i, c_{\sigma_i}) \iff \sigma_j(i) = 1 \iff i < j$.

So $\varphi(x, y)$, with $(a_i)_{i<\omega}$ and $(c_\sigma)_{i<\omega}$, witnesses OP.

**Proposition 2.3.** TP$_2$ $\Rightarrow$ TP.
Proof. [1] Suppose $\varphi(x,y)$, with $(a_{i,j})_{i,j<\omega}$, witnesses TP$_2$. Fix an injection $f: \omega \times \omega \to \omega$. Set $b_0 = a_{0,0}$, and for $i < \omega$, set $b(i,j) = a_{i,j}$. Suppose $0 < n < \omega$ and for all $\eta \in \omega^n$ we have $j < \omega$ such that $b_n = a_{n,j}$. Let $(b_\eta)_{\eta<\omega}$ be an enumeration of $\omega^n$ and for $j < \omega$ define $b_{\eta}^{-1} = a_{n+1,f(i,j)}$.

We claim that $\varphi(x,y)$, with $(b_\eta)_{\eta<\omega}$, witnesses TP with respect to 2. If $\sigma \in \omega^\omega$, then for all $n < \omega$, $b_{\sigma|n} = a_{n,j}$ for some $j < \omega$. So if $\tau: \omega \to \omega$ is such that $\tau(n) = j$, we have that
\[ \{ \varphi(x,b_{\sigma|n}) : n < \omega \} = \{ \varphi(x,a_{n,\tau(n)}) : n < \omega \} \]
is consistent. Furthermore suppose $\eta \in \omega^\omega$ and $j < k < \omega$. If $|\eta| = n$, then $b_n^\eta = a_{n+1,f(i,j)}$ and $b_n^\eta = a_{n+1,f(i,k)}$, where $\eta = \eta_i$ in the enumeration of $\omega^n$. Since $f$ is injective, it follows that $f(i,j) \neq f(i,k)$, and so
\[ \{ \varphi(x,b_n^\eta,j), \varphi(x,b_n^\eta,k) \} = \{ \varphi(x,a_{n+1,f(i,j)}), \varphi(x,a_{n+1,f(i,k)}) \} \]
is inconsistent by assumption. □

Proposition 2.4. TP$_2 \Rightarrow$ IP.

Proof. [1] Suppose $\varphi(x,y)$, with $(a_{i,j})_{i,j<\omega}$, witnesses TP$_2$. Let $\sigma \in 2^\omega$. By assumption,
\[ \{ \varphi(x,a_{i,1}) : \sigma(i) = 1 \} \cup \{ \varphi(x,a_{i,0}) : \sigma(i) = 0 \} \]
is consistent, say satisfied by some $b_\sigma$. Furthermore, $\{ \varphi(x,a_{i,0}), \varphi(x,a_{i,1}) \}$ is inconsistent for all $i < \omega$, and so it follows that $b_\sigma$ satisfies
\[ \{ \varphi(x,a_{i,1}) : \sigma(i) = 1 \} \cup \{ \neg \varphi(x,a_{i,1}) : \sigma(i) = 0 \} . \]
Therefore $\varphi(x,y)$, with $(a_{i,1})_{i<\omega}$ and $(b_\sigma)_{\sigma \in 2^\omega}$, witnesses IP. □

Altogether, we have shown the following:

Theorem 2.5.

\[ sOP \Rightarrow SOP \Rightarrow \ldots \Rightarrow SOP_{n+1} \Rightarrow SOP_n \Rightarrow \ldots \Rightarrow SOP_3 \Rightarrow (TP_1 \Leftrightarrow SOP_2) \Rightarrow SOP_1 \Uparrow \Downarrow TP \Uparrow \Downarrow IP \]

Remark 2.6. In [4], the following equivalences are proved,
\[ OP \Leftrightarrow (IP \text{ or } sOP) \text{ and } TP \Leftrightarrow (TP_1 \text{ or } TP_2) . \]

We detail these proofs in the last section.

Recall again that a theory $T$ is stable if and only if $T$ does not have OP; and $T$ is simple if and only if $T$ does not have TP.
3. Alternate Definitions

In the literature, it is easy to find sources with slightly different definitions of the properties discussed above. While this can sometimes make a nominal difference when considering the property with respect to the formula, it usually does not make any difference when considering the property with respect to a theory.

**Theorem 3.1.** Let $n \geq 3$. Then $T$ has SOP$_n$ if and only if there is a formula $\varphi(x, y)$, with $l(x) = l(y)$, such that for all $k \leq n$,
\[\models \neg \exists x_1, \ldots, x_k(\varphi(x_1, x_2) \land \ldots \land \varphi(x_{k-1}, x_k) \land \varphi(x_k, x_1)),\]
and there are $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$.

**Proof.** Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses SOP$_n$. Then for all $k < n$, there are $\varphi_k(x, y)$ and $(a^k_i)_{k<\omega}$ witnessing SOP$_k$ if $k \geq 3$ and OP (respectively an infinite model) if $k = 2$ (resp. $k = 1$). Define
\[\psi(x_1, \ldots, x_n, y_1, \ldots, y_n) := \varphi(x_n, y_n) \land \bigwedge_{k<n} \varphi_k(x_k, y_k).\]
Clearly, for all $k \leq n$, we have
\[\models \neg \exists x_1, \ldots, x_k(\varphi(x_1, x_2) \land \ldots \land \varphi(x_{k-1}, x_k) \land \varphi(x_k, x_1)),\]
Moreover, if $\bar{a}_i = (a^1_i, \ldots, a^{n-1}_i, a^0_i)$, then $\models \psi(\bar{a}_i, \bar{a}_{i+1})$ for all $i < \omega$.

Conversely, suppose we have $\varphi(x, y)$, with $l(x) = l(y)$ and $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and for all $k \leq n$,
\[\models \neg \exists x_1, \ldots, x_k(\varphi(x_1, x_2) \land \ldots \land \varphi(x_{k-1}, x_k) \land \varphi(x_k, x_1)).\]

**Theorem 3.2.** $T$ has sOP if and only if there is a formula $\psi(x, y)$, with $l(x) = l(y)$, defining a partial order (reflexive, antisymmetric, transitive) with infinite chains.

**Proof.** Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses that $T$ has sOP. Define the formula,
\[\psi(y_1, y_2) := y_1 = y_2 \lor \left( \forall x(\varphi(x, y_1) \rightarrow \varphi(x, y_2)) \land \exists x(\neg \varphi(x, y_1) \land \varphi(x, y_2)) \right).\]
In other words, for all $b,c \in \mathbb{M}$,
\[\models \psi(b, c) \iff b = c \text{ or } \varphi(\mathbb{M}, b) \subsetneq \varphi(\mathbb{M}, c).\]
Therefore $\psi(y_1, y_2)$ defines a partial order. By assumption we have $\varphi(M, a_i) \subseteq \varphi(M, a_j)$ for all $i < j$, so $(a_i)_{i<\omega}$ is an infinite chain with respect to $\psi(y_1, y_2)$.

Conversely, suppose we have $\psi(x, y)$ defining a partial order with infinite chains. Let $(a_i)_{i<\omega}$ be an infinite chain, i.e., $\models \psi(a_i, a_j)$ and $a_i \neq a_j$ for all $i < j$. We claim that $\psi(x, y)$, with $(a_i)_{i<\omega}$ witnesses sOP. Indeed, if $i < j$ then we have $\models \neg \psi(a_j, a_i) \land \psi(a_j, a_j)$. On the other hand, if $c \in M$ such that $\models \neg \psi(c, a_i) \land \psi(c, a_j)$ then $i < j$, since otherwise we would have $\models \psi(c, a_j) \land \psi(a_j, a_i)$, and so $\models \psi(c, a_i)$ by transitivity.

\[ \square \]

4. Equivalence Theorems

**Definition 4.1.** A formula $\varphi(x, y)$ is **unstable** if there is some infinite set $A$ such that $|S_\varphi(A)| > |A|$.

Recall that $T$ is stable if and only if no formula is unstable.

**Lemma 4.2.** A formula $\varphi(x, y)$ is unstable if and only if it has OP.

**Proof.** [4, II] Suppose $\varphi(x, y)$ is unstable. As in the proof of Proposition 1.14, there are $(a_i, b_i, d_i)_{i<\omega}$ such that

$\models \varphi(d_i, a_j) \leftrightarrow \varphi(d_i, b_j)$ for all $i < j$, and $\models \varphi(d_i, a_j) \land \neg \varphi(d_i, b_j)$ for all $j \leq i$.

Let $[\omega] = \{(i, j) : i < j < \omega\}$ and define $f : [\omega] \to \{0, 1\}$ such that $f(i, j) = 0$ if and only if $\models \varphi(d_i, a_j)$. By Ramsey’s Theorem, there is an infinite subset $I \subseteq \omega$ such that $f$ is constant on $\{(i, j) \in I^2 : i < j\}$. By renaming, we may assume $f$ is constant on $[\omega]$. If $f \equiv 0$ then we have $\models \varphi(d_i, b_j)$ if and only if $i < j$, so $\varphi(x, y)$ has OP. If $f \equiv 1$ then we have $\models \neg \varphi(d_i, a_j)$ if and only if $i < j$. Define

$\Delta = T \cup \{\varphi(x_i, y_j) : i < j < \omega\} \cup \{\neg \varphi(x_i, y_j) : j \leq i < \omega\}$.

If $\Delta_0 \subseteq \Delta$ is finite then let $n$ be maximal such that $x_n$ or $y_n$ occurs as a variable in $\Delta_0$. For $i \leq n$, interpret $x_i$ as $d_{n-i}$ and $y_j$ as $a_{n-j}$, which satisfies $\Delta_0$. Therefore $\Delta$ is satisfied by compactness and so $\varphi(x, y)$ has OP.

Suppose $\varphi(x, y)$ has OP. By compactness we may assume OP is witnessed by $(a_q)_{q \in \mathbb{Q}}$ and $(b_q)_{q \in \mathbb{Q}}$. Note that for all $q < r$ we have $\models \varphi(a_q, b_r) \land \neg \varphi(a_q, b_q)$, so if $A = \{b_q : q < \omega\}$ then $A$ is countably infinite. Given $t \in \mathbb{R} \setminus \mathbb{Q}$, define the $\varphi$-type $p_t = \{\varphi(x, b_q) : q > t\} \cup \{\neg \varphi(x, b_q) : q < t\}$. By assumption and compactness, each $p_t$ is consistent. If $s < t$ are irrational and $q \in \mathbb{Q}$ with $s < q < t$ then $\varphi(x, b_q) \in p_s$ and $\neg \varphi(x, b_q) \in p_t$. Therefore $|S_\varphi(A)| > |A|$ and so $\varphi(x, y)$ is unstable. \[ \square \]
Theorem 4.3. A formula $\varphi(x, y)$ is unstable if and only if $\theta(y, x) := \varphi(x, y)$ has IP or, for some $n < \omega$ and $\eta \in 2^n$

$$
\psi_\eta(x, y_0, \ldots, y_{n-1}) := \bigwedge_{\eta(i) = 1} \varphi(x, y_i) \land \bigwedge_{\eta(i) = 0} \neg \varphi(x, y_i)
$$

has sOP.

Proof. [4, II] First, if $\varphi(x, y)$ has IP then it is unstable by Proposition 2.2 and Lemma 3.2. On the other hand suppose there is some $n < \omega$ such that $\psi_\eta(x, \bar{y})$ has sOP, witnessed by $(a_i)_{i < \omega}$. If $b_i$ is such that $\models \neg \psi_\eta(b_i, a_i) \land \psi_\eta(b_i, a_{i+1})$, then $\models \psi_\eta(b_i, a_j)$ if and only if $i < j$, so $\psi_\eta(x, y)$ is unstable by Lemma 3.2. Let $\mathcal{A}$ be infinite such that $|S_{\psi_\eta}(\mathcal{A})| > |\mathcal{A}|$. Given $p \in S_{\psi_\eta}(\mathcal{A})$, let $a_p \models p$ and define

$$
\hat{p} = \{ \varphi(x, a) : a \in \mathcal{A}, \models \varphi(a_p, a) \} \cup \{ \neg \varphi(x, a) : a \in \mathcal{A}, \models \neg \varphi(a_p, a) \}.
$$

Clearly, each $\hat{p}$ is a consistent $\varphi$-type. Furthermore, if $p, q \in S_{\psi_\eta}(\mathcal{A})$ and $\hat{p} = \hat{q}$, then $p = q$. Therefore $|S_p(\mathcal{A})| \geq |S_{\psi_\eta}(\mathcal{A})| > |\mathcal{A}|$, and so $\varphi(x, y)$ is unstable.

Conversely, suppose $\varphi(x, y)$ is unstable. By Lemma 3.2, there are $(a_i)_{i < \omega}$ and $(b_i)_{i < \omega}$ witnessing that $\varphi(x, y)$ has OP. By replacing $(a_i, b_i)_{i < \omega}$ with a realization of $EM((a_i, b_i)_{i < \omega})$, we may assume $(a_i, b_i)_{i < \omega}$ is indiscernible. Suppose that for all $n < \omega$ and $\mu \in 2^n$ we have

$$
\models \exists x \left( \bigwedge_{\mu(i) = 1} \varphi(x, b_i) \land \bigwedge_{\mu(i) = 0} \neg \varphi(x, b_i) \right).
$$

Then for any $\sigma \in 2^\omega$, we have a solution $c_\sigma$ to $\{ \varphi(x, b_i) : \sigma(i) = 0 \} \cup \{ \neg \varphi(x, b_i) : \sigma(i) = 1 \}$ by compactness. Setting $\theta(y, x) = \varphi(x, y)$, it follows that $\theta(y, x)$, with $(b_i)_{i < \omega}$ and $(c_\eta)_{\eta \in 2^\omega}$, witnesses IP. Therefore we may assume that there is some $n < \omega$ and $\mu \in 2^n$ such that

$$
\models \neg \exists x \left( \bigwedge_{\mu(i) = 1} \varphi(x, b_i) \land \bigwedge_{\mu(i) = 0} \neg \varphi(x, b_i) \right).
$$

Let $X_0 = \{ i : \mu(i) = 1 \}$ and set $m = |X_0|$. Note that $0 < m < n$. For some $N < \omega$, we construct sets $X_0, \ldots, X_N$ satisfying the following properties:

(i) $X_N = \{ n - m, n - m + 1, \ldots, n - 1 \}$;

(ii) for all $k \leq N$, $|X_k| = m$ and $X_k \subseteq \{ 0, \ldots, n - 1 \}$;

(iii) for all $k < N$ there is some $l \in X_k$ such that $X_{k+1} = (X_k \setminus \{ l \}) \cup \{ l + 1 \}$ (note that altogether this implies $l \in X_k \setminus X_{k+1}$ and $l + 1 \in X_{k+1} \setminus X_k$).

This can be done in the following way. Let $X_0 = \{ l_1, \ldots, l_m \}$ with $l_1 < \ldots < l_m$. Then $l_i \leq n - 1 + m - i$ for all $i$. The next set in the sequence is obtained from the current one by choosing $i$ maximal with $l_i < n - 1 + m - i$ and replacing $l_i$ with $l_i + 1$. Eventually we find $l_i = n - 1 + m - i$ for all $i$. 

We have

$$\models \exists x \left( \bigwedge_{i \in X_0} \varphi(x, b_i) \land \bigwedge_{i \not\in X_0, i < n} \neg \varphi(x, b_i) \right) \quad \text{and} \quad \models \exists x \left( \bigwedge_{i \in X_N} \varphi(x, b_i) \land \bigwedge_{i \not\in X_N, i < n} \neg \varphi(x, b_i) \right),$$

where the second statement is witnessed with $x = a_{n-m-1}$. Therefore there is some $k < N$ such that

$$\models \exists x \left( \bigwedge_{i \in X_k} \varphi(x, b_i) \land \bigwedge_{i \not\in X_k, i < n} \neg \varphi(x, b_i) \right) \quad \text{and} \quad \models \exists x \left( \bigwedge_{i \in X_{k+1}} \varphi(x, b_i) \land \bigwedge_{i \not\in X_{k+1}, i < n} \neg \varphi(x, b_i) \right),$$

Let $l \in X_k$ be such that $X_{k+1} = (X_k \setminus \{l\}) \cup \{l + 1\}$. Set

$$\psi(x, y, y_0, \ldots, y_{i-1}, y_{i+2}, \ldots, y_{n-1}) := \varphi(x, y) \land \bigwedge_{i \in X_k \setminus \{l\}} \varphi(x, y_i) \land \bigwedge_{i \not\in X_{k+1} \cup \{l\}, i < n} \neg \varphi(x, y_i).$$

For $r < \omega$, let $\bar{b}_r = (b_0, \ldots, b_{i-1}, b_{i+2}, \ldots, b_{n-1+r})$. Then we have $\models \exists x(\psi(x, b_{i+1}, \bar{b}_0) \land \neg \varphi(x, b_{i})).$ Fixing $r < \omega$, for all $i, j < \omega$ with $l \leq i < j < l + 2 + r$, we have by indiscernibility

$$\models \exists x(\psi(x, b_{i}, \bar{b}_{r}) \land \neg \varphi(x, b_{i})).$$

But $\models \exists x(\psi(x, b_{i}, \bar{b}_{0}) \land \neg \varphi(x, b_{i+1}))$ so, similarly, for $r < \omega$ and $l \leq i < j < l + 2 + r$, we have

$$\models \neg \exists x(\psi(x, b_{i}, \bar{b}_{r}) \land \neg \varphi(x, b_{j})).$$

It follows that for all $r < \omega$ and $l \leq i < j < l + 2 + r$,

$$\models \exists x(\psi(x, b_{i}, \bar{b}_{r}) \land \neg \psi(x, b_{i}, \bar{b}_{r})) \quad \text{and} \quad \models \neg \exists x(\psi(x, b_{i}, \bar{b}_{r}) \land \neg \psi(x, b_{j}, \bar{b}_{r})).$$

For $r < \omega$ and $i < r$, let $\bar{a}^r_i = (b_{i+1}, \bar{b}_{r}).$ Then for all $r < \omega$ we have

$$\models \exists x((\neg \psi(x, \bar{a}^r_i) \land \varphi(x, \bar{a}^r_j)) \iff i < j.$$

By compactness, $\psi(x, y)$ has $\text{sOP}$. Clearly, $\psi$ is of the desired form $\psi_\eta$, for some $\eta \in 2^{<\omega}$. \hfill \Box

**Corollary 4.4.** OP $\iff$ (IP or sOP). \hfill \Box

**References**


http://www.ub.edu/modeltheory/documentos/nip.pdf

