Model theory and combinatorics of homogeneous metric spaces

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March 25, 2015
Annual North American Meeting of the ASL
University of Illinois at Urbana-Champaign
Definition

(1) Let $\mathcal{L}_{om} = \{\oplus, \leq, 0\}$ be the language of ordered monoids.
(2) We refer to a totally and positively ordered commutative monoid as a **distance monoid**.

Given a distance monoid $\mathcal{R}$, we have the natural notion of $\mathcal{R}$-**metric spaces**.

**Definition**

Given a countable distance monoid $\mathcal{R}$, an $\mathcal{R}$-**Urysohn space** is a countable $\mathcal{R}$-metric space $(X, d)$ such that:

(i) (**ultrahomogeneity**) every isometry between finite subspaces of $X$ extends to a total isometry of $X$;

(ii) (**universality**) every finite $\mathcal{R}$-metric space is isometric to a subspace of $X$. 
Fact (Sauer)

The existence of $\mathcal{U}_R$ is equivalent to associativity of $\oplus$.

Examples:

- If $Q = (\mathbb{Q}_{\geq 0}, +, \leq, 0)$ then $\mathcal{U}_Q$ is the rational Urysohn space. Cameron-Vershik, Melleray, Solecki, Tent-Ziegler

- If $R = (R, \leq, 0)$ is a countable linear order, and $u \oplus v = \max\{u, v\}$, then $\mathcal{U}_R$ exists and is an ultrametric Urysohn space. Gao-Shao, Nguyen Van Thé

- Fix $0 \in S \subseteq \mathbb{N}$ and, for $u, v \in S$, define

$$u +_S v = \max\{x \in S : x \leq u + v\}.$$ 

Let $S = (S, +_S, \leq, 0)$. $\mathcal{U}_S$ exists if and only if $+_S$ is associative. Casanovas-Wagner, Delhommé-Laflamme-Pouzet-Sauer, Nguyen Van Thé, Sauer

- In the last example, if $S = \{0, 1, 2\}$ then $\mathcal{U}_S$ is the countable random graph.
Given a countable distance monoid \( \mathcal{R} \), we fix a relational language

\[
\mathcal{L}_\mathcal{R} = \{ d(x, y) \leq r : r \in R \},
\]
where each \( d(x, y) \leq r \) is a binary relation.

Let \( \text{Th}(\mathcal{U}_\mathcal{R}) \) be the complete \( \mathcal{L}_\mathcal{R} \)-theory of \( \mathcal{U}_\mathcal{R} \).

**Theorem (C.)**

There is a distance monoid extension \( \mathcal{R}^* \) of \( \mathcal{R} \) such that any model \( M \models \text{Th}(\mathcal{U}_\mathcal{R}) \) is an \( \mathcal{R}^* \)-metric space under a “type-definable” \( \mathcal{R}^* \)-metric.

**Idea:**

- \( \mathcal{R}^* \) is the set of quantifier-free 2-types consistent with \( \text{Th}(\mathcal{U}_\mathcal{R}) \).
- Given \( M \models \text{Th}(\mathcal{U}_\mathcal{R}) \) and \( a, b \in M \), \( d(a, b) \) is the quantifier-free 2-type of \( (a, b) \).
- Given \( \alpha, \beta \in \mathcal{R}^* \), \( \alpha \oplus \beta \) is the largest \( \gamma \in \mathcal{R}^* \) such that a triangle with distances \( \alpha, \beta, \gamma \) is consistent.
Lemma (C.)

If $M$ is a saturated model of $\text{Th}(\mathcal{U}_R)$, of cardinality $\kappa$, then $M$ is a $\kappa^+$-universal $\mathcal{R}^*$-metric space.

In order for $M$ to be $\kappa$-homogeneous as an $\mathcal{R}^*$-metric space, we need quantifier elimination for $\text{Th}(\mathcal{U}_R)$.

Theorem (C.)

$\text{Th}(\mathcal{U}_R)$ has quantifier elimination if and only if for all $r \in R$, $\alpha \in R^*$, if $\alpha$ has no immediate predecessor in $R^*$ then

$$\alpha \oplus r = \sup\{x \oplus r : x < \alpha\}.$$  

Definition

A countable distance monoid $\mathcal{R}$ is **Urysohn** if $\text{Th}(\mathcal{U}_R)$ has quantifier elimination.
Definition

A property $P$ of $\mathcal{R}$-Urysohn spaces is **axiomatizable** if there is an $\mathcal{L}_{\omega_1, \omega}$-sentence $\varphi_P$ in $\mathcal{L}_{om}$ such that, given a countable Urysohn monoid $\mathcal{R}$,

$$\mathcal{U}_R \text{ has property } P \text{ if and only if } \mathcal{R} \models \varphi_P.$$ 

If $\varphi_P$ is an $\mathcal{L}_{\omega, \omega}$-sentence then $P$ is **finitely axiomatizable**.

Definition

The **archimedean rank** of $\mathcal{R}$, $\text{arch}(\mathcal{R})$, is the minimum $n < \omega$ such that, for all $r_0 \leq r_1 \leq \ldots \leq r_n$ in $\mathcal{R}$,

$$r_0 \oplus r_1 \oplus \ldots \oplus r_n = r_1 \oplus \ldots \oplus r_n.$$

If no such $n$ exists, set $\text{arch}(\mathcal{R}) = \omega$. 

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Theorem (C.)

Suppose $\mathcal{R}$ is a Urysohn monoid.

(a) $\text{Th}(\mathcal{U}_\mathcal{R})$ is stable if and only if $\text{arch}(\mathcal{R}) \leq 1$ (finitely axiomatizable). I.e., $\text{Th}(\mathcal{U}_\mathcal{R})$ is stable if and only if $\mathcal{U}_\mathcal{R}$ is ultrametric.

(b) $\text{Th}(\mathcal{U}_\mathcal{R})$ is simple if and only if $\text{arch}(\mathcal{R}) \leq 2$ (finitely axiomatizable).

(c) $\text{Th}(\mathcal{U}_\mathcal{R})$ does not have the strict order property.

(d) $\text{Th}(\mathcal{U}_\mathcal{R})$ is $\text{NSOP}_n$ if and only if $\text{arch}(\mathcal{R}) < n$ (finitely axiomatizable).

(e) $\text{Th}(\mathcal{U}_\mathcal{R})$ is superstable if and only if $\text{arch}(\mathcal{R}) \leq 1$ and $\mathcal{R}$ is well-ordered (not axiomatizable).

(f) "$\text{Th}(\mathcal{U}_\mathcal{R})$ is supersimple" is not axiomatizable.

The proofs of these results use a geometric characterization of nonforking independence, along with a fine analysis of amalgamation of indiscernible sequences.
Definition

(1) Given $n > 0$, let $\text{DM}(n)$ be the number of distance monoids with $n$ nontrivial elements.

(2) Given $n, k > 0$, let $\text{DM}(n, k)$ be the number of distance monoids with $n$ nontrivial elements and archimedean rank $k$.

Proposition (C.)

(a) **Upper bound:** $\text{DM}(n) = O(c^{n^2})$, where $c = \sqrt{\frac{27}{16}} \approx 1.299$. (Uses Zeilberger’s proof of the *Alternating Sign Matrix Conjecture*.)

(b) If $n < k$ then $\text{DM}(n, k) = 0$.

(c) If $n > 0$ then $\text{DM}(n, 1) = 1$ (witness: $\{0, 1, \ldots, n\}, \text{max}, \leq, 0$).

(d) If $n > 0$ then $\text{DM}(n, n) = 1$ (witness: $\{0, 1, \ldots, n\}, +, \leq, 0$).
**Theorem (C., continuation of Nguyen Van Thé)**

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