Model theory and combinatorics of homogeneous metric spaces

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Definition

(1) Let $\mathcal{L}_{\text{om}} = \{\oplus, \leq, 0\}$ be the language of ordered monoids.

(2) We refer to a totally and positively ordered commutative monoid as a **distance monoid**.

Given a distance monoid $\mathcal{R}$, we have the natural notion of $\mathcal{R}$**-metric spaces.**

Definition

Given a countable distance monoid $\mathcal{R}$, an $\mathcal{R}$**-Urysohn space** is a countable $\mathcal{R}$-metric space (X, d) such that:

(i) (*ultrahomogeneity*) every isometry between finite subspaces of X extends to a total isometry of X;

(ii) (*universality*) every finite $\mathcal{R}$-metric space is isometric to a subspace of X.

Fact (Sauer)

The existence of $\mathcal{U}_{\mathcal{R}}$ is equivalent to associativity of $\oplus$.

Examples:

- If $\mathcal{Q} = (\mathbb{Q}^{\geq 0}, +, \leq, 0)$ then $\mathcal{U}_{\mathcal{Q}}$ is the *rational Urysohn space*.
  Cameron-Vershik, Melleray, Solecki, Tent-Ziegler
- If $\mathcal{R} = (R, \leq, 0)$ is a countable linear order, and $u \oplus v = \max\{u, v\}$, then $\mathcal{U}_{\mathcal{R}}$ exists and is an ultrametric Urysohn space.
  Gao-Shao, Nguyen Van Thé
- Fix $0 \in S \subseteq \mathbb{N}$ and, for $u, v \in S$, define

  $$u +_S v = \max\{x \in S : x \leq u + v\}.$$

  Let $\mathcal{S} = (S, +_S, \leq, 0)$. $\mathcal{U}_{\mathcal{S}}$ exists if and only if $+_S$ is associative.
  Casanovas-Wagner, Delhommé-Laflamme-Pouzet-Sauer, Nguyen Van Thé, Sauer
- In the last example, if $S = \{0, 1, 2\}$ then $\mathcal{U}_{\mathcal{S}}$ is the countable *random graph*.

Given a countable distance monoid $\mathcal{R}$, we fix a relational language

$$\mathcal{L}_R = \{d(x,y) \leq r : r \in R\},$$

where each $d(x,y) \leq r$ is a binary relation.

Let $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ be the complete $\mathcal{L}_\mathcal{R}$-theory of $\mathcal{U}_\mathcal{R}$.

Theorem (C.)

There is a distance monoid extension $\mathcal{R}^*$ of $\mathcal{R}$ such that any model $M \models \mathrm{Th}(\mathcal{U}_\mathcal{R})$ is an $\mathcal{R}^*$-metric space under a "type-definable" $\mathcal{R}^*$-metric.

Idea:

- R^* is the set of quantifier-free 2-types consistent with $\mathrm{Th}(\mathcal{U}_\mathcal{R})$.
- Given $M \models \mathrm{Th}(\mathcal{U}_\mathcal{R})$ and $a, b \in M$, $d(a,b)$ is the quantifier-free 2-type of (a,b).
- Given $\alpha, \beta \in R^*$, $\alpha \oplus \beta$ is the largest $\gamma \in R^*$ such that a triangle with distances α, β, γ is consistent.

Lemma (C.)

If *M* is a saturated model of Th($\mathcal{U}_{\mathcal{R}}$), of cardinality κ, then *M* is a κ^+-universal $\mathcal{R}^*$-metric space.

In order for *M* to be κ-homogeneous as an $\mathcal{R}^*$-metric space, we need quantifier elimination for Th($\mathcal{U}_{\mathcal{R}}$).

Theorem (C.)

Th($\mathcal{U}_{\mathcal{R}}$) *has quantifier elimination if and only if for all* $r \in R$, $\alpha \in R^*$, *if* α *has no immediate predecessor in* R^* *then*

$$\alpha \oplus r = \sup\{x \oplus r : x < \alpha\}.$$

Definition

A countable distance monoid $\mathcal{R}$ is **Urysohn** if Th($\mathcal{U}_{\mathcal{R}}$) has quantifier elimination.

Definition

A property P of $\mathcal{R}$-Urysohn spaces is **axiomatizable** if there is an $\mathcal{L}_{\omega_1,\omega}$-sentence φ_P in $\mathcal{L}_{\text{om}}$ such that, given a countable Urysohn monoid $\mathcal{R}$,

$$\mathcal{U}_{\mathcal{R}} \text{ has property } P \text{ if and only if } \mathcal{R} \models \varphi_P.$$

If φ_P is an $\mathcal{L}_{\omega,\omega}$-sentence then P is **finitely axiomatizable.**

Definition

The **archimedean rank of** $\mathcal{R}$, $\text{arch}(\mathcal{R})$, is the minimum $n < \omega$ such that, for all $r_0 \leq r_1 \leq \ldots \leq r_n$ in R,

$$r_0 \oplus r_1 \oplus \ldots \oplus r_n = r_1 \oplus \ldots \oplus r_n.$$

If no such n exists, set $\text{arch}(\mathcal{R}) = \omega$.

Theorem (C.)

Suppose $\mathcal{R}$ is a Urysohn monoid.

(*a*) $\text{Th}(\mathcal{U}_\mathcal{R})$ *is stable if and only if* $\text{arch}(\mathcal{R}) \leq 1$ *(finitely axiomatizable). I.e.,* $\text{Th}(\mathcal{U}_\mathcal{R})$ *is stable if and only if* $\mathcal{U}_\mathcal{R}$ *is ultrametric.*

(*b*) $\text{Th}(\mathcal{U}_\mathcal{R})$ *is simple if and only if* $\text{arch}(\mathcal{R}) \leq 2$ *(finitely axiomatizable).*

(*c*) $\text{Th}(\mathcal{U}_\mathcal{R})$ *does not have the strict order property.*

(*d*) $\text{Th}(\mathcal{U}_\mathcal{R})$ *is* NSOP_n *if and only if* $\text{arch}(\mathcal{R}) < n$ *(finitely axiomatizable).*

(*e*) $\text{Th}(\mathcal{U}_\mathcal{R})$ *is superstable if and only if* $\text{arch}(\mathcal{R}) \leq 1$ *and* $\mathcal{R}$ *is well-ordered (not axiomatizable).*

(*f*) "$\text{Th}(\mathcal{U}_\mathcal{R})$ *is supersimple" is not axiomatizable.*

The proofs of these results use a geometric characterization of nonforking independence, along with a fine analysis of amalgamation of indiscernible sequences.

Definition

(1) Given $n > 0$, let $\mathrm{DM}(n)$ be the number of distance monoids with n nontrivial elements.

(2) Given $n, k > 0$, let $\mathrm{DM}(n, k)$ be the number of distance monoids with n nontrivial elements and archimedean rank k.

Proposition (C.)

(a) **Upper bound**: $\mathrm{DM}(n) = O(c^{n^2})$, where $c = \sqrt{\frac{27}{16}} \approx 1.299$. (Uses Zeilberger's proof of the *Alternating Sign Matrix Conjecture*.)

(b) If $n < k$ then $\mathrm{DM}(n, k) = 0$.

(c) If $n > 0$ then $\mathrm{DM}(n, 1) = 1$ (witness: $(\{0, 1, \dots, n\}, \max, \leq, 0)$).

(d) If $n > 0$ then $\mathrm{DM}(n, n) = 1$ (witness: $(\{0, 1, \dots, n\}, +, \leq, 0)$).

Theorem (C., continuation of Nguyen Van Thé)

size / *rank*	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	4	14	451	202
3	0	0	1	6	33	183
4	0	0	0	1	8	54
5	0	0	0	0	1	10
6	0	0	0	0	0	1
DM(n)	1	2	6	22	94	451

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thank you