Extending Isometries in Generalized Metric Spaces

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June 9, 2015
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Theorem (Hrushovski 1992)

For any finite graph A, there is a finite graph B such that A is an induced subgraph of B and any partial automorphism of A extends to a total automorphism of B.

Theorem (Hodges, Hodkinson, Lascar, Shelah 1993)

Let Γ denote the countable random graph. Then $\mathrm{Aut}(\Gamma)$ has the ***small index property****, i.e., any subgroup of index $< 2^{\aleph_0}$ is open.*

Herwig proves analog of Hrushovski's result for:

- (1995) class of finite $\mathcal{L}$-structures ($\mathcal{L}$ a finite relational language);
- (1995) class of finite triangle-free graphs;
- (1998) class of finite K_n-free graphs.

Definition

Let $\mathcal{K}$ be a class of structures in a relational language.

(1) A structure $A \in \mathcal{K}$ has the **extension property in $\mathcal{K}$** if there is a structure $B \in \mathcal{K}$ such that A is a substructure of B and any partial automorphism of A extends to a total automorphism of B.
If, moreover, B is finite then A has the **finite extension property in $\mathcal{K}$**.

(2) $\mathcal{K}$ has the **extension property for partial automorphisms** if, for any finite $A \in \mathcal{K}$, if A has the extension property in $\mathcal{K}$ then it has the finite extension property in $\mathcal{K}$.

Theorem (Herwig-Lascar 2000)

Let $\mathcal{L}$ be a finite relational language and $\mathcal{F}$ a finite class of finite $\mathcal{L}$-structures. Then the class of $\mathcal{F}$-free $\mathcal{L}$-structures has the extension property for partial automorphisms.

Caution: "$\mathcal{F}$-free $\mathcal{L}$-structures" may not always coincide with structures of the desired type. E.g., if $\mathcal{L} = \{E\}$ is the language of graphs compare "K_n-free $\mathcal{L}$-structure" with "K_n-free graph".

Theorem (Solecki 2005)

For any finite metric space A, there is a finite metric space B such that A is a subspace of B and any partial isometry of A extends to a total isometry of B.

- Fix a finite metric space A. Let $S \subseteq \mathbb{R}^{\geq 0}$ be the set of distances appearing in A.
- Let $\mathcal{L} = \{d_s(x, y) : s \in S\}$.
- Let $\mathcal{F}$ be the class of "bad cycles" $(x_1, \ldots, x_n)$ where

$$d(x_1, x_n) > d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n),$$

  and $d(x_i, x_j) \in S$.
- Interpret metric spaces as $\mathcal{F}$-free $\mathcal{L}$-structures.
- Tricky part: Extract a metric space from an $\mathcal{F}$-free $\mathcal{L}$-structure.

Key tool in Herwig-Lascar result:

Theorem (Ribes-Zalesskiĭ 1993)

Let F be a free group. If $H_1, \ldots, H_n$ are finitely generated subgroups of F then $H_1 H_2 \ldots H_n$ is closed in the profinite topology on F (basic open sets are cosets of finite index subgroups).

A theorem of Rosendal (2011) shows explicitly how to obtain Solecki's result from the Ribes-Zalesskiĭ theorem.

Definition

(1) A **distance monoid** is a totally and positively ordered commutative monoid. Notation $\mathcal{R} = (R, \oplus, \leq, 0)$.

(2) Given a distance monoid $\mathcal{R}$, we have the natural notion of an $\mathcal{R}$**-metric space**.

Examples

- $(\mathbb{R}^{\geq 0}, +, \leq, 0)$ classical metric spaces
- $(\mathbb{R}^{\geq 0}, \max, \leq, 0)$ ultrametric spaces

(3) Let $\mathcal{R}$ be a distance monoid and $\mathcal{K}$ a class of $\mathcal{R}$-metric spaces.

(*a*) An $\mathcal{R}$-metric space $A \in \mathcal{K}$ has the **(finite) extension property** in $\mathcal{K}$ if there is a (finite) $\mathcal{R}$-metric space $B \in \mathcal{K}$ such that A is a subspace of B and any partial isometry of A extends to a total isometry of B.

(*b*) $\mathcal{K}$ has the **extension property for partial isometries** if, for any finite $A \in \mathcal{K}$, if A has the extension property in $\mathcal{K}$ then it has the finite extension property in $\mathcal{K}$.

Definition

A distance monoid $\mathcal{R} = (R, \oplus, \leq, 0)$ is **archimedean** if, for all $r, s \in R^{>0}$, there is some $n > 0$ such that $s \leq nr$.

Theorem (C. (generalization of Solecki))

Fix an archimedean distance monoid $\mathcal{R}$ and a finite class $\mathcal{F}$ of finite $\mathcal{R}$-metric spaces. Then the class of $\mathcal{F}$-free $\mathcal{R}$-metric spaces has the extension property for partial isometries.

The proof is a direct generalization, with some modifications, of Solecki's method for metric spaces (via Herwig-Lascar).

If $\mathcal{R}$ is not archimedean then the class of "bad cycles", with

$$d(x_1, x_n) > d(x_1, x_2) \oplus d(x_2, x_3) \oplus \ldots \oplus d(x_{n-1}, x_n)$$

is necessarily infinite.

Corollary

Fix an archimedean distance monoid $\mathcal{R}$. For any finite $\mathcal{R}$-metric space A, there is a finite $\mathcal{R}$-metric space B such that A is a subspace of B and any partial isometry of A extends to a total isometry of B.

Using results of Kechris & Rosendal (2007):

Corollary

Fix a *countable* archimedean distance monoid $\mathcal{R}$ and let $\mathcal{U}_{\mathcal{R}}$ denote the countable $\mathcal{R}$-Urysohn space (i.e. the Fraïssé limit of finite $\mathcal{R}$-metric spaces). Then:

- $\text{Isom}(\mathcal{U}_{\mathcal{R}})$ has the small index property.
- Any homomorphism from $\text{Isom}(\mathcal{U}_{\mathcal{R}})$ to a separable group is automatically continuous.

Examples

- **Solecki**: $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$
- **Hrushovski**: $\mathcal{R} = (\{0, 1, 2\}, +, \leq, 0)$ with truncated addition, $\mathcal{R}$-metric spaces coincide with graphs.
- Fix $n \geq 3$ odd. Let $\mathcal{R} = (\{0, 1, \ldots, \frac{n+1}{2}\}, +, \leq, 0)$. Let $\mathcal{F}$ be the class of $\mathcal{R}$-triangles with odd perimeter $\leq n$.
  Herwig: If $n = 3$ then $\mathcal{F}$-free $\mathcal{R}$-metric spaces coincide with triangle-free graphs.

Examples of (possibly) non-archimedean distance monoids

- Let $\mathcal{R} = (R, \leq, \max, 0)$, where $(R, \leq, 0)$ is a linear order with least element 0. $\mathcal{R}$-metric spaces are *ultrametric spaces*.

- **Sauer**: Fix $S \subseteq \mathbb{N}$, with $0 \in S$. Let $\mathcal{S} = (S, +_S, \leq, 0)$ where

$$u +_S v = \max\{x \in S : x \leq u + v\}.$$

  We must restrict to the case that $+_S$ is associative (this also characterizes the existence of an $\mathcal{S}$-Urysohn space).

Definition

A distance monoid $\mathcal{R} = (R, \oplus, \leq, 0)$ is **semi-archimedean** if, for all $r, s \in R^{>0}$, if $nr < s$ for all $n > 0$ then $r \oplus s = s$.

Theorem (C.)

Suppose $\mathcal{R}$ is a semi-archimedean distance monoid. For any finite $\mathcal{R}$-metric space A, there is a finite $\mathcal{R}$-metric space B such that A is a subspace of B and any partial isometry of A extends to a total isometry of B.

Idea: Without loss of generality, we may assume $\mathcal{R}$ has finitely many archimedean classes. Proceed by induction on the number of classes. The base case corresponds to $\mathcal{R}$ archimedean.

Since $\mathcal{R}$ is semi-archimedean, $\mathcal{R}$-metric spaces can be partitioned into "quotient spaces", with well-defined distances between pieces.

In the semi-archimedean case, we obtain the same results for $\text{Isom}(\mathcal{U}_{\mathcal{R}})$ (e.g. small index property, automatic continuity) when $\mathcal{R}$ is countable.

Related results

- Sabok (2013): isometry group of complete Urysohn space
- Malicki (2015): isometry groups of Polish ultrametric spaces

Question

Suppose $\mathcal{R}$ is an arbitrary distance monoid. Does the same isometry extension result hold for the class of finite $\mathcal{R}$-metric spaces?

Smallest "open" case: $(S, +_S, \leq, 0)$, where $S = \{0, 1, 3, 4\}$ and $u +_S v = \max\{x \in S : x \leq u + v\}$.

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