Unstable theories without the strict order property

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Motivating Questions

Broad goal: Understand the structure of unstable theories without the strict order property (i.e. unstable NSOP theories).

General questions about dividing lines

- Is there a non-simple NTP_2 theory without the strict order property?
- What is the deal with non-simple NSOP_3 theories?

Specific questions about the generic K_n-free graph

- (Hill) Understand elimination of imaginaries and hyperimaginaries for the generic K_n-free graph.
- (Koponen) Is the generic K_n-free graph rosy?

Question about homogeneous structures

- (Koponen) Suppose $\mathcal{M}$ is a countable ultrahomogeneous structure in a finite relational language, and $\text{Th}(\mathcal{M})$ is simple. Is $\text{Th}(\mathcal{M})$ 1-based?

Relational structures with free amalgamation

Free Amalgamation

Assumption

For the rest of the talk, $\mathcal{L}$ is a finite relational language.

Definition

Let $\mathcal{M}$ be a countable, ultrahomogeneous $\mathcal{L}$-structure.

(1) Given $A, B, C \subseteq \mathcal{M}$, we write $A \mathop{\smile\!\!\!\!\!\mid}^{fa}_{C} B$ **in** $\mathcal{M}$ if $A \cap B = C$ and any relation holding in AB is contained in A or B.

(2) $\mathcal{M}$ is **closed under free amalgamation** if, for all finite $A, B, C \subseteq \mathcal{M}$, with $C \subseteq A \cap B$, there are isomorphic copies A', B', C' in $\mathcal{M}$ such that $A' \mathop{\smile\!\!\!\!\!\mid}^{fa}_{C'} B'$ in $\mathcal{M}$.

Examples

- $\mathcal{L}$ is the language of graphs and $\mathcal{M}$ is the countable random graph.

- Fix $r \geq 2$ and $\mathcal{L}$ contain a single relation of arity r.
  Given $n > r$, let K_n^r be the complete r-hypergraph on n vertices.
  Let $\mathcal{M}$ be the generic K_n^r-free r-hypergraph.
  When $r = 2$, we omit it: $\mathcal{M}$ is the generic K_n-free graph.

- Let $\mathcal{M}$ be the unique, countable, universal, and ultrahomogeneous metric space with distances $\{0, 1, 2, 3\}$, or:
  - *the Urysohn space with spectrum* $\{0, 1, 2, 3\}$,
  - *the free third root of the complete graph* (Casanovas-Wagner).

  Consider $\mathcal{M}$ as a structure in the language $\mathcal{L} = \{d_1(x, y), d_3(x, y)\}$, where $d_r(x, y)$ is interpreted as $d(x, y) = r$. Then $\mathcal{M}$ is closed under free amalgamation.

Free Amalgamation structures are NSOP

Fact (Patel 2006)

If $\mathcal{M}$ is countable, ultrahomogeneous, and closed under free amalgamation then Th($\mathcal{M}$) is NSOP (in fact, NSOP_4).

First Result:

Simplicity

Simplicity

Let $\mathcal{M}$ be a countable, ultrahomogeneous $\mathcal{L}$-structure, which is closed under free amalgamation. Let $T = \text{Th}(\mathcal{M})$, and fix a monster model $\mathbb{M}$.

Theorem (C.)

The following are equivalent.

(*i*) *T is simple.*

(*ii*) *T is* NTP_2*.*

(*iii*) *T is* NSOP_3*.*

(*iv*) *Given $A, B, C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B$ if and only if $A \cap B \subseteq C$.*

Key Lemma

Fix $C \subset \mathbb{M}$, and tuples $a, b \in \mathbb{M}$, with $a \cap b = C$. Suppose $(b_n)_{n<\omega}$ is a sequence of tuples in $\mathbb{M}$ such that, for all $n < \omega$, $b_n \equiv_C b$ and $b_n \mathop{\smile\!\!\!\!\!\mid}^{fa}_{C} b_{<n}$. Then there is $a_* \in \mathbb{M}$ such that $a_* b_n \equiv_C ab$ for all $n < \omega$.

Second Result:

Imaginaries & Hyperimaginaries

Imaginaries and Hyperimaginaries

Definition

Let T be a complete first-order theory.

(1) T has **elimination of hyperimaginaries** (EHI) if, for any $e \in \mathbb{M}^{\text{heq}}$, there is a tuple c in $\mathbb{M}^{\text{eq}}$ such that $e \in \text{dcl}^{\text{heq}}(c)$ and $c \in \text{dcl}^{\text{heq}}(e)$.

(2) T has **weak elimination of imaginaries** (WEI) if, for any $e \in \mathbb{M}^{\text{eq}}$, there is a finite tuple c in $\mathbb{M}$ such that $c \in \text{acl}^{\text{eq}}(e)$ and $e \in \text{dcl}^{\text{eq}}(c)$.

Theorem (C.)

Let $\mathcal{M}$ be countable, homogeneous, and closed under free amalgamation. Let $T = Th(\mathcal{M})$.

(*a*) *T has EHI and WEI.*

(*b*) *T is rosy.*

Corollary:

1-based homogeneous structures

1-based homogeneous structures

Definition

A simple theory T is *1-based* if, for all $A, B \subset \mathbb{M}^{\text{eq}}$, $A \mathop{\smile\!\!\!\!\!\!\,|}^{f}_{\text{bdd}(A)\cap\text{bdd}(B)} B$.

Question (Koponen)

Suppose $\mathcal{M}$ is countable and ultrahomogeneous. If Th($\mathcal{M}$) is simple then is Th($\mathcal{M}$) 1-based?

Fact

Suppose T is simple, with EHI and WEI. Then T is 1-based if and only if $\mathop{\smile\!\!\!\!\!\!\,|}^{f}$ coincides with algebraic independence in $\mathbb{M}$.

Corollary (C.)

Suppose $\mathcal{M}$ is countable, ultrahomogeneous, and *closed under free amalgamation*. If Th($\mathcal{M}$) is simple then Th($\mathcal{M}$) is 1-based.

Nice Application:

Omitted Structures

Fun Fact

Theorem

Fix $n > r \geq 2$. Let T be the theory of the generic K_n^r-free r-hypergraph.

(a) (Shelah 1996) *If $r = 2$ then T is not simple.*

(b) (Hrushovski 2002) *If $r > 2$ then T is simple.*

Recall: $\mathcal{L}$ is a finite relational language.

Definition

Fix $r \geq 2$. An $\mathcal{L}$-structure A is r**-irreducible** if any r distinct elements of A are part of some relation in A.

E.g. K_n^r is r-irreducible, but not $(r+1)$-irreducible.

Explanation of the fun fact

Fix a countable, ultrahomogeneous $\mathcal{L}$-structure $\mathcal{M}$, which is closed under free amalgamation. Let $\mathcal{K}$ be the age of $\mathcal{M}$. Let $\mathcal{F}_*$ be the class of all finite $\mathcal{L}$-structures not in $\mathcal{K}$.

Observation 1: By the hereditary property, $\mathcal{K}$ is precisely the class of finite $\mathcal{L}$-structures *omitting* $\mathcal{F}_*$ (i.e. no element of $\mathcal{F}_*$ embeds in an element of $\mathcal{K}$).

Observation 2: By Zorn's Lemma, there is a subclass $\mathcal{F} \subseteq \mathcal{F}_*$ such that $\mathcal{K}$ is the class of finite $\mathcal{L}$-structures omitting $\mathcal{F}$, and there is no embedding between two distinct elements of $\mathcal{F}$.

Observation 3: Since $\mathcal{M}$ is closed under free amalgamation, every element of $\mathcal{F}$ is 2-irreducible.

Corollary (C.)

Th$(\mathcal{M})$ is simple if and only if every element of $\mathcal{F}$ is 3-irreducible.

thank you

- Enrique Casanovas and Frank O. Wagner, *The free roots of the complete graph*, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1543–1548 (electronic).
- Gabriel Conant, *An axiomatic approach to free amalgamation*, arXiv:1505.00762 [math.LO], 2015.
- Ehud Hrushovski, *Pseudo-finite fields and related structures*, Model theory and applications, Quad. Mat., vol. 11, Aracne, Rome, 2002, pp. 151–212.
- Vera Koponen, *Binary primitive homogeneous one-based structures*, arXiv:1507.07360 [math.LO], 2015.
- Rehana Patel, *A family of countably universal graphs without SOP_4*, preprint, 2006.
- Saharon Shelah, *Toward classifying unstable theories*, Ann. Pure Appl. Logic **80** (1996), no. 3, 229–255.