Group Reducts of Presburger Arithmetic

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Presburger arithmetic is the complete theory of the ordered group of integers \((\mathbb{Z}, +, <, 0)\).

This theory was first axiomatized by Presburger in 1927. An alternate presentation uses the monoid of nonnegative integers \((\mathbb{N}, +, 0)\).

Presburger arithmetic has quantifier elimination in the definitional expansion \(\mathcal{L}_{Pr} = \{+, -, 0, 1, <, (n\mathbb{Z})_{n<\omega}\}\).
Definable sets in Presburger arithmetic

Sets $A \subseteq \mathbb{Z}^n$, which are definable in Presburger arithmetic, have a very nice structure. For example, if $A \subseteq \mathbb{Z}$ is definable in Presburger arithmetic then

$$A = F \cup L_1 \cup \ldots \cup L_k,$$

where $F$ is some finite set and each $L_i$ is an arithmetic progression.

In particular, sets like

- $\Pi_q = \{ q^n : n \in \mathbb{N} \}$, where $q > 1$,
- $\text{Fac} = \{ n! : n \in \mathbb{N} \}$,
- $\text{Perf}(k) = \{ n^k : n \in \mathbb{N} \}$, where $k > 0$,
- Primes,

are not definable in Presburger arithmetic.
Question

Suppose $\mathcal{Z} = (\mathbb{Z}, +, <, 0, \ldots)$ is some expansion of Presburger arithmetic. How easy is it to recognize that $\mathcal{Z}$ is a proper expansion?

Theorem (Michaux-Villémaire 1996)

Suppose $\mathcal{Z} = (\mathbb{Z}, +, <, 0, \ldots)$ is a proper expansion of Presburger arithmetic. Then there is a subset $A \subseteq \mathbb{Z}$, which is definable in $\mathcal{Z}$, but not in $(\mathbb{Z}, +, <, 0)$.

By contrast: $(\mathbb{C}, =)$ and $(\mathbb{C}, +, \cdot, 0, 1)$ define the same subsets of $\mathbb{C}$. 
Expansions of Presburger arithmetic

Theorem (Michaux-Villemaire 1996)

Suppose $\mathbb{Z} = (\mathbb{Z}, +, <, 0, \ldots)$ is a proper expansion of Presburger arithmetic. Then there is a subset $A \subseteq \mathbb{Z}$, which is definable in $\mathbb{Z}$, but not in $(\mathbb{Z}, +, <, 0)$.

Applications:

• (Belegradek-Peterzil-Wagner 2000) There are no proper quasi-o-minimal expansions of Presburger arithmetic.
• (Dolich-Haskell-Macpherson-Starchenko 2011) There are no proper dp-minimal expansions of Presburger arithmetic.
• (Dolich-Goodrick 2015) There are no proper strong expansions of Presburger arithmetic.
Reducts of Presburger arithmetic

There are lots, \textit{for example},

- \((\mathbb{Z}, =)\)
- \((\mathbb{Z}, x \mapsto x + 1)\)
- \((\mathbb{Z}, x \mapsto -x)\)
- \((\mathbb{Z}, x \mapsto x + 1, x \mapsto -x)\)
- \((\mathbb{Z}, +, 0)\)
- \((\mathbb{Z}, <)\)
- \((\mathbb{Z}, <, x \mapsto -x)\)
- \((\mathbb{Z}, +, <, 0)\)

**Definition**

A structure \(\mathcal{Z}\), with universe \(\mathbb{Z}\), is a \textbf{group reduct of Presburger arithmetic} if \(\mathcal{Z}\) is a reduct of \((\mathbb{Z}, +, <, 0)\) and an expansion of \((\mathbb{Z}, +, 0)\).
The Main Result

Theorem (C. 2016)

There are no proper group reducts of Presburger arithmetic. In other words, if $\mathbb{Z}$ is a group reduct of Presburger arithmetic then $\mathbb{Z}$ is interdefinable with either $(\mathbb{Z}, +, 0)$ or $(\mathbb{Z}, +, <, 0)$. 
Motivation: stable expansions of \((\mathbb{Z}, +, 0)\)

Until recently, it was unknown if \((\mathbb{Z}, +, 0)\) had any proper stable expansions.

**Theorem (Palacín-Sklinos; Poizat 2014)**

*For any \(q > 0\), \((\mathbb{Z}, +, 0, \Pi_q)\) is a proper stable expansion of \((\mathbb{Z}, +, 0)\).*

Palacín-Sklinos prove the same for \((\mathbb{Z}, +, 0, \text{Fac})\).

- \((\mathbb{Z}, +, 0, \text{Perf}(2))\) defines the ordering by the four-square theorem (Lagrange 1770).
- \((\mathbb{Z}, +, 0, \text{Primes})\) defines the ordering by Goldbach-type theorems (e.g. Tao 2012 or Helfgott 2015).
Digression on Primes

Let $P = \{ p \in \mathbb{Z} : |p| \text{ is prime} \}$. Then $(\mathbb{Z}, +, 0, P)$ is again unstable (in fact, has the independence property by Kaplan-Shelah 2016). However, it is possible that $(\mathbb{Z}, +, 0, P)$ does not define the ordering.

Dickson’s Conjecture (1904)

Fix finitely many linear forms $(a_i x + b_i)_{i \leq k}$, with $a_i \geq 1$. Then there are infinitely many $n$ such that $a_in + b_i$ is prime for all $i \leq k$, provided that

$$\gcd \left( \prod_{i \leq k} a_in + b_i : n > 0 \right) = 1.$$ 

- The $k = 1$ case is known (Dirichlet 1837).
- The $k = 2$ case would imply infinitely many twin primes: $(x, x + 2)$, and infinitely many Sophie Germain primes: $(x, 2x + 1)$.

Kaplan-Shelah show that if this conjecture is true, then $(\mathbb{Z}, +, 0, P)$ is supersimple (and so, in particular, does not define the ordering).
**Problem**

Characterize the subsets \( A \subseteq \mathbb{Z}^n \) (even just \( A \subseteq \mathbb{Z} \)) such that \((\mathbb{Z}, +, 0, A)\) is stable.

What about expansions of \((\mathbb{Z}, +, 0)\) satisfying other “tameness” properties (e.g. finite dp-rank)?

**Question (Asch-Dol-Hask-Mac-Star 2013)**

Is every dp-minimal expansion of \((\mathbb{Z}, +, 0)\) a reduct of \((\mathbb{Z}, +, <, 0)\)?

The same could be asked about finite dp-rank expansions in general.

**Theorem (C.-Pillay 2016)**

*Any proper finite dp-rank expansion of \((\mathbb{Z}, +, 0)\) is unstable.*
Group reducts of Presburger arithmetic

Question
Is every finite dp-rank expansion of \((\mathbb{Z}, +, 0)\) a reduct of \((\mathbb{Z}, +, <, 0)\)?

Definition
A structure \(\mathcal{Z}\), with universe \(\mathbb{Z}\), is a **group reduct of Presburger arithmetic** if \(\mathcal{Z}\) is a reduct of \((\mathbb{Z}, +, <, 0)\) and an expansion of \((\mathbb{Z}, +, 0)\).

Theorem (C. 2016)
*There are no proper group reducts of Presburger arithmetic.*

So the initial question is equivalent to:
Is \((\mathbb{Z}, +, <, 0)\) the *only* proper finite dp-rank expansion of \((\mathbb{Z}, +, 0)\)?
Fix $A \subseteq \mathbb{Z}^n$.

- $A$ is **Presburger-definable** if it is definable in $(\mathbb{Z}, +, <, 0)$.
- $A$ is **group-definable** if it is definable in $(\mathbb{Z}, +, 0)$.
- $A$ defines the ordering if $\mathbb{N}$ is definable in $(\mathbb{Z}, +, 0, A)$.

**Main Theorem (restated)**

Suppose $A \subseteq \mathbb{Z}^n$ is Presburger-definable. Then either $A$ defines the ordering or $A$ is group-definable.
Structure of Presburger-definable sets

- A partial function \( f : \mathbb{Z}^n \rightarrow \mathbb{Z} \) is \( \mathbb{Z} \)-linear if it is of the form
  \[
  f(\bar{x}) = u + \sum_{i=1}^{n} a_i \left( \frac{x_i - r_i}{m_i} \right)
  \]
  for some \( \bar{m}, \bar{r} \in \mathbb{N}^n, \bar{a} \in \mathbb{Z}^n \), and \( u \in \mathbb{Z} \) (note: \( \text{dom}(f) = \bar{m}\mathbb{Z}^n + \bar{r} \)).

- A subset of \( \mathbb{Z} \) is a **Presburger cell** if it is of the form
  \[
  [a, b]_m := \{ x \in \mathbb{Z} : a \leq x \leq b, \ x \equiv_m r \},
  \]
  where \( r, m \in \mathbb{N} \) and \( a, b \in \mathbb{Z} \cup \{ \pm \infty \} \).

- A subset of \( \mathbb{Z}^{n+1} \) is a **Presburger cell** if it is of the form
  \[
  C[f, g]_m := \{ (\bar{x}, y) \in \mathbb{Z}^{n+1} : \bar{x} \in C, \ f(\bar{x}) \leq y \leq g(\bar{x}), \ y \equiv_m r \},
  \]
  where \( C \subseteq \mathbb{Z}^n \) is a Presburger cell, \( c, m \in \mathbb{N} \), and each of \( f \) and \( g \) is either constant \( \pm \infty \) or a \( \mathbb{Z} \)-linear function whose domain contains \( C \).
Theorem (Cluckers 2003)

\[ A \subseteq \mathbb{Z}^n \text{ is Presburger-definable if and only if it can be written as a finite union of Presburger cells in } \mathbb{Z}^n. \]
Structure of group-definable sets

Fact (folklore)

\( A \subseteq \mathbb{Z}^n \) is group-definable if and only if it is a finite Boolean combination of cosets of subgroups of \( \mathbb{Z}^n \).

Recall that any subgroup \( H \leq \mathbb{Z}^n \) is isomorphic to \( \mathbb{Z}^k \) for some \( 0 \leq k \leq n \). The rank of \( H \) is \( k \).

- If \( C \) is a coset of a rank \( k \) subgroup of \( \mathbb{Z}^n \) then the rank of \( C \) is \( k \).
- The rank of a set \( A \subseteq \mathbb{Z}^n \) is the minimal \( k \leq n \) such that \( A \) is contained in a finite union of cosets of rank at most \( k \).
- \( A \subseteq \mathbb{Z}^n \) is a quasi-coset if \( A = C \setminus Z \) where \( C \) is a coset and \( Z \) is group-definable with \( \text{rk}(Z) < \text{rk}(C) \) (by convention, \( \text{rk}(\emptyset) = -1 \)).

Theorem

\( A \subseteq \mathbb{Z}^n \) is group-definable if and only if it can be written as a finite union of quasi-cosets in \( \mathbb{Z}^n \).
Proof of the main result

Main Theorem (restated)

Suppose $A \subseteq \mathbb{Z}^n$ is Presburger-definable. Then either $A$ defines the ordering or $A$ is group-definable.

We proceed by induction on $n$.

The base case $n = 1$ is straightforward from quantifier elimination.

Assume the main result for Presburger-definable subsets of $\mathbb{Z}^n$, and fix a Presburger-definable subset $A \subseteq \mathbb{Z}^{n+1}$. Assume $A$ does not define the ordering. We want to show $A$ is group-definable.

Let $\pi(A)$ denote the projection of $A$ to $\mathbb{Z}^n$. Then $\pi(A)$ is group-definable by induction and the assumption on $A$. 
Technical reductions

**Notation:** Fix $m > 0$ and $a, b \in \mathbb{Z}$ with $a \leq b$.

- $[a, b]_m = [a, b] \cap m\mathbb{Z}$ (i.e. $[a, b]_m^0$)
- $a \leq_m b$ if $[a, b] \cap m\mathbb{Z} \neq \emptyset$
- $a <_m b$ if $(a, b) \cap m\mathbb{Z} \neq \emptyset$

A set $B \subseteq \mathbb{Z}^{n+1}$ has **sorted fibers** if there is an integer $m > 0$ and tuples of $\mathbb{Z}$-linear functions $\bar{f} = (f_1, \ldots, f_k)$ and $\bar{g} = (g_1, \ldots, g_k)$ on $\mathbb{Z}^n$ satisfying the following properties:

- $\pi(B) \subseteq \text{dom}(f_i) \cap \text{dom}(g_i)$ for all $1 \leq i \leq k$,
- for all $\bar{x} \in \pi(B)$ there are $\sigma, \tau \in S_k$ such that

$$B_{\bar{x}} = \bigcup_{i=1}^k \left[ f_{\sigma(i)}(\bar{x}), g_{\tau(i)}(\bar{x}) \right]_m$$

and

$$f_{\sigma(1)}(\bar{x}) \leq_m g_{\tau(1)}(\bar{x}) <_m \ldots <_m f_{\sigma(k)}(\bar{x}) \leq_m g_{\tau(k)}(\bar{x}).$$
A ⊆ \mathbb{Z}^{n+1} is Presburger-definable and does not define the ordering.

**Main Technical Work**

(a) A is interdefinable with a finite sequence of Presburger-definable sets with sorted fibers.

(b) Suppose A has sorted fibers, witnessed by k-tuples \( \bar{f} \) and \( \bar{g} \). Suppose further that \( f_s = g_t + c \) for some \( s, t \leq k \) and \( c \in \mathbb{Z} \). Then A is interdefinable with a finite sequence of Presburger-definable sets in \( \mathbb{Z}^{n+1} \), each of which has sorted fibers witnessed by some proper subtuples of \( \bar{f} \) and \( \bar{g} \).

Altogether, we may assume A has sorted fibers and induct on the length of the witnessing tuples of \( \mathbb{Z} \)-linear functions.
Technical reductions

\(A \subseteq \mathbb{Z}^{n+1}\) is Presburger-definable, does not define the ordering, and has sorted fibers witnessed by \(\overline{f}, \overline{g}\).

One last reduction: We may assume \(\pi(A)\) is a single quasi-coset.

For simplicity, assume \(\pi(A) = \mathbb{Z}^n \setminus X\), where \(X \subseteq \mathbb{Z}^n\) is group-definable with \(\text{rk}(X) < n\).

*After certain linear transformations, the general case essentially reduces to this.

**Goal:** Prove that some \(f_s\) and \(g_t\) are parallel.
Polyhedra in $\mathbb{R}^n$

- Given non-parallel affine functions $f, g$ on $\mathbb{R}^n$, define the **half-spaces**

  
  \[
  H(f \leq g) := \{ \bar{x} \in \mathbb{R}^n : f(\bar{x}) \leq g(\bar{x}) \}, \\
  H(f < g) := \{ \bar{x} \in \mathbb{R}^n : f(\bar{x}) < g(\bar{x}) \},
  \]

  as well as the **hyperplane** $H(f = g) := \{ \bar{x} \in \mathbb{R}^n : f(\bar{x}) = g(\bar{x}) \}$.

- Set $H(f \leq g)^* = H(g \leq f)$ and $H(f < g)^* = H(g < f)$.

- A **polyhedron** in $\mathbb{R}^n$ is the intersection of finitely many half-spaces.

- If $P = H_1 \cap \ldots \cap H_k$ is a polyhedron, where each $H_i$ is a half-space, then the **opposite polyhedron** is

  \[
  P^* = H_1^* \cap \ldots \cap H_k^*.
  \]

- The **inradius** of a polyhedron $P$ is

  \[
  r(P) := \sup \{ r \geq 0 : P \text{ contains a closed ball of radius } r \}.
  \]
### Polyhedra in $\mathbb{R}^n$

**Theorem (Kadets 2005)**

If $P, Q_1, \ldots, Q_n$ are polyhedra and $P \subseteq Q_1 \cup \ldots \cup Q_n$ then

$$r(P) \leq r(Q_1) + \ldots + r(Q_n).$$

**Corollary**

Suppose $P$ is a polyhedra with infinite inradius.

(a) $P^*$ has infinite inradius.

(b) If $X \subseteq \mathbb{Z}^n$ has $\text{rk}(X) < n$ then $(\mathbb{Z}^n \setminus X) \cap P$ cannot be covered by finitely many polyhedra of finite inradius.
Back to the proof

A \subseteq \mathbb{Z}^{n+1} is Presburger-definable, does not define the ordering, has sorted fibers witnessed by \( \bar{f}, \bar{g} \), and \( \pi(A) = \mathbb{Z}^n \setminus X \) with \( \text{rk}(X) < n \).

Suppose no \( f_s, g_t \) are parallel.

Given \( \sigma, \tau \in S_k \), define the following polyhedron in \( \mathbb{R}^n \),

\[
P(\sigma, \tau) = \bigcap_{i \leq k} H\left( f_{\sigma(i)} \leq g_{\tau(i)} \right) \cap \bigcap_{i < k} H\left( g_{\tau(i)} < f_{\sigma(i+1)} \right).
\]

By assumption, \( \pi(A) \subseteq \bigcup_{\sigma, \tau} P(\sigma, \tau) \).

So we may fix \( \mu, \nu \in S_k \) such that \( P(\mu, \nu) \) has infinite inradius.
End of the proof

- $A \subseteq \mathbb{Z}^{n+1}$ has sorted fibers witnessed by $\vec{f}$, $\vec{g}$,
- $\pi(A) = \mathbb{Z}^n \setminus X$ where $\text{rk}(X) < n$,
- $\pi(A) \subseteq \bigcup_{\sigma, \tau} P(\sigma, \tau)$, and $\mu, \nu \in S_k$ are such that $r(P(\mu, \nu)) = \infty$.

Then $P(\mu, \nu)^*$ has infinite radius. Set

$$C = \pi(A) \cap P(\mu, \nu)^*.$$  

For any $\vec{x} \in C$, we have

- $g_{\nu(k)}(\vec{x}) \leq f_{\mu(k)}(\vec{x}) < \ldots < g_{\nu(1)}(\vec{x}) \leq f_{\mu(1)}(\vec{x})$, and
- $f_{\sigma(1)}(\vec{x}) \leq g_{\tau(1)}(\vec{x}) < \ldots < f_{\sigma(k)}(\vec{x}) \leq g_{\tau(k)}(\vec{x})$ for some $\sigma, \tau \in S_k$.

Chasing inequalities: $f_{\mu(k)}(\vec{x}) = g_{\nu(k)}(\vec{x})$ for all $\vec{x} \in C$.

Then $f_{\mu(k)} = g_{\nu(k)}$, since otherwise $C$ is contained in the hyperplane

$$H(f_{\mu(k)} = g_{\mu(k)}).$$


