Model Theory and Forking Independence

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Types

We fix a first order language $\mathcal{L}$ and a complete $\mathcal{L}$-theory T.
Fix a model $\mathcal{M} \models T$ and a subset $B \subseteq \mathcal{M}$.

Definition

A **partial type over** B, $p(\bar{x}, \bar{b})$, is a consistent set of formulas $\varphi(\bar{x}, \bar{b})$, where the parameters of the formulas are taken from the set B. If $|\bar{x}| = n$, this is also called an n**-type**.

Given $\bar{a} \in \mathcal{M}$, we have the *complete type* over B

$$\text{tp}(\bar{a}/B) := \{\varphi(\bar{x}, \bar{b}) : \mathcal{M} \models \varphi(\bar{a}, \bar{b}),\ \bar{b} \in B\}.$$

If $\text{tp}(\bar{a}/B) = \text{tp}(\bar{b}/B)$ then we write $\bar{a} \equiv_B \bar{b}$.

Saturation

Definition

Given $n > 0$ we define the **Stone space**

$$S_n^{\mathcal{M}}(B) := \{\text{tp}(\bar{a}/B) : \bar{a} \in \mathcal{N} \succ \mathcal{M}\}.$$

This is a compact Hausdorff space with the basis of clopen sets

$$[\varphi(\bar{x}, \bar{b})] := \{p \in S_n^{\mathcal{M}}(B) : \varphi(\bar{x}, \bar{b}) \in p\}.$$

Definition

Fix an infinite cardinal κ. $\mathcal{M}$ is **κ-saturated** if for any subset $B \subseteq \mathcal{M}$ of size less than κ, every type in $S_n^{\mathcal{M}}(B)$ has a realization in $\mathcal{M}$.

Saturation

Example

Consider the theory of *algebraically closed fields of characteristic* 0 in the language $\mathcal{L} = \{+, \cdot, 0, 1\}$.

- $\mathbb{Q}^{alg}$ is not $\aleph_0$-saturated, e.g. the type $p(x)$ of a transcendental element is not realized, where $p(x) := \{f(x) \neq 0 : f(x) \in \mathbb{Z}[x]\}$.
- $\mathbb{C}$ is $\aleph_1$-saturated.

Fact

(*a*) If $\mathcal{M}$ is κ-saturated then $|\mathcal{M}| \geq \kappa$. In particular, the type $\{x \neq a : a \in \mathcal{M}\}$ cannot be realized in $\mathcal{M}$.

(*b*) For any complete theory T we can find κ-saturated models for arbitrarily large κ. In general, these models could be much larger than κ. We can ensure κ-saturated models of size κ by making set theoretic assumptions (e.g. κ is *inaccessible*) or assumptions on T (e.g. *stability*).

The Monster Model

For the rest of this talk, we will work in a very large κ-saturated model $\mathbb{M}$, for some very large cardinal κ.

What this means is that all models $\mathcal{M}$ of our theory, of size smaller than κ, will be *elementary submodels* of $\mathbb{M}$.

In particular, any consistent partial type over a parameter set of size less than κ will be realized in $\mathbb{M}$.

Unless otherwise stated, all models $\mathcal{M}$, $\mathcal{N}$ and sets A, B, C will be of size less than κ.

Another consequence of saturation is that for any set B and tuples $\bar{a}$, $\bar{b}$, we have $\bar{a} \equiv_B \bar{b}$ if and only if there is an automorphism of $\mathbb{M}$, which fixes B pointwise, and sends $\bar{a}$ to $\bar{b}$.

Forking and Dividing

Definition

A formula $\varphi(\bar{x}, \bar{b})$ **divides over** C if there is a sequence of tuples $(\bar{b}_i)_{i<\omega}$ such that:

- $\bar{b}_i \equiv_C \bar{b}$ for all $i < \omega$,
- there is an integer $k \geq 1$ such that every k-element subset of $\{\varphi(\bar{x}, \bar{b}_i) : i < \omega\}$ is inconsistent.

A partial type $p(\bar{x})$ **divides over** C if it contains a formula that divides over C.

A partial type $p(\bar{x})$ **forks over** C if there are finitely many formulas $\varphi_1(\bar{x}, \bar{b}), \ldots, \varphi_n(\bar{x}, \bar{b})$ such that:

- each $\varphi_i(\bar{x}, \bar{b})$ divides over C,
- any realization of $p(\bar{x})$ also realizes some $\varphi_i(\bar{x}, \bar{b})$.

Circular Order on $\mathbb{Q}$

Consider $\mathcal{L} = \{\text{cyc}\}$, where cyc is a ternary relation.

Interpret cyc in $\mathbb{Q}$ by

$$\mathbb{Q} \models \text{cyc}(x, y, z) \;\Leftrightarrow\; x \leq y \leq z \;\text{ or }\; z \leq x \leq y \;\text{ or }\; y \leq z \leq x.$$

Consider the formula $\text{cyc}(0, x, 1)$.
For $n < \omega$, $(2n, 2n+1) \equiv_{\emptyset} (0, 1)$. Moreover any 2-element subset of

$$\{\text{cyc}(2n, x, 2n+1) : n < \omega\}$$

is inconsistent.

So $\text{cyc}(0, x, 1)$ divides over $\emptyset$.

Forking and Independence

We define a ternary relation $\mathop{\smile\!\!\!\!\!\!\;\;|}^{f}$ on subsets of $\mathbb{M}$. In particular,

$$A \mathop{\smile\!\!\!\!\!\!\;\;|}^{f}_{C} B \;\;\Leftrightarrow\;\; \text{for all } \bar{a} \in A, \;\; \text{tp}(\bar{a}/BC) \text{ does not fork over } C.$$

This is supposed to capture some kind of notion of "freeness" or "independence". If $A \mathop{\smile\!\!\!\!\!\!\;\;|}^{f}_{C} B$, we often say "$A$ is free from B over C."

If $A \not\mathop{\smile\!\!\!\!\!\!\;\;|}^{f}_{C} B$, we say "$A$ forks with B over C".

Another common slogan:

$$A \not\mathop{\smile\!\!\!\!\!\!\;\;|}^{f}_{C} B \;\;\Leftrightarrow\;\; BC \text{ knows more about } A \text{ than } C \text{ knows alone.}$$

The Random Graph

Let $\mathcal{L} = \{R\}$, where R is a binary relation. A **random graph** is a nonempty graph G with the property that for any finite, disjoint subsets $A, B \subset G$, there is a vertex in G that is connected to every point in A and no point in B.

This can be axiomatized in the language $\mathcal{L}$.

Theorem

In the theory of the random graph,

$$A \mathop{\smile\!\!\!\!\!\!\!\;|}^{f}_{C} B \iff A \cap B \subseteq C$$

Additive Group of the Integers

Consider the theory of $\mathbb{Z}$ in the language $\mathcal{L} = \{+, 0, 1\}$.

Given a set A, let $\text{cl}(A)$ be the divisible hull of the subgroup generated by $A \cup \mathbb{Z}$.

In other words, $x \in \text{cl}(A)$ if and only if there is some integer $n > 0$ such that nx is in the subgroup generated by $A \cup \mathbb{Z}$.

Theorem

In $\text{Th}(\mathbb{Z}, +, 0, 1)$,

$$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{f}_{C} B \iff \text{cl}(AC) \cap \text{cl}(BC) = \text{cl}(C)$$

Vector spaces over $\mathbb{Q}$

Consider the theory of $\mathbb{Q}$ in the language $\mathcal{L} = \{+, 0, 1\}$. A model of $\mathrm{Th}(\mathbb{Q}, +, 0, 1)$ can be thought of as a torsion free divisible abelian group or, equivalently, as a vector space over $\mathbb{Q}$.

Given sets A and C, let $\dim(A/C)$ be the cardinality of a basis for A over C. Let $\langle C \rangle$ be the vector space span of C.

Note that if $C \subseteq D$ then $\dim(A/D) \leq \dim(A/C)$.

Theorem

In $\mathrm{Th}(\mathbb{Q}, +, 0, 1)$*,*

$$\begin{aligned} A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{f}_{C} B &\Leftrightarrow \dim(A/BC) = \dim(A/C) \\ &\Leftrightarrow \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle \end{aligned}$$

Algebraically Closed Fields

Consider the theory ACF_0 of algebraically closed fields of characteristic 0.

Given a set B, let K_B be the algebraically closed field generated by B. Given $\bar{a} \in \mathbb{M}$, let $\text{trdg}(\bar{a}/B)$ be the transcendence degree of $\bar{a}$ over K_B.

Note that if $C \subseteq D$ then $\text{trdg}(\bar{a}/D) \leq \text{trdg}(\bar{a}/C)$.

Theorem

In algebraically closed fields,

$$\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B \quad \Leftrightarrow \quad \begin{array}{l} \text{for every ideal } I \subseteq K_{BC}[\bar{x}], \\ \text{if } \bar{a} \in V(I) \text{ then } V(I) \text{ contains a point in } K_C^n \end{array}$$

$$\Leftrightarrow \quad \text{trdg}(\bar{a}/BC) = \text{trdg}(\bar{a}/C)$$

Remark: $A \mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B \;\Rightarrow\; K_{AC} \cap K_{BC} = K_C$ is still true, but the implication is strict.

The Random K_n-free Graph

Let $\mathcal{L} = \{R\}$, where R is a binary relation. Fix $n \geq 3$ and let K_n be the complete graph on n vertices.

A K_n**-free random graph** is a nonempty graph G with the property that for any finite, disjoint subsets $A, B \subset G$, if A is K_{n-1}-free then there is a vertex in G that is connected to every point in A and no point in B.

Given disjoint sets B and C, we say B **is** n**-bound to** C if there is a graph $X \subseteq BC$ of size n, intersecting both B and C, such that the only edges missing in X are between two points in B.

Theorem (C.)

In the theory of the K_n-free random graph,

$$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{f}_{C} B \quad \Leftrightarrow \quad \begin{array}{l} A \cap B \subseteq C \text{ and for all } \bar{b} \subseteq B \backslash C, \\ \bar{b} \text{ is either } n\text{-bound to } C \text{ or not } n\text{-bound to } AC. \end{array}$$

The Urysohn Sphere

The Urysohn sphere is a *countably universal and homogeneous metric space* in the sense that:

- every finite metric space (with distances bounded by 1) can be isometrically embedded into it.
- every isometry between finite subspaces can be extended to an isometry of the whole space.

This theory can be quantified in a generalization of classical first order logic called **continuous logic**. In this setting, we let $\mathbb{U}$ be a κ-saturated model of the theory of the Urysohn sphere. Then $\mathbb{U}$ has the two properties above, where we can replace "finite" with "size less than κ".

The Urysohn Sphere

Fix $C \subset \mathbb{U}$ and $b_1, b_2 \in \mathbb{U}$. Define

$$d_{\min}(b_1, b_2/C) = \max\left\{\frac{1}{3}d(b_1, b_2), \sup_{c \in C} |d(b_1, c) - d(b_2, c)|\right\} \text{ and}$$
$$d_{\max}(b_1, b_2/C) = \inf_{c \in C}(d(b_1, c) \dot{+} d(b_2, c)).$$

Interpretation: Considering $C \cup \{b_1\}$ and $C \cup \{b_2\}$ as individual metric spaces, we can amalgamate them into a new metric space $C \cup \{b_1, b_2\}$ by choosing $d(b_1, b_2)$. We only need

$$\sup_{c \in C} |d(b_1, c) - d(b_2, c)| \ \leq \ d(b_1, b_2) \ \leq \ \inf_{c \in C}(d(b_1, c) \dot{+} d(b_2, c)).$$

The $\frac{1}{3}d(b_1, b_2)$ term is due to more technical issues.

The Urysohn Sphere

Note that if $C \subseteq D$ then

$$d_{\min}(b_1, b_2/C) \;\leq\; d_{\min}(b_1, b_2/D) \;\leq\; d_{\max}(b_1, b_2/D) \;\leq\; d_{\max}(b_1, b_2/C).$$

Theorem (C., Terry)

In the Urysohn sphere,

$$A \mathop{\smile\!\!\!\!\!\!\!\;\,|}\nolimits^{f}_{C} B \;\Leftrightarrow\; \begin{array}{l} \text{for every } b_1, b_2 \in B, \\ d_{\min}(b_1, b_2/C) = d_{\min}(b_1, b_2/AC) \text{ and} \\ d_{\max}(b_1, b_2/C) = d_{\max}(b_1, b_2/AC). \end{array}$$

Forking and Dividing

Except for the random K_n-free graph and the Urysohn sphere, all of the previous examples have the property that forking and dividing are the same for formulas.

Theorem (C.)

In the random K_n-free graph, forking and dividing are the same for complete types. However, there is a formula that forks and does not divide.

Theorem (C.,Terry)

In the Urysohn sphere, forking and dividing are the same for complete types.

Question

Are forking and dividing the same for formulas in the Urysohn sphere?

Good Behavior of Forking

The property of forking equaling dividing for formulas can be considered an example of good or desirable behavior for a theory. It is also stronger than forking equaling dividing for complete types.

Even better behavior is when forking is symmetric, i.e., $A \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B \;\Leftrightarrow\; B \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} A$. This is stronger than forking equaling dividing for formulas, and such theories are called **simple**.

Except for the random K_n-free graph and Urysohn sphere, all of the previous examples (random graph, Th$(\mathbb{Z}, +, 0, 1)$, Th$(\mathbb{Q}, +, 0, 1)$, ACF) are simple.

Bad Behavior of Forking

Recall the slogan:

$$A \not\mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B \iff BC \text{ knows more about } A \text{ than } C \text{ knows alone.}$$

With this in mind, we see that $A \not\mathop{\smile\!\!\!\!\!\!|}^{f}_{C} C$ should be considered bad behavior.

This cannot happen in a simple theory, or in a theory where forking and dividing are the same (even just for complete types).

$A \not\mathop{\smile\!\!\!\!\!\!|}^{f}_{C} C$ can also be described as a type "forking over its own set of parameters". It is not hard to see that a type can never *divide* over its own set of parameters.

Circular Order on $\mathbb{Q}$

Recall

$$\mathbb{Q} \models \mathrm{cyc}(x, y, z) \;\Leftrightarrow\; x \leq y \leq z \text{ or } z \leq x \leq y \text{ or } y \leq z \leq x.$$

We showed that $\mathrm{cyc}(0, x, 1)$ divides over $\emptyset$. A similar argument shows that $\mathrm{cyc}(1, x, 0)$ divides over $\emptyset$.

Note that $\mathrm{cyc}(0, x, 1) \vee \mathrm{cyc}(1, x, 0)$ holds for any element of $\mathbb{M}$.

Therefore the partial type $\{x = x\}$ forks over $\emptyset$ by definition.

$a \not\mathop{\smile\hskip-0.9ex\raise0.4ex\hbox{$|$}}^{f}_{\emptyset} \emptyset$ for any $a \in \mathbb{M}$.

Classifying Lines Among Theories

Definition

(*a*) A theory T has the **independence property** if there is a formula $\varphi(\bar{x}, \bar{y})$ and tuples $(\bar{b}_i)_{i<\omega}$ such that if $A_i := \{\bar{a} : \mathbb{M} \models \varphi(\bar{a}, \bar{b}_i)\}$ then for all $I \subseteq \omega$

$$\bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} \neg A_i \neq \emptyset.$$

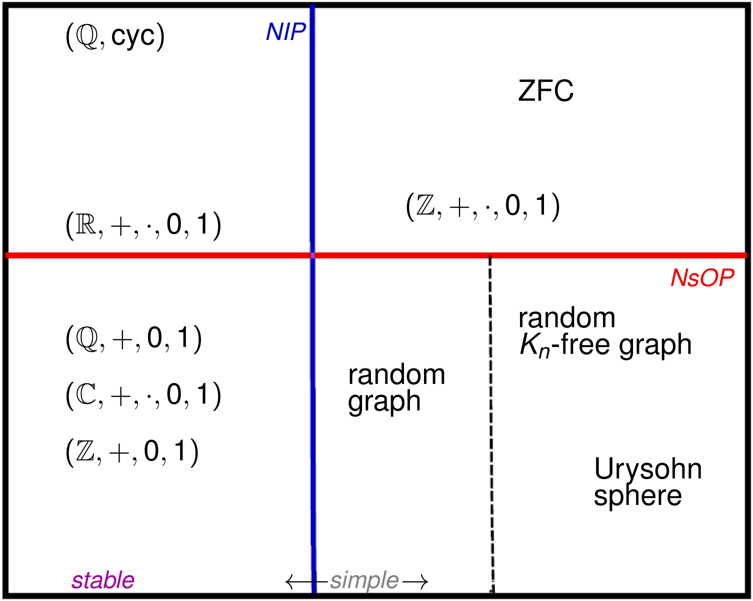
(*b*) A theory T has the **strict order property** if there is a formula $\varphi(\bar{x}, \bar{y})$ and tuples $(\bar{b}_i)_{i<\omega}$ such that

$$A_0 \subsetneq A_1 \subsetneq A_2 \ldots.$$

If T does not have the independence property then T is NIP.

If T does not have the strict order property then T is NsOP.

Map of the Universe



A Coincidental Question

Theorem (Chernikov, Kaplan)

Suppose T is an NIP theory in which no type forks over its own parameter set. Then forking and dividing are the same for formulas.

Question (Chernikov, Kaplan)

Can the same result be shown for NsOP theories?

Answer (C.)

No. The random K_n-free graph is an NsOP theory in which no type forks over its parameter set, but there is a formula that forks and does not divide.

References

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