Forking and Dividing in Random Graphs

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UIC

Graduate Student Conference in Logic
University of Notre Dame
April 28-29, 2012

Definitions

(a) A formula $\varphi(\bar{x}, \bar{b})$ **divides** over a set $A \subset \mathbb{M}$ if there is a sequence $(\bar{b}^l)_{l<\omega}$ and $k \in \mathbb{Z}^+$ such that $\bar{b}^l \equiv_A \bar{b}$ for all $i < \omega$ and $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$ is k-inconsistent.

(b) A partial type **divides** over A if it proves a formula that divides over A.

(c) A partial type **forks** over A if it proves a finite disjunction of formulas that divide over A.

(d) Given a theory T and $\mathbb{M} \models T$, we define the ternary relations $\mathop{\smile\!\!\!\!\!\!|}^d$ and $\mathop{\smile\!\!\!\!\!\!|}$ on $\{\bar{c} \in \mathbb{M}\} \times \{A \subset \mathbb{M}\}^2$ by

$$\bar{c} \mathop{\smile\!\!\!\!\!\!|}^d_A B \quad \Leftrightarrow \quad \text{tp}(\bar{c}/AB) \text{ does not divide over } A,$$
$$\bar{c} \mathop{\smile\!\!\!\!\!\!|}_A B \quad \Leftrightarrow \quad \text{tp}(\bar{c}/AB) \text{ does not fork over } A.$$

Facts

Theorem

A partial type $\pi(\bar{x}, \bar{b})$ (with $\pi(\bar{x}, \bar{y})$ a type over A) divides over A if and only if there is a sequence $(\bar{b}^l)_{l<\omega}$ of indiscernibles over A with $\bar{b}^0 = \bar{b}$ and $\bigcup_{l<\omega} \pi(\bar{x}, \bar{b}^l)$ inconsistent.

Definition

A theory T is **simple** if there is some formula $\varphi(\bar{x}, \bar{y})$ with the tree property.

Theorem

A theory T is simple if and only if for all $B \subset \mathbb{M}$ and $p \in S_n(B)$ there is some $A \subseteq B$ with $|A| \leq |T|$ such that p does not divide over A.

Setting

Let $\mathcal{L} = \{R\}$, where R is a binary relation symbol. We write xRy, rather than $R(x, y)$. If A and B are sets, we write ARB to mean that every point in A is connected to every point in B; and $A\not{R}B$ to mean that no point in A is connected to any point in B.

We write $A \subset \mathbb{M}$ to mean that A is a subset of $\mathbb{M}$ with $|A| < |\mathbb{M}|$.

Given $A \subseteq \mathbb{M}$, let $\mathcal{L}^0_A \subseteq \mathcal{L}_A$ be the set of conjunctions of atomic and negated atomic formulas such that no conjunct is of the form $x_i = a$, for some variable x_i and $a \in A$. When we refer to formulas $\varphi(\bar{x}, \bar{y})$ in $\mathcal{L}^0_A$, we will assume no conjunct is of the form $x_i = x_j$ or $y_i = y_j$ for distinct i, j.

Let $\mathcal{L}^R_A$ be the set of $\mathcal{L}^0_A$-formulas $\varphi(\bar{x}, \bar{y})$ containing no conjunct of the form $x_i = y_j$ for any i, j.

The Random Graph

We let T_0 be the theory of the random graph, i.e., T_0 contains

1. the axioms for nonempty graphs,

$$\forall x \neg xRx \qquad \forall x \forall y (xRy \leftrightarrow yRx) \qquad \exists x\, x = x;$$

2. "extension axioms" asserting that if A and B are finite disjoint sets of vertices then there is some c such that cRA and $c \not R B$.

T_0 is a complete, $\aleph_0$-categorical theory with quantifier elimination and no finite models.

We fix an uncountable saturated monster model $\mathbb{M} \models T_0$ and define the ternary relation $\mathop{\smile\!\!\!\!\!|}^{\cap}$ on $\{\bar{c} \in \mathbb{M}\} \times \{A \subset \mathbb{M}\}^2$ such that

$$\bar{c} \mathop{\smile\!\!\!\!\!|}^{\cap}_A B \;\Leftrightarrow\; \bar{c} \cap B \subseteq A.$$

The Random Graph is Simple (first proof)

Theorem (Kim & Pillay)

A theory T is simple if and only if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{o}$ satisfying automorphism invariance, extension, local and finite character, symmetry, transitivity, monotonicity, and independence over models. Moreover, in this case $\mathop{\smile\!\!\!\!\!\!|}^{o}$, $\mathop{\smile\!\!\!\!\!\!|}^{d}$, and $\mathop{\smile\!\!\!\!\!\!|}$ are all the same.

Independence over models: Let $M \models T$, $M \subseteq A, B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{o}_{M} B$. Suppose $\bar{a}, \bar{b} \in \mathbb{M}$ such that $\bar{a} \equiv_M \bar{b}$, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{o}_{M} A$, and $\bar{b} \mathop{\smile\!\!\!\!\!\!|}^{o}_{M} B$. Then there is $\bar{c} \in \mathbb{M}$ such that $\text{tp}(\bar{a}/A) \cup \text{tp}(\bar{b}/B) \subseteq \text{tp}(\bar{c}/AB)$ and $\bar{c} \mathop{\smile\!\!\!\!\!\!|}^{o}_{M} AB$.

Proposition

The ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{\cap}$ satisfies all the necessary properties to witness that T_0 is simple. In particular, we have

$$\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{A} B \quad \Leftrightarrow \quad \bar{c} \cap B \subseteq A.$$

The Random Graph is Simple (second proof)

Lemma

Let $A \subset \mathbb{M}$, $\bar{b}, \bar{c} \in \mathbb{M}$ such that $\bar{c} \mathop{\smile\!\!\!\!\!\mid}^{\cap}_{A} \bar{b}$. If $(\bar{b}^l)_{l<\omega}$ is A-indiscernible, with $\bar{b}^0 = \bar{b}$, then there is $\bar{d} \in \mathbb{M}$ with $\bar{d} \mathop{\smile\!\!\!\!\!\mid}^{\cap}_{A} \bigcup_{l<\omega} \bar{b}^l$ such that $\bar{d}\bar{b}^l \equiv_A \bar{c}\bar{b}$ for all $l < \omega$.

Theorem

Suppose $B \subset \mathbb{M} \models T_0$ and $p \in S_n(B)$. Then p does not divide over $A = \{b \in B : \exists i \; x_i = b \in p(\bar{x})\}$.

1. A theory T is **supersimple** if for all $p \in S_n(B)$ there is some finite $A \subseteq B$ such that p does not divide over A. Therefore T_0 is supersimple.
2. (Kim) If $\mathop{\smile\!\!\!\!\!\mid}^{\cap}$ is replaced by $\mathop{\smile\!\!\!\!\!\mid}$ in the above lemma, the statement remains true in any simple theory.

Dividing in the Random Graph

Theorem

Let $A \subset \mathbb{M} \models T_0$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^0$, *and* $\bar{b} \in \mathbb{M}$ *such that* $\varphi(\bar{x}, \bar{b})$ *is consistent. Then* $\varphi(\bar{x}, \bar{b})$ *divides over* A *if and only if* $\varphi(\bar{x}, \bar{b}) \rhd x_j = b$ *for some* $b \in \bar{b} \backslash A$.

The K_n-free Random Graph

For $n \geq 3$, we let T_n be the theory of the K_n-free random graph, i.e., T_n contains

1. the axioms for nonempty graphs;
2. a sentence asserting that the graph is K_n-free;
3. "extension axioms" asserting that if A and B are finite disjoint sets of vertices, and A is K_{n-1}-free, then there is some c such that cRA and $c \not R B$.

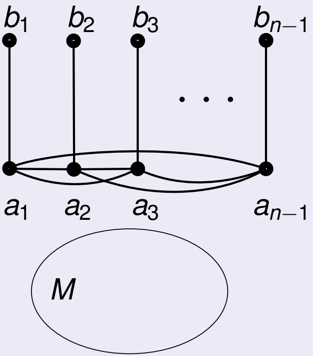
T_n is a complete, $\aleph_0$-categorical theory with quantifier elimination and no finite models.

The K_n-free Random Graph is Not Simple (first proof)

Theorem

$\mathop{\smile\!\!\!\!\!\mid}^{d}$-independence over models fails in T_n.

Proof.



Claim:
If $\bar{b} \in \mathbb{M} \backslash A$, $\bar{b} \not R A$, $l(\bar{b}) < n-1$, and p is in $S_1(A\bar{b})$, then p does not divide over A.

So $a_i \mathop{\smile\!\!\!\!\!\mid}^{d}_{M} \{a_j : j < i\}$ and $b_i \mathop{\smile\!\!\!\!\!\mid}^{d}_{M} a_i$.

Clearly, $\{\mathrm{tp}(b_i/Ma_i) : i < n\}$ cannot be amalgamated.

□

The K_n-free Random Graph is Not Simple (second proof)

Definition

Let $k \geq 3$. A theory T has the **k-strong order property**, SOP_k, if there is a formula $\varphi(\bar{x}, \bar{y})$, with $l(\bar{x}) = l(\bar{y})$, and $(\bar{b}^l)_{l<\omega}$ such that

$$\mathbb{M} \models \varphi(\bar{b}^l, \bar{b}^m) \qquad \forall\, l < m < \omega;$$
$$\mathbb{M} \models \neg\exists \bar{x}^1 \ldots \exists \bar{x}^k (\varphi(\bar{x}^1, \bar{x}^2) \wedge \ldots \wedge \varphi(\bar{x}^{k-1}, \bar{x}^k) \wedge \varphi(\bar{x}^k, \bar{x}^1)).$$

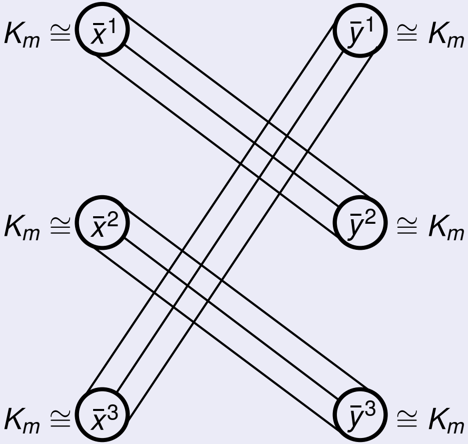
Fact: $\ldots \Rightarrow \mathrm{SOP}_{k+1} \Rightarrow \mathrm{SOP}_k \Rightarrow \ldots \Rightarrow \mathrm{SOP}_3 \Rightarrow$ tree property.

Theorem

For all $n \geq 3$, T_n has SOP_3.

The K_n-free Random Graph is Not Simple (second proof)

Proof (T_n has SOP_3).



Assume $n \neq 4$.
Let $m = \lceil \frac{n}{3} \rceil$. Then $2m < n$.

Let $\varphi(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{y}^1, \bar{y}^2, \bar{y}^3)$ describe this configuration.

We can construct an infinite chain in $\mathbb{M} \models T_n$.

No 3-cycle is possible.

□

The K_n-free Random Graph is Not Simple (second proof)

Definition

Let $k \geq 3$. A theory T has the *k***-strong order property**, SOP_k, if there is a formula $\varphi(\bar{x}, \bar{y})$ and $(\bar{b}^l)_{l<\omega}$ such that

$$\mathbb{M} \models \varphi(\bar{b}^l, \bar{b}^m) \qquad \forall\, l < m < \omega;$$
$$\mathbb{M} \models \neg\exists\bar{x}^1 \ldots \exists\bar{x}^k(\varphi(\bar{x}^1, \bar{x}^2) \wedge \ldots \wedge \varphi(\bar{x}^{k-1}, \bar{x}^k) \wedge \varphi(\bar{x}^k, \bar{x}^1)).$$

Fact: $\ldots \Rightarrow \mathrm{SOP}_{k+1} \Rightarrow \mathrm{SOP}_k \Rightarrow \ldots \Rightarrow \mathrm{SOP}_3 \Rightarrow$ tree property.

Theorem

For all $n \geq 3$, T_n has SOP_3. □

Fact: For all $n \geq 3$, T_n does not have SOP_4.

Recall: Dividing in the Random Graph

Theorem

Let $A \subset \mathbb{M} \models T_0$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^0$, and $\bar{b} \in \mathbb{M}$ such that $\varphi(\bar{x}, \bar{b})$ is consistent. Then $\varphi(\bar{x}, \bar{b})$ divides over A if and only if $\varphi(\bar{x}, \bar{b}) \rhd x_j = b$ for some $b \in \bar{b} \backslash A$.

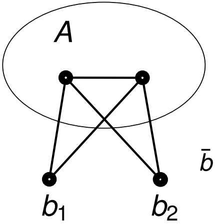
Dividing in T_n

Definition

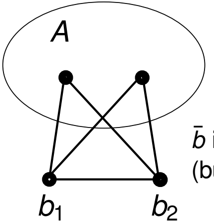
Suppose $A, B \subset \mathbb{M}$ are disjoint. Then B is ***n*-bound to** A if there is $B_0 \subseteq A \cup B$, $|B_0| = n$, $B_0 \cap A \neq \emptyset \neq B_0 \cap B$, such that

1. $(B_0 \cap A) R (B_0 \cap B)$,
2. $(B_0 \cap A) \cong K_m$, where $m = |B_0 \cap A|$.

Informally, B is n-bound to A if there is a subgraph $B_0 \subseteq AB$ of size n, such that the only thing preventing B_0 from being isomorphic to K_n is a possible lack of edges between points in B.



$\bar{b}$ is 4-bound to A



$\bar{b}$ is not 4-bound to A (but A is 4-bound to $\bar{b}$)

Dividing in T_n

Theorem

Let $A \subset \mathbb{M}$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^R$, and $\bar{b} \in \mathbb{M} \backslash A$ such that $\varphi(\bar{x}, \bar{b})$ is consistent. Then $\varphi(\bar{x}, \bar{b})$ divides over A if and only if

1. *$\bar{b}$ is not n-bound to A,*
2. *$\bar{b}$ is n-bound to $A\bar{c}$ for any $\bar{c} \models \varphi(\bar{x}, \bar{b})$.*

Proof.

$(\Leftarrow)$: $\bar{b}$ not n-bound to A allows construction of a sequence $(\bar{b}^l)_{l<\omega}$, with enough edges, so that being n-bound to $A\bar{c}$ will force $(n-1)$-inconsistency in $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$.
$(\Rightarrow)$: Let $(\bar{b}^l)_{l<\omega}$, indiscernible over A, witness dividing. Since $\varphi(\bar{x}, \bar{b})$ does not divide over A in T_0, there must be a copy of K_n in $A\bar{c}(\bar{b}^l)_{l<\omega}$, where $\bar{c}$ is an "optimal" solution of $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$ in T_0. By indiscernibility, this K_n will "project" to the required conditions in $A\bar{b}$. □

Dividing in T_n - Examples

Corollary

Suppose $A \subset \mathbb{M}$ and $b_1, \ldots, b_{n-1} \in \mathbb{M} \backslash A$ are distinct. Then the formula

$$\varphi(x, \bar{b}) := \bigwedge_{i=1}^{n-1} xRb_i$$

divides over A if and only if $\bar{b}$ is not n-bound to A.

Corollary

Let $A \subset \mathbb{M}$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^R$. Suppose $\bar{b} \in \mathbb{M} \backslash A$ such that $\varphi(\bar{x}, \bar{b})$ is consistent and divides over A. Define

$$R^\varphi = \{u \in A\bar{b} : \exists i \ \varphi(\bar{x}, \bar{b}) \rhd x_i Ru\} \cup \{x_i : \exists u \in A\bar{b}, \ \varphi(\bar{x}, \bar{b}) \rhd x_i Ru\}.$$

Then $|R^\varphi| \geq n$ and $|\bar{b} \cap R^\varphi| > 1$.

Recall: $\mathop{\smile\hskip -0.9em ^{\vert}}$ in T_0

Theorem

Let $A, B, \bar{c} \subset \mathbb{M} \models T_0$. *Then*

$$\bar{c} \mathop{\smile\hskip -0.9em ^{\vert}}_A B \;\Leftrightarrow\; \bar{c} \mathop{\smile\hskip -0.9em ^{\vert}}_A^d B \;\Leftrightarrow\; \bar{c} \cap B \subseteq A.$$

$\mathop{\smile\!\!\!\!\!\!|}\,$ in T_n

Lemma

Let $A \subset \mathbb{M} \models T_n$ and $p \in S_m(A)$. If $p \vdash x = b$ for some $b \in \mathbb{M}$ then $b \in A$. If $p \vdash xRb$ for some $b \in \mathbb{M}$, then either $b \in A$ or $p \vdash x = a$ for some $a \in A$.

Theorem

Let $A, B, \bar{c} \subset \mathbb{M} \models T_n$. Then

$\bar{c} \mathop{\smile\!\!\!\!\!\!|}_A B \Leftrightarrow \bar{c} \mathop{\smile\!\!\!\!\!\!|}^d_A B \Leftrightarrow$ $\bar{c} \cap B \subseteq A$ and for all $\bar{b} \in B \backslash A$, $\bar{b}$ is either n-bound to A or not n-bound to $A\bar{c}$.

Corollary

Let $A \subset \mathbb{M} \models T_n$. If $\pi(\bar{x})$ is a partial type over A, then $\pi(\bar{x})$ does not fork over A.

Forking in T_n

Lemma

Suppose $(\bar{b}^l)_{l<\omega}$ is an indiscernible sequence in $\mathbb{M} \models T_n$ such that $I(\bar{b}^0) = 4$ and $\bar{b}^0$ is K_2-free. Then either there are $i < j$ such that $\{b^l_i, b^l_j : l < \omega\}$ is K_2-free, or $\bigcup_{l<\omega} \bar{b}^l$ is not K_3-free.

Forking in T_n

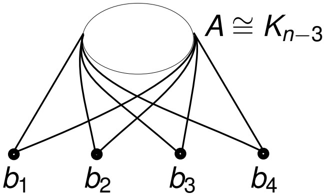
Theorem

Let $\mathbb{M} \models T_n$ and $A\bar{b} \subset \mathbb{M}$ such that $l(\bar{b}) = 4$, $\bar{b}RA$, $A \cong K_{n-3}$, and $\bar{b}$ is K_2-free. For $i \neq j$, let

$$\varphi_{i,j}(x, b_i, b_j) = xRb_i \wedge xRb_j \wedge \bigwedge_{a \in A} xRa,$$

$$\text{and set} \qquad \varphi(x, \bar{b}) := \bigvee_{i \neq j} \varphi_{i,j}(x, b_i, b_j).$$

Then $\varphi(x, \bar{b})$ forks over A but does not divide over A.



$$\varphi(x, \bar{b}) = \text{“}xRA \wedge |\{b_i : xRb_i\}| = 2\text{”}$$

Higher Arity Graphs

Now suppose R is a relation of arity $r \geq 2$. An r-**graph** is an $\mathcal{L}$-structure satisfying the following sentence

$$\forall x_1 \ldots \forall x_r \left(R(x_1, \ldots, x_r) \to \left(\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{\sigma \in S_r} R(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) \right) \right).$$

So R can be thought of as a collection of r-element subsets of the r-graph.

Let T_0^r and T_n^r, for $n > r$, be the natural analogs of T_0 and T_n, respectively, to r-graphs.

Call T_0^r the **theory of the random r-graph**, and T_n^r the **theory of the random K_n^r-free r-graph**, where K_n^r is the complete r-graph of size n.

T_n^r can be thought of as the theory of the unique (countable) Fraïssé limit of the class of finite K_n^r-free r-graphs.

Recall - Dividing in T_0

Lemma

Let $A \subset \mathbb{M} \models T_0$, $\bar{b}, \bar{c} \in \mathbb{M}$ such that $\bar{c} \mathop{\smile\!\!\!\!\!\!|}^{\cap}_A \bar{b}$. If $(\bar{b}^l)_{l<\omega}$ is A-indiscernible, with $\bar{b}^0 = \bar{b}$, then there is $\bar{d} \in \mathbb{M}$ with $\bar{d} \mathop{\smile\!\!\!\!\!\!|}^{\cap}_A \bigcup_{l<\omega} \bar{b}^l$ such that $\bar{d}\bar{b}^l \equiv_A \bar{c}\bar{b}$ for all $l < \omega$.

Lemma

Let $A \subset \mathbb{M} \models T_0$ and $p(\bar{x}, \bar{y}) \in S(A)$ such that $x_i \neq y_j \in p(\bar{x}, \bar{y})$ for all i, j. Suppose $p(\bar{x}, \bar{b})$ is consistent and $(\bar{b}^l)_{l<\omega}$ is indiscernible over A with $\bar{b}^0 = \bar{b}$. Then there is a solution $\bar{c}$ of $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Dividing in T_0^r

Lemma

Let $A \subset \mathbb{M} \models T_0^r$ and $p(\bar{x}, \bar{y}) \in S(A)$ such that $x_i \neq y_j \in p(\bar{x}, \bar{y})$ for all i, j. Suppose $p(\bar{x}, \bar{b})$ is consistent and $(\bar{b}^l)_{l<\omega}$ is indiscernible over A with $\bar{b}^0 = \bar{b}$. Then there is a solution $\bar{c}$ of $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

In fact, we can take $\bar{c}$ to be an **optimal solution**, in particular if $A_0 \subseteq A(\bar{b}^l)_{l<\omega}$, then for any $i_1, \ldots, i_k$

$$\mathbb{M} \models R(c_{i_1}, \ldots, c_{i_k}, A_0) \quad \Leftrightarrow \quad R(x_{i_1}, \ldots, x_{i_k}, A_0) \in \bigcup_{l<\omega} p(\bar{x}, \bar{b}^l).$$

Corollary (T_0^r is simple.)

Suppose $B \subset \mathbb{M} \models T_0^r$ and $p \in S(B)$. Then p does not fork over $A = \{b \in B : \exists i \; x_i = b \in p(\bar{x})\}$.

Dividing in T_n^r, $r > 2$

Lemma

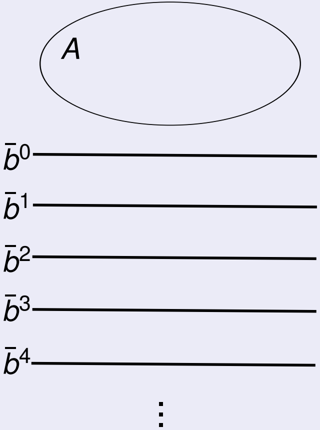
Assume $r > 2$. Let $A \subset \mathbb{M} \models T_n^r$ and $p(\bar{x}, \bar{y}) \in S(A)$ such that $x_i \neq y_j \in p(\bar{x}, \bar{y})$ for all i, j. Suppose $p(\bar{x}, \bar{b})$ is consistent and $(\bar{b}^l)_{l<\omega}$ is indiscernible over A with $\bar{b}^0 = \bar{b}$. Then there is a solution $\bar{c}$ of $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Corollary (T_n^r is simple if $r > 2$.)

Suppose $r > 2$ and $B \subset \mathbb{M} \models T_n^r$ and $p \in S(B)$. Then p does not fork over $A = \{b \in B : \exists i \; x_i = b \in p(\bar{x})\}$.

Dividing in T_n^r, $r > 2$

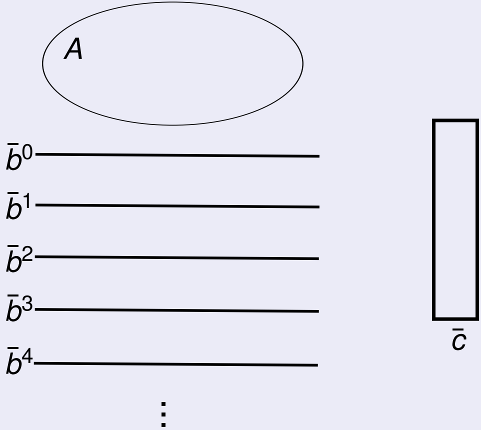
Proof of Lemma.



Dividing in T_n^r, $r > 2$

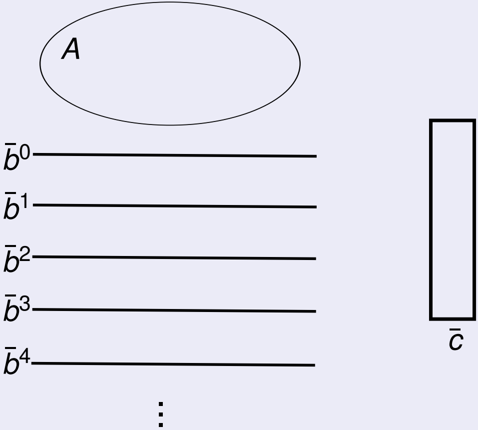
Proof of Lemma.

$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.



Dividing in T_n^r, $r > 2$

Proof of Lemma.

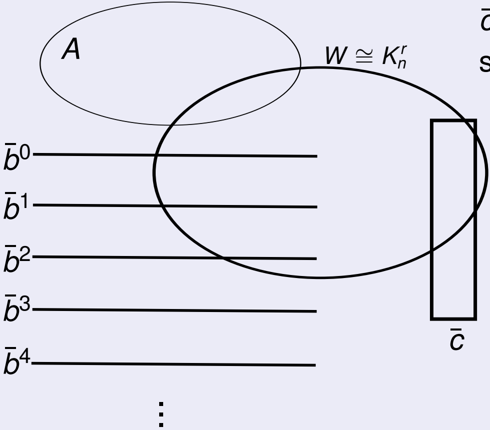


$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K_n^r-free.

Dividing in T_n^r, $r > 2$

Proof of Lemma.



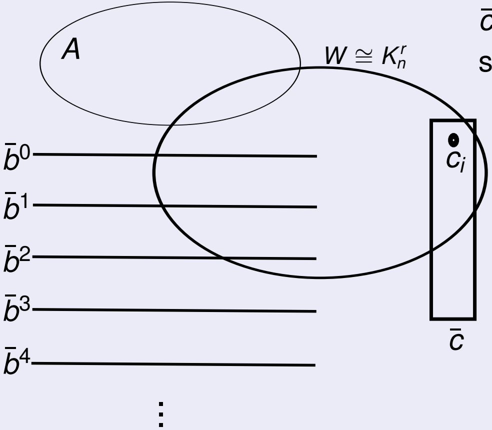
$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K_n^r-free.

Let $K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

Dividing in T_n^r, $r > 2$

Proof of Lemma.



$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

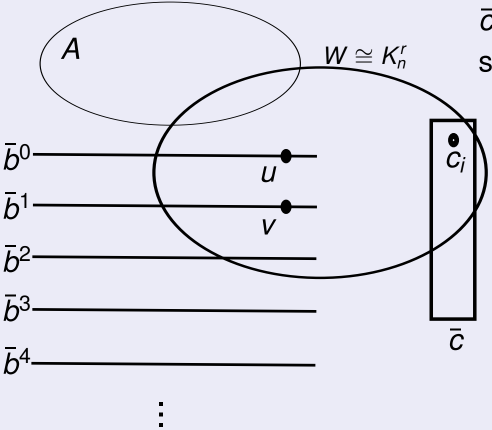
Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K_n^r-free.

Let $K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

There is $c_i \in W \cap \bar{c}$.

Dividing in T^r_n, $r > 2$

Proof of Lemma.



$\bar{c} \in \mathbb{M}' \models T^r_0$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K^r_n-free.

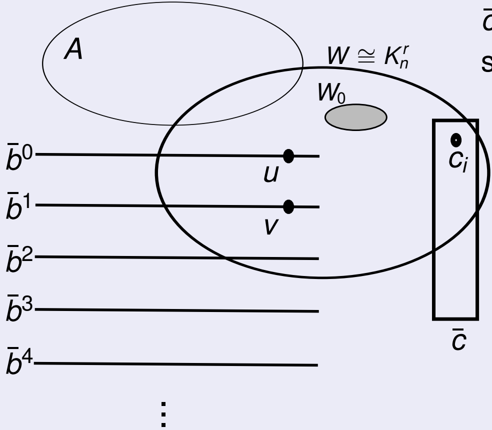
Let $K^r_n \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

There is $c_i \in W \cap \bar{c}$.

If $|l : W \cap \bar{b}^l \neq \emptyset\}| > 1$

Dividing in T_n^r, $r > 2$

Proof of Lemma.



$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

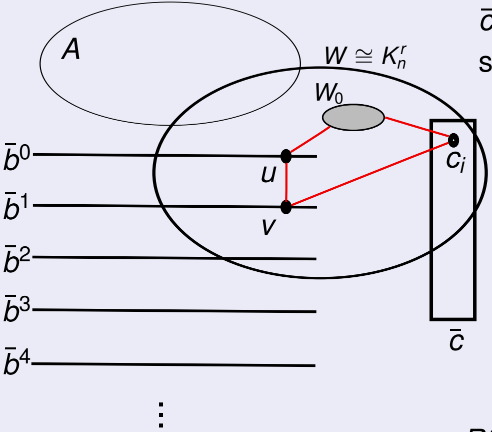
Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K_n^r-free.

Let $K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

There is $c_i \in W \cap \bar{c}$.
If $|l : W \cap \bar{b}^l \neq \emptyset\}| > 1$
pick $W_0 \subseteq W \backslash \{c_i, u, v\}$,
with $|W_0| = r - 3$.

Dividing in T_n^r, $r > 2$

Proof of Lemma.



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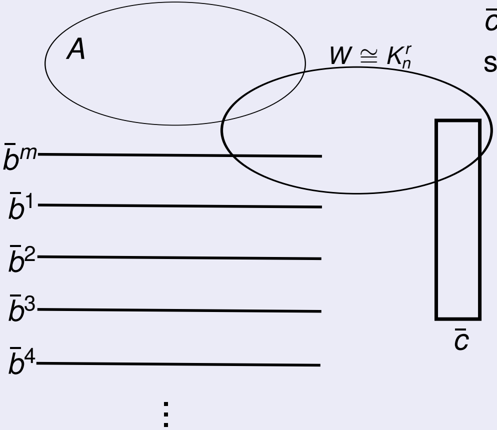
$R(c_i, u, v, W_0) \in \bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Dividing in T_n^r, $r > 2$

Proof of Lemma.



$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K_n^r-free.

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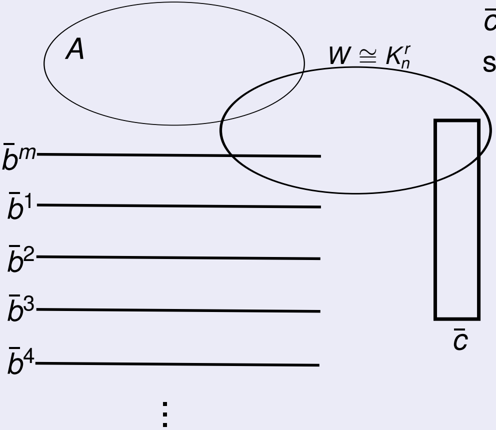
$|\{l : W \cap \bar{b}^l \neq \emptyset\}| \leq 1$

Dividing in T^r_n, $r > 2$

Proof of Lemma.



$\bar{c} \in \mathbb{M}' \models T^r_0$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

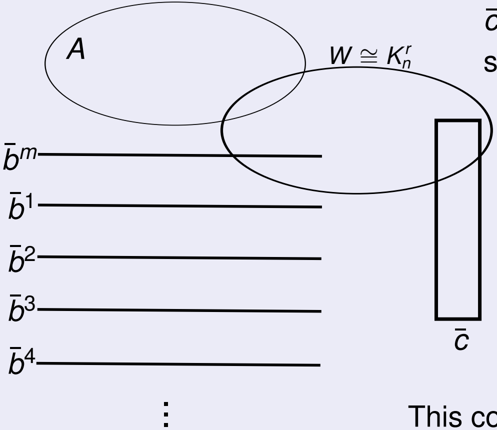
Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K^r_n-free.

Let $K^r_n \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

$|\{l : W \cap \bar{b}^l \neq \emptyset\}| \leq 1$
"$W \cong K^r_n$" $\in p(\bar{c}, \bar{b}^m)$

Dividing in T_n^r, $r > 2$

Proof of Lemma.



$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is K_n^r-free.

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"$W \cong K_n^r$" $\in p(\bar{c}, \bar{b}^m)$

This contradicts $\mathbb{M} \models \exists \bar{x} p(\bar{x}, \bar{b}^m)$. □

References

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