Forking and Dividing in Random Graphs

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Definitions

(a) A formula $\varphi(\bar{x}, \bar{b})$ **divides** over a set $A \subset \mathcal{M}$ if there is a sequence $(\bar{b}'_l)_{l<\omega}$ and $k \in \mathbb{Z}^+$ such that $\bar{b}'_l \equiv_A \bar{b}$ for all $i < \omega$ and

$$\{\varphi(\bar{x}, \bar{b}'_l) : l < \omega\}$$

is $k$-inconsistent.

(b) A partial type **divides** over $A$ if it proves a formula that divides over $A$.

(c) A partial type **forks** over $A$ if it proves a finite disjunction of formulas that divide over $A$.

(d) Given a theory $T$ and $\mathcal{M} \models T$, we define the ternary relations $\downarrow^d$ and $\downarrow$ on $\{\bar{c} \in \mathcal{M}\} \times \{A \subset \mathcal{M}\}^2$ by

$$\bar{c} \downarrow^d_A B \iff \text{tp}(\bar{c}/AB) \text{ does not divide over } A,$$

$$\bar{c} \downarrow_A B \iff \text{tp}(\bar{c}/AB) \text{ does not fork over } A.$$
Facts

Theorem

A partial type $\pi(\bar{x}, \bar{b})$ (with $\pi(\bar{x}, \bar{y})$ a type over $A$) divides over $A$ if and only if there is a sequence $(\bar{b}^l)_{l<\omega}$ of indiscernibles over $A$ with $\bar{b}^0 = \bar{b}$ and $\bigcup_{l<\omega} \pi(\bar{x}, \bar{b}^l)$ inconsistent.

Definition

A theory $T$ is **simple** if there is some formula $\varphi(\bar{x}, \bar{y})$ with the tree property.

Theorem

A theory $T$ is simple if and only if for all $B \subset M$ and $p \in S_n(B)$ there is some $A \subseteq B$ with $|A| \leq |T|$ such that $p$ does not divide over $A$. 
Setting

Let $\mathcal{L} = \{R\}$, where $R$ is a binary relation symbol. We write $xRy$, rather than $R(x, y)$. If $A$ and $B$ are sets, we write $ARB$ to mean that every point in $A$ is connected to every point in $B$; and $A \not\subset R B$ to mean that no point in $A$ is connected to any point in $B$.

We write $A \subset M$ to mean that $A$ is a subset of $M$ with $|A| < |M|$.

Given $A \subseteq M$, let $\mathcal{L}_A^0 \subseteq \mathcal{L}_A$ be the set of conjunctions of atomic and negated atomic formulas such that no conjunct is of the form $x_i = a$, for some variable $x_i$ and $a \in A$. When we refer to formulas $\varphi(\bar{x}, \bar{y})$ in $\mathcal{L}_A^0$, we will assume no conjunct is of the form $x_i = x_j$ or $y_i = y_j$ for distinct $i, j$.

Let $\mathcal{L}_A^R$ be the set of $\mathcal{L}_A^0$-formulas $\varphi(\bar{x}, \bar{y})$ containing no conjunct of the form $x_i = y_j$ for any $i, j$. 
The Random Graph

We let \( T_0 \) be the theory of the random graph, i.e., \( T_0 \) contains

1. the axioms for nonempty graphs,

\[
\forall x \neg xRx \quad \forall x \forall y (xRy \leftrightarrow yRx) \quad \exists x \ x = x;
\]

2. “extension axioms” asserting that if \( A \) and \( B \) are finite disjoint sets of vertices then there is some \( c \) such that \( cRA \) and \( c \not\in RB \).

\( T_0 \) is a complete, \( \aleph_0 \)-categorical theory with quantifier elimination and no finite models.

We fix an uncountable saturated monster model \( M \models T_0 \) and define the ternary relation \( \downarrow \cap \) on \( \{\bar{c} \in M\} \times \{A \subseteq M\}^2 \) such that

\[
\bar{c} \downarrow_A B \Leftrightarrow \bar{c} \cap B \subseteq A.
\]
The Random Graph is Simple  (first proof)

Theorem (Kim & Pillay)

A theory $T$ is simple if and only if there is a ternary relation $\downarrow^o$ satisfying automorphism invariance, extension, local and finite character, symmetry, transitivity, monotonicity, and independence over models. Moreover, in this case $\downarrow^o$, $\downarrow^d$, and $\downarrow$ are all the same.

Independence over models: Let $M \models T$, $M \subseteq A$, $B$ and $A \downarrow^o_M B$.
Suppose $\bar{a}, \bar{b} \in M$ such that $\bar{a} \equiv_M \bar{b}$, $\bar{a} \downarrow^o_M A$, and $\bar{b} \downarrow^o_M B$. Then there is $\bar{c} \in M$ such that $\text{tp}(\bar{a}/A) \cup \text{tp}(\bar{b}/B) \subseteq \text{tp}(\bar{c}/AB)$ and $\bar{c} \downarrow^o_M AB$.

Proposition

The ternary relation $\downarrow^\cap$ satisfies all the necessary properties to witness that $T_0$ is simple. In particular, we have

$$\bar{c} \downarrow^\cap B \iff \bar{c} \cap B \subseteq A.$$
The Random Graph is Simple  (second proof)

**Lemma**

Let $A \subseteq \mathbb{M}$, $\bar{b}, \bar{c} \in \mathbb{M}$ such that $\bar{c} \downarrow \bigcap_A \bar{b}$. If $(\bar{b}'_{l<\omega})$ is $A$-indiscernible, with $\bar{b}^0 = \bar{b}$, then there is $\bar{d} \in \mathbb{M}$ with $\bar{d} \downarrow \bigcap_A \bigcup_{l<\omega} \bar{b}'_{l}$ such that $\bar{d}b'_{l} \equiv_A \bar{c}b$ for all $l < \omega$.

**Theorem**

Suppose $B \subseteq \mathbb{M} \models T_0$ and $p \in S_n(B)$. Then $p$ does not divide over $A = \{b \in B : \exists i x_i = b \in p(\bar{x})\}$.

1. A theory $T$ is **supersimple** if for all $p \in S_n(B)$ there is some finite $A \subseteq B$ such that $p$ does not divide over $A$. Therefore $T_0$ is supersimple.

2. (Kim) If $\downarrow \bigcap$ is replaced by $\downarrow$ in the above lemma, the statement remains true in any simple theory.
Dividing in the Random Graph

Theorem

Let \( A \subset \mathbb{M} \models T_0, \, \varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^0, \) and \( \bar{b} \in \mathbb{M} \) such that \( \varphi(\bar{x}, \bar{b}) \) is consistent. Then \( \varphi(\bar{x}, \bar{b}) \) divides over \( A \) if and only if \( \varphi(\bar{x}, \bar{b}) \triangleleft x_j = b \) for some \( b \in \bar{b} \setminus A \).
The $K_n$-free Random Graph

For $n \geq 3$, we let $T_n$ be the theory of the $K_n$-free random graph, i.e., $T_n$ contains

1. the axioms for nonempty graphs;
2. a sentence asserting that the graph is $K_n$-free;
3. “extension axioms” asserting that if $A$ and $B$ are finite disjoint sets of vertices, and $A$ is $K_{n-1}$-free, then there is some $c$ such that $c \mathcal{R} A$ and $c \not\mathcal{R} B$.

$T_n$ is a complete, $\aleph_0$-categorical theory with quantifier elimination and no finite models.
The $K_n$-free Random Graph is Not Simple (first proof)

**Theorem**

$\downarrow d$-independence over models fails in $T_n$.

**Proof.**

Claim:

If $\bar{b} \in M \setminus A$, $\bar{b} \not\models RA$, $l(\bar{b}) < n - 1$, and $p$ is in $S_1(A\bar{b})$, then $p$ does not divide over $A$.

So $a_i \downarrow_M \{a_j : j < i\}$ and $b_i \downarrow_M a_i$.

Clearly, $\{\text{tp}(b_i/\text{Ma}_i) : i < n\}$ cannot be amalgamated.
The $K_n$-free Random Graph is Not Simple (second proof)

**Definition**

Let $k \geq 3$. A theory $T$ has the $k$-**strong order property**, $SOP_k$, if there is a formula $\varphi(\bar{x}, \bar{y})$, with $l(\bar{x}) = l(\bar{y})$, and $(\bar{b}^l)_{l<\omega}$ such that

\[
\mathbb{M} \models \varphi(\bar{b}^l, \bar{b}^m) \quad \forall \ l < m < \omega;
\]

\[
\mathbb{M} \models \neg \exists \bar{x}^1 \ldots \exists \bar{x}^k (\varphi(\bar{x}^1, \bar{x}^2) \land \ldots \land \varphi(\bar{x}^{k-1}, \bar{x}^k) \land \varphi(\bar{x}^k, \bar{x}^1)).
\]

Fact: \ldots \Rightarrow SOP_{k+1} \Rightarrow SOP_k \Rightarrow \ldots \Rightarrow SOP_3 \Rightarrow tree\ property.

**Theorem**

*For all $n \geq 3$, $T_n$ has SOP$_3$.***
The $K_n$-free Random Graph is Not Simple (second proof)

Proof ($T_n$ has SOP$_3$).

Assume $n \neq 4$.
Let $m = \lceil \frac{n}{3} \rceil$. Then $2m < n$.

Let $\varphi(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{y}^1, \bar{y}^2, \bar{y}^3)$ describe this configuration.

We can construct an infinite chain in $\mathbb{M} \models T_n$.

No 3-cycle is possible.
The $K_n$-free Random Graph is Not Simple (second proof)

**Definition**

Let $k \geq 3$. A theory $T$ has the $k$-**strong order property**, SOP$_k$, if there is a formula $\varphi(\vec{x}, \vec{y})$ and $(\vec{b}^l)_{l<\omega}$ such that

\[
\mathcal{M} \models \varphi(\vec{b}^l, \vec{b}^m) \quad \forall l < m < \omega;
\]

\[
\mathcal{M} \models \neg \exists \vec{x}^1 \ldots \exists \vec{x}^k (\varphi(\vec{x}^1, \vec{x}^2) \land \ldots \land \varphi(\vec{x}^{k-1}, \vec{x}^k) \land \varphi(\vec{x}^k, \vec{x}^1)).
\]

**Fact:** $\ldots \Rightarrow$ SOP$_{k+1} \Rightarrow$ SOP$_k \Rightarrow \ldots \Rightarrow$ SOP$_3 \Rightarrow$ tree property.

**Theorem**

*For all $n \geq 3$, $T_n$ has SOP$_3$.***

**Fact:** For all $n \geq 3$, $T_n$ does not have SOP$_4$.  

Recall: Dividing in the Random Graph

Theorem

Let \( A \subset M \models T_0 \), \( \varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^0 \), and \( \bar{b} \in M \) such that \( \varphi(\bar{x}, \bar{b}) \) is consistent. Then \( \varphi(\bar{x}, b) \) divides over \( A \) if and only if \( \varphi(\bar{x}, \bar{b}) \models x_j = b \) for some \( b \in \bar{b} \setminus A \).
Dividing in $T_n$

**Definition**

Suppose $A, B \subset M$ are disjoint. Then $B$ is $n$-bound to $A$ if there is $B_0 \subseteq A \cup B$, $|B_0| = n$, $B_0 \cap A \neq \emptyset \neq B_0 \cap B$, such that

1. $(B_0 \cap A) R (B_0 \cap B)$,
2. $(B_0 \cap A) \cong K_m$, where $m = |B_0 \cap A|$.

Informally, $B$ is $n$-bound to $A$ if there is a subgraph $B_0 \subseteq AB$ of size $n$, such that the only thing preventing $B_0$ from being isomorphic to $K_n$ is a possible lack of edges between points in $B$.

- $\bar{b}$ is 4-bound to $A$
- $\bar{b}$ is not 4-bound to $A$ (but $A$ is 4-bound to $\bar{b}$)
Dividing in $T_n$

**Theorem**

Let $A \subset \mathbb{M}$, $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_A^R$, and $\bar{b} \in \mathbb{M} \setminus A$ such that $\varphi(\bar{x}, \bar{b})$ is consistent. Then $\varphi(\bar{x}, \bar{b})$ divides over $A$ if and only if

1. $\bar{b}$ is not $n$-bound to $A$,
2. $\bar{b}$ is $n$-bound to $A\bar{c}$ for any $\bar{c} \models \varphi(\bar{x}, \bar{b})$.

**Proof.**

($\Leftarrow$): $\bar{b}$ not $n$-bound to $A$ allows construction of a sequence $(\bar{b}^l)_{l<\omega}$, with enough edges, so that being $n$-bound to $A\bar{c}$ will force $(n−1)$-inconsistency in $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$.

($\Rightarrow$): Let $(\bar{b}^l)_{l<\omega}$, indiscernible over $A$, witness dividing. Since $\varphi(\bar{x}, \bar{b})$ does not divide over $A$ in $T_0$, there must be a copy of $K_n$ in $A\bar{c}(\bar{b}^l)_{l<\omega}$, where $\bar{c}$ is an “optimal” solution of $\{\varphi(\bar{x}, \bar{b}^l) : l < \omega\}$ in $T_0$. By indiscernibility, this $K_n$ will “project” to the required conditions in $A\bar{b}$. □
Dividing in $T_n$ - Examples

**Corollary**

Suppose $A \subset \mathbb{M}$ and $b_1, \ldots, b_{n-1} \in \mathbb{M} \setminus A$ are distinct. Then the formula

$$\varphi(x, \bar{b}) := \bigwedge_{i=1}^{n-1} xRb_i$$

divides over $A$ if and only if $\bar{b}$ is not $n$-bound to $A$.

**Corollary**

Let $A \subset \mathbb{M}$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}^R_A$. Suppose $\bar{b} \in \mathbb{M} \setminus A$ such that $\varphi(\bar{x}, \bar{b})$ is consistent and divides over $A$. Define

$$R^\varphi = \{ u \in A\bar{b} : \exists i \varphi(\bar{x}, \bar{b}) \triangleright x_iRu \} \cup \{ x_i : \exists u \in A\bar{b}, \varphi(\bar{x}, \bar{b}) \triangleright x_iRu \}.$$ 

Then $|R^\varphi| \geq n$ and $|\bar{b} \cap R^\varphi| > 1$. 
Recall: $\downarrow$ in $T_0$

**Theorem**

Let $A, B, \bar{c} \subset \mathcal{M} \models T_0$. Then

$$\bar{c} \downarrow^A_B \iff \bar{c} \downarrow^d_A B \iff \bar{c} \cap B \subseteq A.$$
Lemma

Let $A \subseteq \mathbb{M} \models T_n$ and $p \in S_m(A)$. If $p \vdash x = b$ for some $b \in \mathbb{M}$ then $b \in A$. If $p \vdash xRb$ for some $b \in \mathbb{M}$, then either $b \in A$ or $p \vdash x = a$ for some $a \in A$.

Theorem

Let $A, B, \bar{c} \subseteq \mathbb{M} \models T_n$. Then

$$\bar{c} \downarrow_A B \iff \bar{c} \downarrow^d_A B \iff \bar{c} \cap B \subseteq A \text{ and for all } \bar{b} \in B \setminus A, \bar{b} \text{ is either } n\text{-bound to } A \text{ or not } n\text{-bound to } A\bar{c}.$$  

Corollary

Let $A \subseteq \mathbb{M} \models T_n$. If $\pi(\bar{x})$ is a partial type over $A$, then $\pi(\bar{x})$ does not fork over $A$.  

Lemma

Suppose \((\bar{b}^l)_{l<\omega}\) is an indiscernible sequence in \(M \models T_n\) such that \(l(\bar{b}^0) = 4\) and \(\bar{b}^0\) is \(K_2\)-free. Then either there are \(i < j\) such that \(\{b^i_l, b^j_l : l < \omega\}\) is \(K_2\)-free, or \(\bigcup_{l<\omega} \bar{b}^l\) is not \(K_3\)-free.
Forking in $T_n$

**Theorem**

Let $\mathbb{M} \models T_n$ and $A\overline{b} \subset \mathbb{M}$ such that $l(\overline{b}) = 4$, $\overline{b}RA$, $A \cong K_{n-3}$, and $\overline{b}$ is $K_2$-free. For $i \neq j$, let

$$\varphi_{i,j}(x, b_i, b_j) = xRb_i \land xRb_j \land \bigwedge_{a \in A} xRa,$$

and set

$$\varphi(x, \overline{b}) := \bigvee_{i \neq j} \varphi_{i,j}(x, b_i, b_j).$$

Then $\varphi(x, \overline{b})$ forks over $A$ but does not divide over $A$.

$$\varphi(x, \overline{b}) = "xRA \land |\{b_i : xRb_i\}| = 2"$$
Higher Arity Graphs

Now suppose $R$ is a relation of arity $r \geq 2$. An $r$-graph is an $\mathcal{L}$-structure satisfying the following sentence

$$\forall x_1 \ldots \forall x_r \left( R(x_1, \ldots, x_r) \rightarrow \left( \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_{\sigma \in S_r} R(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) \right) \right).$$

So $R$ can be thought of as a collection of $r$-element subsets of the $r$-graph.

Let $T^r_0$ and $T^r_n$, for $n > r$, be the natural analogs of $T_0$ and $T_n$, respectively, to $r$-graphs.

Call $T^r_0$ the **theory of the random $r$-graph**, and $T^r_n$ the **theory of the random $K^r_n$-free $r$-graph**, where $K^r_n$ is the complete $r$-graph of size $n$.

$T^r_n$ can be thought of as the theory of the unique (countable) Fraïssé limit of the class of finite $K^r_n$-free $r$-graphs.
Recall - Dividing in \( T_0 \)

**Lemma**

Let \( A \subset M \models T_0, \bar{b}, \bar{c} \in M \) such that \( \bar{c} \downarrow \bigcap_A \bar{b} \). If \((\bar{b}^l)_{l<\omega}\) is \( A \)-indiscernible, with \( \bar{b}^0 = \bar{b} \), then there is \( \bar{d} \in M \) with \( \bar{d} \downarrow \bigcap_A \bigcup_{l<\omega} \bar{b}^l \) such that \( \bar{d} \bar{b}^l \equiv_A \bar{c} \bar{b} \) for all \( l < \omega \).

**Lemma**

Let \( A \subset M \models T_0 \) and \( p(\bar{x}, \bar{y}) \in S(A) \) such that \( x_i \neq y_j \in p(\bar{x}, \bar{y}) \) for all \( i, j \). Suppose \( p(\bar{x}, \bar{b}) \) is consistent and \((\bar{b}^l)_{l<\omega}\) is indiscernible over \( A \) with \( \bar{b}^0 = \bar{b} \). Then there is a solution \( \bar{c} \) of \( \bigcup_{l<\omega} p(\bar{x}, \bar{b}^l) \).
Dividing in $T'_0$

Lemma

Let $A \subseteq M \models T'_0$ and $p(\bar{x}, \bar{y}) \in S(A)$ such that $x_i \neq y_j \in p(\bar{x}, \bar{y})$ for all $i, j$. Suppose $p(\bar{x}, \bar{b})$ is consistent and $(\bar{b}'_l)_{l<\omega}$ is indiscernible over $A$ with $b^0 = \bar{b}$. Then there is a solution $\bar{c}$ of $\bigcup_{l<\omega} p(\bar{x}, \bar{b}'_l)$.

In fact, we can take $\bar{c}$ to be an optimal solution, in particular if $A_0 \subseteq A(\bar{b}'_l)_{l<\omega}$, then for any $i_1, \ldots, i_k$

$$M \models R(c_{i_1}, \ldots, c_{i_k}, A_0) \iff R(x_{i_1}, \ldots, x_{i_k}, A_0) \in \bigcup_{l<\omega} p(\bar{x}, \bar{b}'_l).$$

Corollary ($T'_0$ is simple.)

Suppose $B \subseteq M \models T'_0$ and $p \in S(B)$. Then $p$ does not fork over $A = \{b \in B : \exists i \ x_i = b \in p(\bar{x})\}$. 

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**Lemma**

Assume $r > 2$. Let $A \subseteq \mathcal{M} \models T'_n$ and $p(\bar{x}, \bar{y}) \in S(A)$ such that $x_i \neq y_j \in p(\bar{x}, \bar{y})$ for all $i, j$. Suppose $p(\bar{x}, \bar{b})$ is consistent and $(\bar{b}^l)_{l < \omega}$ is indiscernible over $A$ with $\bar{b}^0 = \bar{b}$. Then there is a solution $\bar{c}$ of $\bigcup_{l < \omega} p(\bar{x}, \bar{b}^l)$.

**Corollary ($T'_n$ is simple if $r > 2.$)**

Suppose $r > 2$ and $B \subseteq \mathcal{M} \models T'_n$ and $p \in S(B)$. Then $p$ does not fork over $A = \{b \in B : \exists i \ x_i = b \in p(\bar{x})\}$. 


Proof of Lemma.

Let $K_r \sim = W \subseteq A(\bar{b}_l)$ where $|\{l: W \cap \bar{b}_l \neq \emptyset\}| > 1$.

If $|\{l: W \cap \bar{b}_m \neq \emptyset\}| \leq 1$,

This contradicts $M| = \exists \bar{x}p(\bar{b}_m)$.
Proof of Lemma.

\[ \bar{c} \in \mathbb{M}' \models T_0' \text{ is an optimal solution to } \bigcup_{l<\omega} p(\bar{x}, \bar{b}'). \]
Proof of Lemma.

\[ \bar{c} \in M' \models T_0^r \text{ is an optimal solution to } \bigcup_{l < \omega} p(\bar{x}, \bar{b}^l). \]

Show \( A\bar{c}(\bar{b}^l)_{l < \omega} \) is \( K_n^r \)-free.
Dividing in $T^r_n$, $r > 2$

Proof of Lemma.

$\bar{c} \in \mathbb{M}' \models T^r_0$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is $K^r_n$-free.

Let $K^r_n \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.
Dividing in $T_r^n$, $r > 2$

Proof of Lemma.

$\bar{c} \in M' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is $K_n^r$-free.

Let $K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

There is $c_i \in W \cap \bar{c}$.
Proof of Lemma.

\[ \bar{c} \in \mathbb{M'} \models T_0^r \text{ is an optimal solution to } \bigcup_{l<\omega} p(\bar{x}, \bar{b}^l). \]

Show \( A\bar{c}(\bar{b}^l)_{l<\omega} \) is \( K_n^r \)-free.

Let \( K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega} \).

There is \( c_i \in W \cap \bar{c} \).

If \( |l : W \cap \bar{b}^l \neq \emptyset| > 1 \)
Dividing in $T_n^r$, $r > 2$

**Proof of Lemma.**

$\bar{c} \in \mathbb{M}' \models T_0^r$ is an optimal solution to $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$.

Show $A\bar{c}(\bar{b}^l)_{l<\omega}$ is $K_n^r$-free.

Let $K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega}$.

There is $c_i \in W \cap \bar{c}$.

If $|l : W \cap \bar{b}^l \neq \emptyset| > 1$ pick $W_0 \subseteq W \setminus \{c_i, u, v\}$, with $|W_0| = r - 3$. 

Proof of Lemma.

\[ \overline{c} \in \mathbb{M}' \models T_0' \text{ is an optimal solution to } \bigcup_{l < \omega} \rho(\overline{x}, \overline{b}^l). \]

Show \( A\overline{c}(\overline{b}^l)_{l < \omega} \) is \( K_n^r \)-free.

Let \( K_n^r \cong W \subseteq A\overline{c}(\overline{b}^l)_{l < \omega} \).

There is \( c_i \in W \cap \overline{c} \).

If \( |l : W \cap \overline{b}^l \neq \emptyset| > 1 \)

pick \( W_0 \subseteq W \setminus \{c_i, u, v\} \),

with \( |W_0| = r - 3 \).

\[ R(c_i, u, v, W_0) \in \bigcup_{l < \omega} \rho(\overline{x}, \overline{b}^l). \]
Proof of Lemma.

\[ \bar{c} \in M' \models T_0' \text{ is an optimal solution to } \bigcup_{l<\omega} p(\bar{x}, \bar{b}^l). \]

Show \( A\bar{c}(\bar{b}^l)_{l<\omega} \) is \( K_n^r \)-free.

Let \( K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega} \).

\[ |\{l : W \cap \bar{b}^l \neq \emptyset\}| \leq 1 \]
Proof of Lemma.

\[ \bar{c} \in \mathbb{M}' \models T_0^r \text{ is an optimal solution to } \bigcup_{l < \omega} p(\bar{x}, \bar{b}^l). \]

Show \( A\bar{c}(\bar{b}^l)_{l < \omega} \) is \( K_n^r \)-free.

Let \( K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l < \omega} \).

\[ |\{l : W \cap \bar{b}^l \neq \emptyset\}| \leq 1 \]

“\( W \cong K_n^r \)” \( \in p(\bar{c}, \bar{b}^m) \)
Proof of Lemma.

\( \bar{c} \in \mathbb{M} \models T_0^r \) is an optimal solution to \( \bigcup_{l<\omega} p(\bar{x}, \bar{b}^l) \).

Show \( A\bar{c}(\bar{b}^l)_{l<\omega} \) is \( K_n^r \)-free.

Let \( K_n^r \cong W \subseteq A\bar{c}(\bar{b}^l)_{l<\omega} \).

\( |\{l : W \cap \bar{b}^l \neq \emptyset\}| \leq 1 \)

"\( W \cong K_n^r \)" \( \in p(\bar{c}, \bar{b}^m) \)

This contradicts \( \mathbb{M} \models \exists \bar{x} p(\bar{x}, \bar{b}^m) \).
References

