Computing strength of structures related to the field of real numbers

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1 Introduction

The behavior of structures in generic extensions of the universe has been studied from a number of different angles; for example, Baldwin, Laskowski, and Shelah [2] studied the conditions under which non-isomorphic structures may become isomorphic, and Knight, Montalban, and Schweber [8] (and independently Kaplan and Shelah [7]) studied structures existing in every generic extension of the universe by some forcing. In the latter example, general results about such “generically presentable” structures led to a new proof of a result of Harrington saying that if \( T \) is a counterexample to Vaught’s Conjecture, then \( T \) has models of cardinality \( \aleph_1 \) with arbitrarily large Scott ranks less than \( \omega_2 \). (There are now at least three new proofs of this result. In addition to the one in [8], there is one by Baldwin, S.-D. Friedman, Koerwien, and Laskowski [1] and one by Larson [10]; these other proofs do not use generically presentable structures directly, but do use related ideas.)

We can do more with generic extensions. In [8], the third author defined a notion that lets us compare the computing power of structures of any cardinality.

**Definition 1** (Schweber). Let \( A \) and \( B \) be structures in \( V \) (of any cardinality). We say that \( A \leq^* w B \) if in a generic extension \( V(G) \) in which both \( A \) and \( B \) are countable, every copy of \( B \) computes a copy of \( A \).

In [8], there are a few examples comparing familiar structures. In particular, it is shown that \( W \leq^* w R \), where \( R \) is the ordered field of real numbers, and \( W \) represents the power set of \( \omega \). We have \( W = (P(\omega) \cup \omega, P(\omega), \omega, \in, S) \), where \( S \) is the successor relation on \( \omega \). In computability, the structures \( R \) and \( W \) are sometimes identified; both are referred to as “the reals”. Of course, they are not the same structure: \( R \) is an archimedean ordered field, while \( W \) is just a family of subsets of \( \omega \).

Let \( R^* \) be an \( \omega \)-saturated extension of \( R \). In [6], it is shown that \( R^* \equiv^*_w W \) and that \( R \not\leq^*_w R^* \), so \( R \not\leq^*_w W \). We recall a little of the proof. First, \( R \) is a residue field section of \( R^* \). After collapse, \( R^* \) is no longer \( \omega \)-saturated, but it is recursively saturated, and it realizes just the types in the Scott set.
that is the old \( P(\omega) \). It is shown that for a countable recursively saturated real closed field \( K \), with residue field \( k \), some copy of \( K \) does not compute a copy of \( k \). The proof of this involves a reduction. It is shown that if every copy of \( K \) computes a copy of \( k \), then the set \( FT(K) \) consisting of finite elements that are not infinitesimally close to any algebraic element must be defined in \( K \) by a computable \( \Sigma_2 \) formula. It is then shown that \( FT(K) \) has no such definition.

In the present paper, we consider further structures related to the reals. Let \( R^{\text{exp}} = (\mathbb{R}, \exp) \). We show that \( R^{\text{exp}} \leq^*_w R \). This was a surprise to the authors. We had expected \( R^{\text{exp}} \) to be strictly more powerful than \( R \). We generalize this result, replacing \( R \) by an apparently very weak structure \( R_Q \) consisting of the real numbers with the ordering and constants naming the rationals, and replacing \( R^{\text{exp}} \) by certain expansions of \( R_Q \).

**General Theorem.** Let \( M \) be an \( o \)-minimal expansion of \( R_Q \), in a countable language, with definable Skolem functions. Then \( M \leq^*_w R_Q \). In fact, after collapse, every copy \( K \) of \( R_Q \) computes the complete diagram of a copy of \( M \).

Using the General Theorem, we prove that if \( R_f = (\mathbb{R}, f) \), where \( f \) is total analytic on \( R \), then \( R_f \equiv^*_w R \). As further applications, we have the fact that \( (R_n)_{n \in \omega} \equiv^*_w R \), where \( (r_n)_{n \in \omega} \) is an arbitrary countable sequence of reals, and \( R^+ \equiv^*_w R \), where \( R^+ \) is the reduct of \( R \) in which multiplication is dropped. This last result was obtained independently by Downey, Greenberg, and Miller [3].

It is clear that \( W \equiv^*_w C = (2^\omega, (R_n)_{n \in \omega}) \), where \( f \in R_n \) iff \( f(n) = 1 \). The structure \( C \) represents Cantor space. To represent Baire space, we may take \( B = (\omega^\omega, (R_{n,k})_{n,k \in \omega}) \), where \( f \in R_{n,k} \) iff \( f(n) = k \). Downey, Greenberg, and Miller [3] showed that \( B \equiv^*_w R \). From Baire space, we derive a structure \( R_{\text{int}} \), consisting of the real numbers, with the ordering, and predicates for the half-open intervals \( [q, q') \), where \( q, q' \) are dyadic rationals with \( q < q' \). It is not difficult to show that \( B \equiv^*_w R_{\text{int}} \).

On its face, \( R_{\text{int}} \) is even weaker than \( R_Q \). Modifying the proof of the General Theorem, we could show that \( R_Q \leq^*_w R_{\text{int}} \). Hence, \( R \equiv^*_w R_{\text{int}} \). Alternatively, we could modify our General Theorem, replacing \( R_Q \) by \( R_{\text{int}} \). Applying this variant of the General Theorem, we would get the fact that \( R \leq^*_w R_{\text{int}} \), so \( R_Q \leq^*_w R_{\text{int}} \). Before knowing the results in [3], the authors used this approach to show that \( R \equiv^*_w B \).

In Section 2, we show that \( R^{\text{exp}} \equiv^*_w R \). The proof combines ideas from computability (jumps and effective guessing strategies), computable structure theory (definability by computable infinitary formulas), and model theory (\( o \)-minimality). In Section 3, we generalize the result from Section 2 to prove the General Theorem. In Section 4, we apply the General Theorem from Section 3 to show that the expansions of \( R \) by an analytic function or an arbitrary sequence of constants, and the reduct \( R^+ \) are equivalent to \( R \). In Section 5, we show that \( B \equiv^*_w R_{\text{int}} \). We indicate briefly what would be involved in proving
the modified version of the General Theorem, with $R_Q$ replaced by $R_{int}$. In the remainder of the introduction, we give some background on $o$-minimality.

Remark. If $A$ is an expansion of $B$ such that $A \equiv^e_{in} B$, it may not be the case that (in an appropriate generic extension) every copy of $B$ computes a copy of $A$ together with an isomorphism between the copy of $B$ and the reduct of the copy of $A$. Indeed, that is the case with expansions of $R$: for example, the functions that $R$ can compute in this sense are precisely the piecewise algebraic functions.

1.1 $o$-minimality

Definition 2. A structure $M$ with a dense linear ordering on the universe is $o$-minimal if each set definable by an elementary first order formula (with parameters) is a finite union of intervals (possibly trivial) with endpoints in $M$.

The following is well-known [9].

Proposition 1.1. If $T$ is the elementary first order theory of an $o$-minimal structure $M$, then all models of $T$ are $o$-minimal.

We say that $T$ is an $o$-minimal theory if some/all models of $T$ are $o$-minimal.

Examples.

1. $R$ is $o$-minimal. Tarski [14] proved that $Th(R)$ is decidable. In the proof, Tarski gave an effective elimination of quantifiers. There is an algorithm (familiar to every school child) for deciding the truth of the quantifier-free sentences. As a side result, Tarski stated the fact that in $R$, and the other models of the theory, the definable sets are finite unions of intervals.

2. $R^+$ is $o$-minimal. It is clear from the definition that any reduct of an $o$-minimal structure that includes the ordering is $o$-minimal.

3. $R_{sin}$ is not $o$-minimal—think of the set of zeroes of $sin(x)$.

4. $R_{exp}$ is $o$-minimal. Wilkie [17] showed that $T_{exp} = Th(R_{exp})$ is model complete; i.e., if $M_1$ and $M_2$ are models of the theory, with $M_1 \subseteq M_2$, then $M_1 \prec M_2$. By results of Khovanskii [5], it follows that the theory is $o$-minimal. Ressayre [13] gave another proof of model completeness.

5. $R_{an}$ is $o$-minimal, where this is the expansion of $R$ with the restrictions $f_I$ of analytic functions $f$ to compact intervals $I = [a, b]$. More precisely, $f_I$ is the total function that agrees with $f$ on $I$ and has value 0 otherwise. By results of van den Dries [15], building on work of Gabrielov [4], $R_{an}$ is $o$-minimal.

We will use the following three facts. The first is due to van den Dries [16].

Proposition 1.2. Any $o$-minimal expansion of $R^+$ has definable Skolem functions.
The second fact is due to Pillay [9].

**Proposition 1.3.** For an o-minimal theory with definable Skolem functions, definable closure is a good closure notion, satisfying the Exchange Property—if $a$ is definable from $b,c$ and not from $b$, then $c$ is definable from $b,a$.

This means that independence, basis, and dimension are well-defined. The third fact is also due to Pillay [9].

**Proposition 1.4.** For an independent tuple $\bar{b}$ in an o-minimal structure $A$, if $\varphi(\bar{x})$ is a finitary formula true of $\bar{b}$, then there is an open box $B$ around $\bar{b}$, with vertices having coordinates in $A$, such that $\varphi(\bar{x})$ is valid on $B$.

**Remark.** If $A$ is an Archimedean model of $Th(R)$ (or $Th(R^+)$), and $\bar{a}$ is a tuple in an open box $B$ with vertices having coordinates in $A$, then there is another open box $B^* \subseteq B$ such that $\bar{a} \in B^*$ and $B^*$ has vertices with rational coordinates. We refer to $B^*$ as a rational box.

2 $\mathcal{R}_{exp} \equiv^*_w \mathcal{R}$

In this section, our goal is to prove the following.

**Theorem 2.1.** $\mathcal{R}_{exp} \equiv^*_w \mathcal{R}$.

Here is a brief overview of the proof. Clearly, $\mathcal{R} \leq^*_w \mathcal{R}_{exp}$. We show that $\mathcal{R}_{exp} \leq^*_w \mathcal{R}$. Let $T_{exp}$ be the elementary first order theory of $\mathcal{R}_{exp}$. The theory $T_{exp}$ may or may not be decidable, depending on Schanuel’s Conjecture see [12]. In any case, the theory is coded in a real parameter. After collapse, let $K$ be a copy of $\mathcal{R}$ with universe a subset of $\omega$. Since $K$ is isomorphic to $\mathcal{R}$, there is an expansion $K_{exp}$ satisfying $T_{exp}$. We show that this expansion is unique. Next, we show that independence is defined in a way that yields a basis for $K_{exp}$ that is $\Delta^0_2$ relative to $K$. Finally, we use a computable approximation to this basis in a finite injury priority construction in order to construct a copy of $K_{exp}$ that is computable in $K$.

2.1 Expanding $K$ to a model of $T_{exp}$

We first show that for a countable Archimedean real closed ordered field $K$ with an added function $f$ satisfying $T_{exp}$, the expansion is unique. Moreover, if the function $exp^K$ is defined by a computable $\Pi_1$ formula. The same is true if we substitute for $exp$ an arbitrary continuous function.

**Lemma 2.2** (Uniqueness). Let $f$ be a continuous function on the reals, and let $T_f = Th(\mathcal{R}, f)$. If $K$ is a copy of $\mathcal{R}$ (after collapse), then there is a unique expansion $(K, f^K)$ satisfying $T_f$. Moreover, the function $f^K$ is defined by a computable $\Pi_1$ formula with a real parameter $r$. Hence, it is $\Delta^0_2$ relative to $K$. 

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Proof. Let \( a \in K \). For each open interval \( I \) containing \( f(a) \), and having rational endpoints, there is an open interval \( J \) containing \( a \), also with rational endpoints, such that \( f \) (as a function on \( R \)) maps \( J \) to \( I \). For each such pair of intervals \( I \) and \( J \), with rational endpoints, there is a sentence in \( T_f \) saying that \( f \) maps \( J \) into \( I \). Then the function \( f^K \) must map \( J \) to \( I \) in \( K \). This implies that \( f^K(a) \) in \( K \) must match \( f(a) \) in \( R \). This proves uniqueness.

Suppose \( r \) is a parameter coding \( T_f \). This means that \( r \in [0,1] \), and in its “preferred” binary expansion, \( r \) has 1 in the \( k^{th} \) place iff \( k \) is the Gödel number of a sentence of \( T_{exp} \). The preferred binary expansion has infinitely many 0’s.

Most reals in the interval \([0,1]\) have a unique binary expansion, with infinitely many 0’s and infinitely many 1’s. However, the dyadic rationals in the interval \([0,1]\) have two binary expansions, one ending in an infinite string of 1’s, with only finitely many 0’s, and the other ending in an infinite string of 0’s, with only finitely many 1’s.

We show that there is a computable \( \Pi_1 \) formula, with parameter \( r \), defining \( f^K \) in \( K \). We have \( y = f(x) \) iff for all pairs of rational intervals \( J \) and \( I \) such that \( T_f \) contains the sentence saying that \( f : J \to I \), if \( x \in J \), then \( y \in I \). This is naturally expressed as the conjunction of finitary quantifier-free formulas over a set that is c.e. relative to \( T_f \). We can replace this by a c.e. conjunction involving the parameter \( r \in [0,1] \). Let \( c_k(u) \) be a finitary quantifier-free formula saying of \( u \in [0,1] \) that its preferred binary expansion has 1 in the \( k^{th} \) place. For all \( k \), we define a finitary quantifier-free formula \( \rho_k(u, x, y) \). If \( k \) is the Gödel number of a sentence saying that \( f : J \to I \), then \( \rho_k(u, x, y) \) says \( c_k(u) \rightarrow (x \in I \rightarrow y \in J) \), and if \( k \) is not the Gödel number of such a sentence, then \( \rho_k(u, x, y) = \top \). Then the computable \( \Pi_1 \) formula \( \bigwedge_k \rho_k(r, x, y) \) holds just in case \( y = f(x) \).

Remark. Not all Archimedean real closed ordered fields can be expanded to models of \( T_{exp} \). In particular, since \( e = exp(1) \) is transcendental, the ordered field of real algebraic numbers cannot be expanded in this way.

By Proposition 1.4, for a tuple \( \bar{a} \) that is independent in \( R_{exp} \), each formula \( \varphi(\bar{x}) \) true of \( \bar{a} \) is valid on a rational box \( B \) around \( \bar{a} \). We need the converse of this.

Lemma 2.3. Let \( \bar{a} \) be a tuple of reals. Suppose that for every formula \( \varphi(\bar{x}) \) true of \( \bar{a} \) in \( R_{exp} \), there is a rational box \( B \) around \( \bar{a} \) such that \( T_{exp} \) contains the sentence saying that \( \varphi(\bar{x}) \) is valid on \( B \). Then \( \bar{a} \) is independent in \( R_{exp} \).

Proof. Suppose not. Say \( a_k \) is defined from \( a_1, \cdots, a_{k-1} \) in \( R_{exp} \). Let \( \varphi(\bar{x}) \) be a formula saying that \( x_k \) is defined in this way from \( x_1, \cdots, x_{k-1} \). This cannot be valid on an open box.

2.2 Independence relations on \( R_{exp} \)

Definition 3. Suppose \( K \) is an Archimedean real closed ordered field with an expansion \((K, exp)\) satisfying \( T_{exp} \). Let \( IND_n(K) \) be the set of \( n \)-tuples in \( K \) that are independent in \((K, exp)\).
We show that the relations $IND_n(R)$ are defined in $R$ by computable sequences of computable $\Pi_2$ and computable $\Sigma_2$ formulas. The computable $\Pi_2$ definitions are easy.

**Lemma 2.4** (Computable $\Pi_2$ definition of $IND_n$). For each $n$, we can effectively find a computable $\Pi_2$ definition of $IND_n$, with a parameter $r$ coding $\text{T}_{\text{exp}}$.

**Proof.** We have $\bar{a} \in IND_n$ iff for each formula $\varphi(\bar{x})$, there is a rational box $B$ around $\bar{a}$ such that $\text{T}_{\text{exp}}$ contains one of the sentences $(\forall \bar{x} \in B)\varphi(\bar{x})$ or $(\forall \bar{x} \in B)\neg \varphi(\bar{x})$. We can express this as a computable $\Pi_2$ formula. Let $c_k(u)$ be the formula saying that the $k$th place in the preferred binary expansion of $u$ is 1. For each formula $\varphi$ in the appropriate variables, and each rational box $B$, let $k(\varphi, B)$ be the Gödel number of the sentence saying $(\forall \bar{x} \in B)\varphi(\bar{x})$. We have $\bar{a} \in IND_n$ iff

$$\bigwedge_{\varphi} \bigvee_B (\bar{a} \in B \land (c_k(\varphi,B)(r) \lor c_k(\neg \varphi,B)(r)))$$

where the conjunction is over all $\varphi$ with appropriate variables, and the disjunction is over all rational boxes $B$. This is computable $\Pi_2$, with the parameter $r$, as required. 

The computable $\Sigma_2$ definition for the relation $IND_n$ is less obvious.

**Lemma 2.5** (Computable $\Sigma_2$ definition of $IND_n$). For each $n$, we can effectively find a computable $\Sigma_2$ definition of $IND_n$, with a parameter $r$ coding $\text{T}_{\text{exp}}$.

**Proof.** Fix $n$. Let $(\varphi_m(\bar{x}))_{m\in\omega}$ be a computable list of formulas in the variables $\bar{x} = (x_1, \ldots, x_n)$, in the language of $\text{R}_{\text{exp}}$. We build a tree $T$, computable in $\text{T}_{\text{exp}}$, consisting of finite sequences of rational boxes $B_1, B_2, \ldots, B_s$ such that $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_s$ and for each $k < s$, one of the sentences $(\forall \bar{x} \in B_k)\varphi_k(\bar{x})$ or $(\forall \bar{x} \in B_k)\neg \varphi_k$ is in $\text{T}_{\text{exp}}$. By Proposition 1.4 and Lemma 2.3, $\bar{a} \in IND_n$ iff there is a path $\pi = B_1, B_2, \ldots$ through $T$ such that for each $s$, $\bar{a}$ is in the box $B_s$. We show that this definition can be expressed by a computable $\Sigma_2$ formula in the language of real closed ordered fields, with the parameter $r$.

**Claim:** There is a computable $\Pi_1$ formula, with parameter $r$, saying that $x$ codes a path through $T$.

**Proof of Lemma.** The preferred binary expansion of $x \in [0,1]$ is the characteristic function $f_x$ of a set $S_x \subseteq \omega$. The set $S_x$ may be finite, but it cannot be co-finite. We consider a path through $T$ to be a set $S$ with the following properties.

1. each element of $S$ is a code for a finite sequence $(B_1, \ldots, B_s)$ in $T$,
2. if $(B_1, \ldots, B_s, B_{s+1})$ is in $S$, then so is $(B_1, \ldots, B_s)$,
3. if two sequences in $S$ have length $s$, then they are equal,
4. \( S \) is infinite.

We show that each of the four properties above can be expressed by a computable \( \Pi_1 \) formula. For Property 1, we say that \( S_x \) has no elements not in \( T \). Since \( T \) is computable in \( T_x \exp \) and \( T_x \exp \) is coded by \( r \), there is a c.e. set \( C \) of pairs \((\sigma, k)\), with \( \sigma \in 2^{<\omega} \), such that \( k \notin T \) iff for some \((\sigma, k) \in C\), \( r \) agrees with \( \sigma \), where this means that the preferred binary expansion of \( r \) extends \( \sigma \); i.e., \( c_k(r) \) holds for \( \sigma(k) = 1 \) and \( \neg c_k(r) \) holds for \( \sigma(k) = 0 \). To say that \( S_x \) has no elements not in \( T \), we take the c.e. conjunction over \((\sigma, k) \in C\) of formulas saying that if \( r \) agrees with \( \sigma \), then \( \neg c_k(x) \). This is computable \( \Pi_1 \) with parameter \( r \).

For Property 2, we say that \( S_x \) (a set of codes for finite sequences), is closed under initial segments. We take the conjunction of formulas \( c_k(x) \rightarrow c_k'(x) \), for the pairs \((k, k')\) such that for some \( s \), \( k \) is the code for a sequence of length \( s + 1 \) and \( k' \) is the code for the initial segment of length \( k \). This is computable \( \Pi_1 \), with no parameter.

For Property 3, we say that if two sequences in \( S_x \) have length \( s \), then they are equal. We take the conjunction of formulas \( \neg (c_k(x) \& c_k'(x)) \), for the pairs \((k, k')\) coding distinct sequences of the same length. This is computable \( \Pi_1 \), with no parameter.

For Property 4, we must say that \( S_x \) is infinite. We recall that the elements \( x \) of \([0, 1)\) that code finite sets are just the dyadic rationals. We have a computable \( \Pi_1 \) formula saying that \( r \) is not equal to any of these rationals.

Putting the four statements together, we have a computable \( \Pi_1 \) formula, with parameter \( r \), saying that \( x \) codes a path through \( T \). This proves the Claim.

Knowing that \( x \) codes a path through \( T \), we want a computable \( \Pi_1 \) formula saying that an \( n \)-tuple \( \bar{u} \) lies in the boxes on this path. We take the conjunction over \( k \) coding a finite sequence of rational boxes \((B_1, \ldots, B_s)\) of the formulas saying \( c_k(x) \rightarrow \bar{u} \in B_s \). To say that \( \bar{u} \) is independent, we have a computable \( \Sigma_2 \) formula saying that there exists \( x \) such that \( S_x \) is a path through \( T \) and \( \bar{u} \) lies in the boxes corresponding to this path.

Thanks to the computable \( \Pi_2 \) and computable \( \Sigma_2 \) definitions, we know that for any copy \( K \) of \( R \), the relations \( IND_n(K) \) are \( \Delta^0_2 \) relative to \( K \), uniformly in \( n \).

Lemma 2.6 (Basis). Suppose \( K \) is a copy of \( R \). Then we have a sequence \( b_1, b_2, \ldots \), \( \Delta^0_2 \) relative to \( K \), and forming a basis for \( K_x \exp \).

Proof. Applying a procedure that is \( \Delta^0_2 \) relative to \( K \), we run through the elements, and we use the relations \( IND_n \) to choose a basis. We let \( b_1 \) be first satisfying \( IND_1(u_1) \), we let \( b_2 \) be first such that \( (b_1, b_2) \) satisfies \( IND_2(u_1, u_2) \), etc.

To complete the proof that \( R_x \exp \leq^*_w R \), we show the following.
Proposition 2.7 (Enumerating the complete diagram of the expansion). After collapse, let $K \cong R$. Then there is a a copy $C$ of $K_{\exp}$ with complete diagram computable in $K$.

Proof. Let $b_1, b_2, \ldots$ be a basis for $K_{\exp}$ that is $\Delta^0_2$ relative to $K$, determined as in the previous lemma. Guessing at this basis, and using $T_{\exp}$, we enumerate the complete diagram of a copy $C$ of $K_{\exp}$. The universe of $C$ will be $\omega$, which we think of as a set of constants. We fix a computable enumeration of the sentences $\varphi(\bar{c})$, where $\varphi(\bar{x})$ is a formula in the language of $R_{\exp}$ and $\bar{c}$ is a tuple of constants. We suppose that the language includes symbols for the definable Skolem functions. We fix a computable enumeration of terms $\tau(\bar{c})$, where $\tau(\bar{x})$ is a term in variables $\bar{x}$ and $\bar{c}$ is a corresponding tuple of constants.

At each stage $s$, we have enumerated into the complete diagram of $C$ a finite set $\delta_s$ of sentences. The set $\delta_s$ includes sentences saying that the constants mentioned are all distinct. We start with $\delta_0 = \emptyset$, and $\delta_s \subseteq \delta_{s+1}$. We will arrange that for each sentence $\varphi(\bar{c})$, one of $\pm \varphi(\bar{c})$ is in $\delta_s$ for some $s$. We will also arrange that for each term $\tau(\bar{c})$, a sentence of the form $\tau(\bar{x}) = \bar{c}'$ appears in $\delta_s$ for some $s$. To determine an isomorphism $f$ from $C$ onto $K_{\exp}$, it is enough to determine $f^{-1}(b_n)$ for all $n$, since the rest of the elements are definable from the basis. We have the following requirements.

$R_n$: Determine $f^{-1}(b_n)$.

At stage $s$, we have tentatively mapped some constants $\bar{d}_s$ to a tuple $\bar{v}_s$ in $K$ which we believe to be an initial segment of the basis $b_1, b_2, \ldots$. In $\delta_s$, we have mentioned the constants $\bar{d}_s$, plus some further constants $\bar{c}_s$. Each $c_i \in \bar{c}_s$ has been given a definition $\tau_i(\bar{d}_s)$, and the sentence $c_i = \tau_i(\bar{d}_s)$ is in $\delta_s$. We will maintain the condition that what we have said in $\delta_s$ about $\bar{d}_s$ is valid on a rational box $B_s$ around $\bar{v}_s$. We must make this precise.

Let $\chi_s(\bar{d}_s, \bar{c}_s)$ be the conjunction of the sentences in $\delta_s$. Let $\bar{u}$ be a tuple of variables corresponding to $\bar{d}_s$. We suppose that these variables do not appear in the sentences of $\delta_s$. Let $\chi_s^*(\bar{u})$ be the formula obtained from $\chi_s(\bar{d}_s, \bar{c}_s)$ by replacing each $d_i \in \bar{d}_s$ by the corresponding variable $u_i$, and replacing each $c_i \in \bar{c}_s$ by $\tau_i(\bar{u})$, where $c_i$ is defined to be $\tau_i(\bar{d}_s)$. Note that $\chi_s^*(\bar{u})$ has conjuncts saying that the terms $u_i$ and $\tau_i(\bar{u})$ are all distinct. Now, $\chi_s^*(\bar{u})$ expresses what we have said about $\bar{d}_s$ in $\delta_s$. We say how to check that this is true on a rational box $B_s$ around $\bar{v}_s$. We write $\chi^*_s(B_s)$ for the sentence saying $(\forall u \in B_s) \chi_s^*(\bar{u})$. We check that $\bar{v}_s \in B_s$ and that $\chi_s^*(B_s) \in T_{\exp}$. We can check that $\bar{v} \in B_s$ using $K$. We can check, using the real coding $T_{\exp}$, that the sentence $\chi^*_s(B_s) \in T_{\exp}$.

At stage $s+1$, if our stage $s$ guess $\bar{v}_s$ at the initial segment $b_s$ of the basis seems correct, then $\bar{v}_{s+1} = \bar{v}_s, v'$, where $v'$ appears to be the next element of the basis. If at stage $s$ our guess $\bar{v}_s$ at $b_s$ changes, then $\bar{v}_{s+1}$ is the restriction of $\bar{v}_s$ to the part that still seems to be an initial segment of the basis. In this case, any constants in $\bar{d}_s$ which were tentatively mapped to elements of $\bar{v}_s$ that are not in $\bar{v}_{s+1}$ will now be given definitions in terms of $\bar{d}_{s+1}$. At all future stages $t$, these elements will be part of the definable tuple $\bar{c}_t$. In the event that at a
later stage some elements of \( \bar{v}_s \) appear to return to the basis, we will map new constants to those elements.

At stage \( s + 1 \), assuming that our stage \( \tilde{v}_{s+1} \) has a new element \( v' \), we map a new constant \( d' \) to it. We put into \( \delta_{s+1} \) sentences saying that \( d' \) is not equal to any element of \( \tilde{d}_s \) or \( \tilde{c}_s \). We decide the next sentence \( \phi \) that mentions only the constants from \( \tilde{d}_s, \tilde{c}_s \). Also, for the next term \( \tau(\tilde{d}_s) \) not already given a name, we add a sentence \( c = \tau(\tilde{d}_s) \), where \( c \) is either in \( \tilde{c}_s \) or the first constant not yet mentioned. The lemmas below guarantee that we can do all of this, while maintaining the condition that what we have said in \( \delta_{s+1} \) about \( \tilde{d}_{s+1} \) is valid on a rational box around \( \bar{v}_{s+1} \). We need some terminology.

**Definition 4.** We say that \((\delta; \tilde{d}; \tilde{c})\) is a good triple if

1. \( \delta \) is a finite set of sentences with constants split into disjoint sets \( \tilde{d} \) and \( \tilde{c} \),
2. \( \delta \) includes sentences saying that the constants are all distinct,
3. for each \( c \in \tilde{c} \), \( \delta \) includes a sentence \( \tau(c) = c \).

For a good triple \((\delta; \tilde{d}; \tilde{c})\), a test formula \( \chi^*(\bar{u}) \) is obtained in the way we obtained \( \chi^*_s(\bar{u}) \) from \( \delta_s \) above.

**Definition 5.** For a good triple \((\delta; \tilde{d}; \tilde{c})\), we say that \( \chi^*(\bar{u}) \) is a test formula if it is obtained by the following steps.

1. Let \( \chi \) be the conjunction of \( \delta \), and let \( \bar{u} \) be a sequence of new variables corresponding to \( \tilde{d} \).
2. Let \( \chi^*(\bar{u}) \) be the formula obtained from \( \chi \) by replacing \( d_i \) by \( u_i \) and replacing \( c_i \) by \( \tau_i(\bar{u}) \), where \( c_i = \tau_i(\tilde{d}) \) is a sentence of \( \delta \) defining \( c_i \) in terms of \( \tilde{d} \).

For our construction, at stage \( s \), we will have a good triple \((\delta_s; \tilde{d}_s; \tilde{c}_s)\) with a test formula \( \chi^*_s(\bar{u}) \) that is valid on a rational box \( B_s \), so that the sentence \( \chi^*_s(B_s) \) is in \( T_{\text{exp}} \). Moreover, we will have \( f \) tentatively mapping \( \tilde{d}_s \) to \( \bar{v}_s \in K \), where \( \bar{v}_s \in B_s \). We believe that \( \bar{v}_s \) is an initial segment of the basis.

**Lemma 2.8.** Let \((\delta; \tilde{d}; \tilde{c})\) be a good triple with test formula \( \chi^*(\bar{u}) \) valid on a rational box \( B \) containing an independent tuple \( \bar{b} \). Let \( \bar{b}' \) be a further element independent over \( \bar{b} \). Let \( \delta' \) be the result of adding to \( \delta \) sentences saying of a new constant \( d \) that it is not equal to any mentioned in \( \delta \). Then \((\delta'; \tilde{d}; \tilde{c})\) is a good triple, with test formula \( \chi^*(\bar{u}, u') \) that is valid on a rational box \( B' \) around \( \bar{b}, \bar{b}' \).

(We may suppose that the projection of \( B' \) on the initial coordinates, omitting the last one, is contained in \( B \).)

**Lemma 2.9.** Let \((\delta; \tilde{d}; \tilde{c})\) be a good triple with test formula \( \chi^*(\bar{u}) \) valid on a rational box \( B \) containing an independent tuple \( \bar{b} \). Let \( \varphi \) be a sentence with constants among \( \tilde{d}, \tilde{c} \). There is a good triple \((\delta'; \tilde{d}; \tilde{c})\), where \( \delta' \) is the result of adding \( \pm \varphi \) to \( \delta \), with test formula \( \chi^*(\bar{u}) \) valid on a rational box \( B' \subseteq B \) containing \( \bar{b} \).
Lemma 2.10. Let \((\delta; d; c)\) be a good triple with test formula \(\chi^*(u)\) valid on a rational box \(B\) containing an independent tuple \(b\). For a term \(\tau(d)\), there is a good triple \((\delta'; d'; c')\), with a test formula \(\chi^*(u)\) valid on a rational box \(B' \subseteq B\) containing \(b\), where \(d'\) and \(c'\) satisfy one of the following:

1. \(d'\) is the result of adding to \(d\) a sentence \(c = \tau(d)\), for some \(c \in c\), and \(c' = c\).
2. \(d'\) is the result of adding to \(d\) a sentence \(c' = \tau(d)\), where \(c'\) is new, along with sentences saying that \(c'\) is not equal to any of the constants in \(d, c\), and \(c'\) is \(c, c'\).

In our construction, it may be that at stage \(s + 1\), our guess at the initial segment of the basis changes. Then \(v_{s+1}\) is the restriction of \(v_s\) to the part that seems correct. We must give the extra elements of \(d_s\) definitions in terms of \(d_{s+1}\). The following lemma says that we can do this.

Lemma 2.11. Let \((\delta; d; d'; c)\) be a good triple with test formula \(\chi^*(u, u')\) valid on a rational box \(B\) containing a tuple \(b, b'\), where \(b\) is independent. There is a good triple \((\delta'; d'; v, d')\) with test formula \(\chi(u, u')\) valid on a rational box containing \(b\), where \(\delta'\) is the result of adding to \(\delta\) a sentence \(d' = \tau(d)\). We may take \(B'\) to be the projection of \(B\) on the initial coordinates, omitting the one that corresponds to \(u'\).

Proof of Lemma. The box \(B\) is a cross product of rational intervals. Say that \(I\) is the interval corresponding to the coordinate \(u'\), and take \(q \in I\). There is a term \(\tau\) in our language naming \(q\). Let \(\delta'\) be the result of adding to \(\delta\) the defining sentence \(d' = \tau\), and modifying the definitions \(c_i = \tau(u, u')\), by replacing \(u'\) by \(\tau\). We have in \(\delta\) sentences saying that \(d'\) is distinct from all constants in \(d, c\).

The formulas of \(\delta\) are valid on \(B\), and they guarantee that for \(u \in B'\), nothing in \(I\) can be equal to any \(u_i\). Also, for \(c_i\) with a definition \(c_i = \tau(d)\) in \(\delta\), for \(u \in B'\), nothing in \(I\) can be equal to \(\tau(u)\).

We begin at stage 0 with the good triple \((0, 0, 0)\). Our guess at an initial segment of the basis is \(0\), and \(f\) is not defined on any elements. Suppose at stage \(s\), our guess at an initial segment of the basis is \(v_s\), we have the good triple \((\delta_s, d_s, c_s)\), with test formula \(\chi^*(u)\) valid on a box \(B_s\) around \(v_s\), and we have \(f\) mapping \(d_s\) to \(v_s\).

We must say what happens at stage \(s + 1\). Supposing \(v_s\) still appears to be an initial segment of the basis, and that \(v'\) is the next element of the basis, we consider letting \(v_{s+1} = v_s, v'\) and extending \(f\) to map a new constant \(d'\) to \(v'\), and letting \(d_{s+1} = d_s, d'\). Assuming that we can find an appropriate rational box on which the test formula is valid, we let \(\delta_{s+1}\) be an extension of \(\delta_s\), with some sentences added as follows:

Step 1 We add sentences saying that \(d'\) is not equal to anything in \(d_s, c_s\).

Step 2 We add one of the sentences \(\pm \varphi\), where \(\varphi\) is the first sentence on our list that involves only constants from \(d_s, c_s\),

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Step 3 For the first term $\tau(\bar{d}_s)$ such that $\delta_s$ does not include a defining sentence $c = \tau(\bar{d})$, we add such a sentence. Here $c$ may be an element of $\bar{d}_s$, or $\bar{c}_s$ or a new constant.

Lemma 2.8 says that we can carry out Step 1, finding a rational box on which the appropriate test formula is valid, provided that our guess the initial segment of the basis is correct. Lemma 2.9 says that we can carry out Step 2, provided that our guess at the initial segment of the basis is correct. Lemma 2.10 says that we can carry out Step 3, provided that our guess at the initial segment of the basis is correct.

Running our approximations ahead, either $\bar{v}_s$ will no longer seem to be an initial segment of the basis, or else we will arrive at $\bar{v}_{s+1}$, the result of adding a single element to $\bar{v}_s$ and a good triple $(\delta_{s+1}, \bar{d}_{s+1}, \bar{c}_{s+1})$, carrying out all three steps, with a test formula that is valid on an appropriate rational box $B_{s+1}$ containing $\bar{v}_{s+1}$. We do not add to the diagram unless this happens.

If it appears that $\bar{v}_s$ is not an initial segment of the basis, then we apply Lemma 2.11 finitely many times, to give definitions to the elements of $\bar{d}_s$ that are mapped to the elements of $\bar{v}_s$ that are not in $\bar{v}_{s+1}$. This lemma tells us how to arrive at an appropriate next good triple and a rational box $B_{s+1}$. If those elements of $\bar{v}_s$ later return to our approximation for $\bar{b}$, the construction will create new elements that will be mapped to those elements.

Eventually, our guess at the initial segment of the basis of length $n$ is correct. Say this happens at stage $s$. The initial segment of the basis of length $n$ is $\bar{v}_s$, and for all stages $t \geq s$, the stage $t$ version of $f$ will map $\bar{d}_s$ to $\bar{v}_s$. What we say about $\bar{d}_s$ is true about $\bar{v}_s$. Taking the limit, $f$ gives pre-images to all elements of the basis. Each element of our $C$ that is not the pre-image of a basis element under $f$ has a definition in terms of some elements that pre-images of the basis elements.

We have arranged that if $\varphi(\bar{d}, \bar{c})$ is in $\delta_s$, where $f(\bar{d}) = \bar{v}$ is part of the basis, and $c_i$ has been given a definition $\tau_i(\bar{d})$ in $\delta_s$, then $\varphi(\bar{v}, \tau(\bar{v}))$ is true in $K_{exp}$. Thus, $f$ is an isomorphism. This completes the proof of Proposition 2.7. □

3 Generalizing

In the previous section, we proved that $\mathcal{R}_{exp} \leq^*_w \mathcal{R}$. In this section, we use essentially the same proof to obtain the General Theorem stated in the introduction. Recall that $\mathcal{R}_Q$ consists of the set of real numbers, with the ordering and constants for the rational numbers. This is a very weak structure in terms of elementary first order definability. The theorem says that it has as much computing power as various expansions of the reals, including $\mathcal{R}_{exp}$.

**Theorem 3.1** (General Theorem). Let $\mathcal{M}$ be an o-minimal expansion of $\mathcal{R}_Q$, in a countable language, and with definable Skolem functions. Then $\mathcal{M} \leq^*_w \mathcal{R}_Q$. In fact, after collapse, every copy $K$ of $\mathcal{R}_Q$ computes the complete diagram of a copy of $\mathcal{M}$.

We split the proof of Theorem 3.1 into a sequence of lemmas, following the outline from the previous section. The greatest difference is in the lemma below.
In the previous section, the proof of Lemma 2.2, on uniqueness of the expansion, did not use o-minimality, just the fact that $\text{exp}$ is a continuous function, together with the fact that the theory $T_{\text{exp}}$ is coded by a real. Let $T_M = \text{Th}(M)$.

**Lemma 3.2** (Uniqueness). For $K \cong R_Q$, there is a unique expansion $K_M$ to a model of $T_M$.

*Proof.* Since $M$ is o-minimal, with definable Skolem functions, definable closure is a good closure notion. Since $K \cong R_Q$, there is at least one expansion of $K$ to a model of $T_M$, say $K_1$. Let $b_1, b_2, \ldots$ be a basis for $K_1$. Suppose $K_2$ is another expansion of $K$ to a model of $T_M$. To show that $K_1 = K_2$, we first show that for all tuples $\bar{b}$ in the $K_1$-basis, $\bar{b}$ satisfies the same formulas in $K_1$ and $K_2$. Suppose $\phi(\bar{x})$ is true in $K_1$ of a basis tuple $\bar{b}$. Since $\bar{b}$ is independent in $K_1$, by Proposition 1.4 there is a rational box $B$ around $\bar{b}$ such that the sentence $(\forall \bar{x} \in B)\phi(\bar{x})$ is in $T_M$. Then $\phi(\bar{x})$ must be true of $\bar{b}$ in $K_2$.

To complete the proof that $K_1 = K_2$, we show that every element $c$ has a definition from a tuple in the $K_1$-basis that is good in both $K_1$ and $K_2$. For an element $c$, we know that there is a definition in $K_1$ from a tuple $\bar{b}$ of basis elements. Say $c = \tau(\bar{b})$. Let $c'$ be the element satisfying the definition $\tau(\bar{b})$ in $K_2$. We can show that $c = c'$. If $c$ is in a rational interval $I$, then the formula saying $\tau(\bar{x}) \in I$ is true of $\bar{b}$ in $K_1$. This formula is also true of $\bar{b}$ in $K_2$, so $c' \in I$. This shows that $c = c'$, completing the proof that $K_1 = K_2$. □

We write $K_M$ for the unique expansion of $K$ to a model of $T_M$. Let $IND_n$ be the set of $n$-tuples in $K$ that are independent in the expansion $K_M$. The next lemma is the analogue of Lemma 2.4.

**Lemma 3.3** (Computable $\Pi_2$ definition of $IND_n$). For each $n$, we can effectively find a computable $\Pi_2$ definition of $IND_n$, with a real parameter $r$ coding $T_M$.

*Proof.* We have $\bar{b}$ in $IND_n$ iff for all formulas $\phi(\bar{x})$ in the appropriate variables, there is a rational box $B$ around $\bar{b}$ such that $T_M$ contains one of the sentences $(\forall \bar{x} \in B)\phi(\bar{x})$ or $(\forall \bar{x} \in B)\neg \phi(\bar{x})$. We express this as a computable $\Pi_2$ formula, just as we did in the proof of Lemma 2.4. □

The next lemma is the analogue of Lemma 2.5.

**Lemma 3.4** (Computable $\Sigma_2$ definition of $IND_n$). For each $n$, we can effectively find a computable $\Sigma_2$ definition of $IND_n$, with a real parameter $r$ coding $T_M$.

*Proof.* For a fixed tuple of variables $\bar{x}$, let $(\varphi_n(\bar{x}))_{n \in \omega}$ be a computable list of the formulas in the language of $M$ with free variables $\bar{x}$. Let $T$ be the tree of finite sequences of rational boxes $(B_1, B_2, \ldots, B_s)$ such that $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_s$ such that for each $k$, one of the sentences $(\forall \bar{x} \in B_k)\varphi_k(\bar{x})$ or $(\forall \bar{x} \in B_k)\neg \varphi_k(\bar{x})$ is in $T_M$. The tree is $T$ computable in $T_M$. We have $\bar{b}$ in $IND_n$ iff there is a path $\pi$ through $T$ such that $\bar{b}$ is in all of the boxes associated with $\pi$.

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The computable \( \Sigma_2 \) definition of \( IND_\mu(\bar{u}) \) says that there exists \( x \) coding a path through \( \mathcal{T} \) such that \( \bar{u} \) is in all of the boxes associated with the path. As in the proof of Lemma 2.5, we have a computable \( \Pi_1 \) formula (with a parameter for \( T_M \)) saying that \( x \) codes a path through \( \mathcal{T} \), and another computable \( \Pi_1 \) formula saying that \( \bar{u} \) lies in the boxes corresponding to the path coded by \( x \).

The next lemma is the analogue of Lemma 2.6.

**Lemma 3.5 (Basis).** There is a basis \( b_1, b_2, \ldots \) for \( K_M \) that is \( \Delta^0_2 \) relative to \( K \).

**Proof.** We proceed exactly as in the proof of Lemma 2.6. Since the relations \( IND_\mu \) are \( \Delta^0_2 \) relative to \( K \) (uniformly), we apply a procedure \( \Delta^0_2 \) relative to \( K \) to run through the elements of \( K \) in order, adding a given element to our basis just in case it is independent of those previously added.

The next lemma is the analogue of Proposition 2.7. This will complete the proof of Theorem 3.1.

**Lemma 3.6 (Enumerating the complete diagram of the expansion).** Any copy of \( K \) computes the complete diagram of a copy of the expansion \( K_M \).

**Proof.** Let \( b_1, b_2, \ldots \) be a basis for \( K_M \) that is \( \Delta^0_2 \) relative to \( K \). Guessing at the basis, we enumerate the complete diagram of a copy of \( K_M \). We maintain the condition that if we have tentatively mapped \( \bar{d} \) to \( \bar{v} \) in \( K \) which we believe to be an initial segment \( \bar{b} \) of the basis, and we have defined other elements \( \bar{c} \) in terms of \( \bar{d} \), then the theory \( T_M \) guarantees that what we have said about \( \bar{d} \) is true on a rational box around \( \bar{v} \).

**Corollary 3.7.** \( \mathcal{R}_Q \equiv^*_w \mathcal{R} \)

**Proof.** It is easy to see that \( \mathcal{R}_Q \leq^*_w \mathcal{R} \leq^*_w (\mathcal{R}, (q)_{q \in Q}) \), where the structure on the right is the expansion of the ordered field of reals with constants for the rationals. By Theorem 3.1, \( (\mathcal{R}, (q)_{q \in Q}) \leq^*_w \mathcal{R}_Q \).

### 4 Applying the general result

In this section, we apply Theorem 3.1 to show that various structures are equivalent to \( \mathcal{R} \) in computing power. We begin with \( \mathcal{R}_f = (\mathcal{R}, f) \), where \( f \) is analytic.

In Section 2, we considered the case where \( f \) is the exponential function. In this case, \( \mathcal{R}_f \) is \( \alpha \)-minimal, but in general, in particular, if \( f \) is the sine function, \( \mathcal{R}_f \) will not be \( \alpha \)-minimal.

**Proposition 4.1.** Let \( f \) be analytic on \( \mathcal{R} \). Then \( \mathcal{R}_f \equiv^*_w \mathcal{R} \).
Proof. Clearly, $\mathcal{R} \leq^*_w \mathcal{R}_f$. Let $\mathcal{R}_{\text{bounded } f}$ be the expansion of $\mathcal{R}$ by the family of functions $f_z$, for $z \in \mathbb{Z}$, where

$$f_z(x) = \begin{cases} 
    f(x) & \text{if } x \in [z, z + 1] \\
    0 & \text{otherwise}
\end{cases}$$

Since $\mathcal{R}_{\text{an}}$ is $o$-minimal, so is $\mathcal{R}_{\text{bounded } f}$.

Lemma 4.2. $\mathcal{R}_f \leq^*_w \mathcal{R}_{\text{bounded } f}$.

Proof. Let $(K, (f^K_z)_{z \in \omega})$ be a copy of $\mathcal{R}_{\text{bounded } f}$. We define $f^K_z$ such that $(K, f^K_z)$ is isomorphic to $\mathcal{R}_f$. Given $a \in K$, we can find, effectively in the field $K$, the integer $z$ that is the “floor” of $a$; i.e., $z \leq a < z + 1$. Then $f^K_z(a) = f^K_z(x)$.

We can now complete the proof of Proposition 4.1. It is clear that $\mathcal{R}_{\text{bounded } f} \leq^*_w (\mathcal{R}_{\text{bounded } f}, \langle q \rangle_{q \in \mathbb{Q}})$. Using Theorem 3.1, we get $(\mathcal{R}_{\text{bounded } f}, \langle q \rangle_{q \in \mathbb{Q}}) \leq^*_w \mathcal{R}_Q \leq^*_w \mathcal{R}$. This shows that $\mathcal{R}_f \leq^*_w \mathcal{R}$. \qed

Recall that $\mathcal{R}^+$ is the reduct of $\mathcal{R}$ without multiplication, but including addition, the ordering, and the constants $0$ and $1$. The result below is also proved in [3].

Proposition 4.3. $\mathcal{R} \equiv^*_w \mathcal{R}^+$.

Proof. By Theorem 3.1, we have $\mathcal{R} \leq^*_w \mathcal{R}_Q$. It is easy to see that $\mathcal{R}_Q \leq^*_w (\mathcal{R}^+, \langle q \rangle_{q \in \mathbb{Q}}) \leq^*_w \mathcal{R}^+ \leq^*_w \mathcal{R}$. \qed

The final example of this section is simple. Let $(r_n)_{n \in \omega}$ be any sequence of elements of $\mathcal{R}$, and consider $(\mathcal{R}, \langle r_n \rangle_{n \in \omega})$, the expansion of the ordered field $\mathcal{R}$ with constants for those elements.

Proposition 4.4. $\mathcal{R} \equiv^*_w (\mathcal{R}, \langle r_n \rangle_{n \in \omega})$.

Proof. Let $\mathcal{M}$ be the expansion of $\mathcal{R}_Q$ with all of the structure of $\mathcal{R}$ and the constants $r_n$. We have $\mathcal{R} \leq^*_w \mathcal{M} \leq^*_w \mathcal{R}_Q \leq^*_w \mathcal{R}$. We use Theorem 3.1 for the second reduction. The other reductions are clear. \qed

5 The structure $\mathcal{R}_{\text{int}}$

Recall that the structure $\mathcal{R}_{\text{int}}$ has just the set of real numbers, with the ordering, and unary predicates for the half-open intervals $[q, q')$ with dyadic rational endpoints. This can be thought of as the minimal structure that is able to recover the (preferred) binary expansions of the real numbers, as each initial segment of the binary expansion of a number corresponds exactly to the number being in a half-open interval of this sort. For instance, knowing that the binary expansion of $x$ begins 0.10 corresponds exactly to knowing that $x \in (\frac{1}{2}, \frac{3}{4})$.

In this section, we will show that $\mathcal{R}_{\text{int}} \equiv^*_w \mathcal{B}$, where $\mathcal{B}$ is the structure representing Baire space. In [3], Downey, Greenberg, and Miller showed that
\[ B \equiv^*_w R, \] by a proof which resembles our proofs from Sections 2 and 3, although the work was done independently. This, together with the fact that \( R \equiv^*_w B, \) implies that \( R_{\text{int}} \equiv^*_w R. \) Independently of Downey, Greenberg, and Miller, the authors had arrived at the fact that \( B \equiv^*_w R. \) We end the section by saying just a little about our reasoning. In particular, we state a variant of Theorem 3.1 in which the \( R_Q \) is replaced by \( R_{\text{int}}. \)

**Proposition 5.1.** \( B \equiv^*_w R_{\text{int}}. \)

We split the proof of Proposition 5.1 into two lemmas.

**Lemma 5.2.** \( B \leq^*_w R_{\text{int}}. \)

**Proof.** Given a copy \( K \) of \( R_{\text{int}} \), we can enumerate the preferred binary expansions of the reals in the interval \([0,1). \) For each such real, we get a function \( f \in 2^\omega \) such that \( f \) has infinitely many 0’s. Given such an \( f \), we pass to a function \( g \in \omega^\omega \) where \( g(0) \) is the number of 1’s before the first 0, and for \( k > 0 \), \( g(k) \) is the number of 1’s between the \( k^{th} \) 0 and the \((k + 1)^{st}\). This gives a copy of \( B. \) \( \square \)

**Lemma 5.3.** \( R_{\text{int}} \leq^*_w B. \)

**Proof.** Given a copy of \( B, \) we can enumerate the functions \( g \in \omega^\omega. \) From each \( g, \) we pass effectively to a function \( f \in 2^\omega \) such that \( g(0) \) is the number of 1’s before the first 0 in \( f, \) and \( g(k + 1) \) is the number of 1’s between the \((k + 1)^{st}\) and \((k + 2)^{nd}\) 0’s in \( f. \) The functions \( f \in 2^\omega \) are just the preferred binary expansions of reals in the interval \([0,1). \) The ordering on these reals corresponds to the lexicographic ordering on the functions \( f. \) For each dyadic rational \( q, \) we give a name in which we mark the first in the infinite sequence of 0’s.

For a function \( f \) that is the preferred binary expansion of a real \( r \) in the interval \([0,1), \) we cannot effectively determine whether \( r = q. \) However, for a pair \( q < q' \in D, \) we can effectively determine whether \( r \in [q, q'). \) We have a copy of the restriction of \( R_{\text{int}} \) to the interval \([0,1). \) For the full structure, we take pairs \((z, f), \) where \( z \in Z \) and \( f \in 2^\omega \) has infinitely many 0’s. We take the lexicographic ordering on these pairs. The full set of dyadic rationals consists of the elements \( z + q, \) for \( q \) with a special name. We can determine membership in intervals with these endpoints. This gives a copy of \( R_{\text{int}}. \) \( \square \)

**Corollary 5.4.** \( R_Q \equiv^*_w R_{\text{int}}. \)

**Proof.** By Corollary 3, \( R_Q \equiv^*_w R. \) As we said above, Downey, Greenberg, and Miller [3] showed that \( R \equiv^*_w B. \) By Proposition 5.1, \( B \equiv^*_w R_{\text{int}}. \) \( \square \)

### 5.1 Alternative reasoning

Here we say a few words about alternative reasoning for Corollary 5.4. It is clear that \( R_{\text{int}} \leq^*_w R_Q. \) We could prove that \( R_Q \leq^*_w R_{\text{int}}, \) independent of [3], in two slightly different ways.
1. We could argue directly that $\mathcal{R}_Q \leq^*_w \mathcal{R}_{\text{int}}$.

2. Alternatively, we could prove a variant of Theorem 3.1, with $\mathcal{R}_{\text{int}}$ replacing $\mathcal{R}_Q$.

For the direct approach, the proof follows the same outline as in Sections 2 and 3, with analogous lemmas. We note that the structure $\mathcal{R}_Q$ does not have definable Skolem functions, although definable closure still forms a good closure notion. In fact, the definable closure of a set $S$ in $\mathcal{R}_Q$ is exactly $S \cup \mathbb{Q}$. Thus, a basis for $\mathcal{R}_Q$ is simply the entire set of irrationals of $\mathbb{R}$. So, instead of defining the sets $IND_n$ of independent $n$-tuples, we just define the set of irrationals.

Here is the variant of Theorem 3.1.

**Theorem 5.5** (Variant of Theorem 3.1). *Let $\mathcal{M}$ be an $o$-minimal expansion of $\mathcal{R}_{\text{int}}$, in a countable language, with definable Skolem functions. Then $\mathcal{M} \leq^*_w \mathcal{R}_{\text{int}}$. In fact, after collapse, every copy $K$ of $\mathcal{R}_{\text{int}}$ computes the complete diagram of a copy of $\mathcal{M}$.***

Comments on the Proof. The proof of Theorem 5.5 involves a sequence of lemmas like those in the proof of Theorem 3.1, but with modifications. We mention the main points briefly. The first modification is to Lemma 3.4. The dyadic rationals are definable without quantifiers in $\mathcal{R}_Q$, but in $\mathcal{R}_{\text{int}}$, they are $\Pi_1$ definable. We have that $x = 1/2$ iff $(x \in [1/2, 1)) \land (\forall y \in [1/2, 1))(x \leq y)$. Thus, in $\mathcal{R}_{\text{int}}$ we do not have a computable $\Pi_1$ formula saying that $x$ is not a dyadic rational, so to ensure that $x$ codes an infinite set, we instead use the complement of the set that $x$ would normally code. Every number has infinitely many zeroes in its binary expansion, so this ensures that every $x$ codes an infinite set. The second modification is simple in conception but more cumbersome in the details. Every use of open rational intervals and open rational boxes needs to be replaced with half-open dyadic rational intervals and boxes, and then every lemma needs to be reformulated, and every proof must be re-written.

To prove that $\mathcal{R}_Q \leq^*_w \mathcal{R}_{\text{int}}$, using Theorem 5.5, we would apply Theorem 5.5 to show that $\mathcal{R} \leq^*_w \mathcal{R}_{\text{int}}$, and note that $\mathcal{R}_Q \leq^*_w \mathcal{R}$.

**References**


