Goal Programming

Multicriteria Decision Problems

- Goal Programming
- Goal Programming: Formulation

Goal Programming ??

How do you find optimal solutions to the following?

• Multiple criterion for measuring performance (car with low cost, good gas mileage, stylish, etc.. / attend school with good reputation, low tuition, close to home, right program…)

• Multiple objectives / goals
(e.g. Minimize service cost, maximize customer satisfaction)

Answer: Use Goal Programming

Goal Programming

- Goal programming may be used to solve linear programs with multiple objectives, with each objective viewed as a "goal".
- In goal programming, d_i^+ and d_i^- , deviation variables, are the amounts a targeted goal i is overachieved or underachieved, respectively.
- The goals themselves are added to the constraint set with d_i^+ and d_i^- acting as the surplus and slack variables.
- One approach to goal programming is to satisfy goals in a priority sequence. Second-priority goals are pursued without reducing the first-priority goals, etc.

Goal Programming

- For each priority level, the objective function is to minimize the (weighted) sum of the goal deviations.

Goal Programming Approach

Step 1: Decide the priority level of each goal.

Step 2: Decide the weight on each goal.

If a priority level has more than one goal, for each goal i decide the weight, w_i, to be placed on the deviation(s), d_i^+ and/or d_i^-, from the goal.

Step 3: Set up a linear program.

Consider new objectives (minimize deviations), subject to all Functional Constraints, and Goal Constraints

Step 4: Solve the current linear program.

Example: Conceptual Products

Conceptual Products is a computer company that produces the CP400 and the CP500 computers. The computers use different mother boards produced in abundant supply by the company, but use the same cases and disk drives. The CP400 models use two floppy disk drives and no zip disk drives whereas the CP500 models use one floppy disk drive and one zip disk drive.

Example: Conceptual Products

The disk drives and cases are bought from vendors. There are 1000 floppy disk drives, 500 zip disk drives, and 600 cases available to Conceptual Products on a weekly basis. It takes one hour to manufacture a CP400 and its profit is $200 and it takes one and one-half hours to manufacture a CP500 and its profit is $500.

Example: Conceptual Products

The company has four goals which are given below:

Priority 1: Meet a state contract of 200 CP400 machines weekly. (Goal 1)

Priority 2: Make at least 500 total computers weekly. (Goal 2)

Priority 3: Make at least $250,000 weekly. (Goal 3)

Priority 4: Use no more than 400 man-hours per week. (Goal 4)

Example: Conceptual Products

- Variables

  x_1 = number of CP400 computers produced weekly

  x_2 = number of CP500 computers produced weekly

  d_i^- = amount the right hand side of goal i is deficient

  d_i^+ = amount the right hand side of goal i is exceeded

- Functional Constraints

  Availability of floppy disk drives: $2x_1 + x_2 \le 1000$

  Availability of zip disk drives: $x_2 \le 500$

  Availability of cases: $x_1 + x_2 \le 600$

Example: Conceptual Products

- Goals

(1) 200 CP400 computers weekly:

$$x_1 + d_1^- - d_1^+ = 200$$

(2) 500 total computers weekly:

$$x_1 + x_2 + d_2^- - d_2^+ = 500$$

(3) $250(in thousands) profit:

$$.2x_1 + .5x_2 + d_3^- - d_3^+ = 250$$

(4) 400 total man-hours weekly:

$$x_1 + 1.5x_2 + d_4^- - d_4^+ = 400$$

Non-negativity:

$$x_1, x_2, d_i^-, d_i^+ \geq 0 \text{ for all } i$$

Example: Conceptual Products

- Objective Functions

Priority 1: Minimize the amount the state contract is not met: Min d_1^-

Priority 2: Minimize the number under 500 computers produced weekly: Min d_2^-

Priority 3: Minimize the amount under $250,000 earned weekly: Min d_3^-

Priority 4: Minimize the man-hours over 400 used weekly: Min d_4^+

Example: Conceptual Products

- Formulation Summary

Min $P_1(d_1^-) + P_2(d_2^-) + P_3(d_3^-) + P_4(d_4^+)$

$$
\begin{array}{lllllll}
\text{s.t.} & 2x_1 + x_2 & & & & & \leq 1000 \\
& \quad + x_2 & & & & & \leq 500 \\
& x_1 + x_2 & & & & & \leq 600 \\
& x_1 & + d_1^- - d_1^+ & & & & = 200 \\
& x_1 + x_2 & & + d_2^- - d_2^+ & & & = 500 \\
& .2x_1 + .5x_2 & & & + d_3^- - d_3^+ & & = 250 \\
& x_1 + 1.5x_2 & & & & + d_4^- - d_4^+ & = 400
\end{array}
$$

$$x_1,\ x_2,\ d_1^-,\ d_1^+,\ d_2^-,\ d_2^+,\ d_3^-,\ d_3^+,\ d_4^-,\ d_4^+ \geq 0$$

Objective Function and Constraints

Maximize $Z = \$40\, x_1 + 50\, x_2$

Subject to

$$
\begin{array}{ll}
x_1 + 2x_2 \leq 40 \text{ hr} & \text{(labor constraint)} \\
4x_1 + 3x_2 \leq 120 \text{ lb} & \text{(clay constraint)} \\
x_1,\ x_2 \geq 0 &
\end{array}
$$

Decision variables
***x*1 = number of bowls to produce**
***x*2 = number of mugs to produce**

Goals

Instead of having one objective, the pottery company has several objectives that are listed ***<u>in order of importance</u>***:

- To avoid layoffs, the company does not want to use fewer than 40 hours of labor per day;
- The company would like to achieve a satisfactory profit level of $1,600 per day;
- Because the clay must de stored in a special place so that it does not dry out, the company prefers not to keep more than 120 pounds on hand per day;
- Because high overhead costs results when the plant is kept open past normal hours, the company would like to minimize the amount of overtime.

Labor Goal

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$$

$X_1 = 5$; $x_2 = 10$ 25 hours are used in production $x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$ $(5) + 2(10) + d_1^- - d_1^+ = 40\,hr$ $25 + d_1^- - d_1^+ = 40\,hr$ $If \quad d_1^- = 15 \quad and \quad d_1^+ = 0 \quad then$ $25 + 15 - 0 = 40\,hr$	$X_1 = 10$; $x_2 = 20$ 50 hours are used in production $x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$ $(10) + 2(20) + d_1^- - d_1^+ = 40\,hr$ $50 + d_1^- - d_1^+ = 40\,hr$ $Thus \quad d_1^- = 0 \quad and \quad d_1^+ = 10$ $(50 + 0 - 10 = 40\,hr)$

$$Minimize \quad P_1 d_1^- \; , \; P_4 d_1^+$$

Profit Goal

$$40x_1 + 50x_2 + d_2^- - d_2^+ = \$1{,}600$$

$$Minimize \quad P_1 d_1^-, \; P_2 d_2^-, \; P_4 d_1^+$$

Material Goal

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120 \; lb$$

$$Minimize \quad P_1 d_1^-, \; P_2 d_2^-, \; P_3 d_3^+, \; P_4 d_1^+$$

The model

$$Minimize \quad P_1 d_1^-,\ P_2\ d_2^-,\ P_3\ \ d_3^+, P_4\ d_1^+$$

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40\,hr$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = \$1{,}600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120\ lb$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Goal Programming – Example Problem

You work for an Advertising agency. A customer has identified three primary target audiences they are trying to reach, and has an Advertising budget of $600,000. They have expressed their targets in the form of three goals:

Goal 1 – Ads should be seen by at least 40 million high-income men (HIM)
Goal 2 – Ads should be seen by at least 60 million low-income people (LIP)
Goal 3 – Ads should be seen by at least 35 million high-income women (HIW)

You recognize that advertising during football games and soap operas will cover the target audience. The table below indicates the number of viewers from the different categories that will be viewing these types of programming.

	HIM	LIP	HIW	Cost
Football ad (per min.)	7 million	10 million	5 million	$100,000
Soap Opera ad (/min)	3 million	5 million	4 million	$60,000

Goal Programming – Example Problem

Expressing the goals as an equation.

Let: x_1 – minutes of football ad
x_2 – minutes of soap opera ad

Goal 1 - HIM) $7 x_1 + 3 x_2 \geq 40$
Goal 2 - LIP) $10 x_1 + 5 x_2 \geq 60$
Goal 3 - HIW) $5 x_1 + 4 x_2 \geq 35$

Ad Budget) $100 x_1 + 60 x_2 \leq 600$

Goal Programming – Example Problem

Formulating the problem as an LP:

Min (or Max) Z = something
s.t.
HIM) $7 x_1 + 3 x_2 > 40$
LIP) $10 x_1 + 5 x_2 > 60$
HIW) $5 x_1 + 4 x_2 \geq 35$
Ad Bud.) $100 x_1 + 60 x_2 \leq 600$
$x_1 , x_2 \geq 0$

Which constraints are real constraints versus “desired” constraints?
Which constraints are “hard” constraints versus “soft” constraints?

Goal Programming – Example Problem

Since the first three constraints are really goals, and not "hard" constraints, express these constraints in terms of **deviational variables.**

$$
\begin{aligned}
&\text{HIM)}\quad 7x_1 + 3x_2 + d_1^- - d_1^+ = 40\\
&\text{LIP)}\quad 10x_1 + 5x_2 + d_2^- - d_2^+ = 60\\
&\text{HIW)}\quad 5x_1 + 4x_2 + d_3^- - d_3^+ = 35
\end{aligned}
$$

$$d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Suppose each shortfall 0f 1,000,000 viewers from the goal translates to a cost of $200,000 for HIM, $100,000 for LIP and $50,000 for HIW.
Then the objective function would be:

$$\text{Min } Z = 200\, d_1^- + 100\, d_2^- + 50\, d_3^-$$

Goal Programming – Example Problem

Then in order to minimize the penalty for not reaching the viewing audience goal can be expressed as the following LP:

$$
\begin{aligned}
&\text{Min}\quad Z = 200\, d_1^- + 100\, d_2^- + 50\, d_3^-\\
&\text{s.t.}\\
&\text{HIM)}\quad 7x_1 + 3x_2 + d_1^- - d_1^+ = 40\\
&\text{LIP)}\quad 10x_1 + 5x_2 + d_2^- - d_2^+ = 60\\
&\text{HIW)}\quad 5x_1 + 4x_2 + d_3^- - d_3^+ = 35\\
&\text{Ad Bud.)}\ 100x_1 + 60x_2 \leq 600
\end{aligned}
$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

The optimal solution to the above LP is:
$Z = 250$, $x_1 = 6$, $x_2 = 0$, $d_1^+ = 0$, $d_1^- = 0$, $d_2^+ = 0$, $d_2^- = 0$, $d_3^+ = 0$, $d_3^- = 5$.

Goal Programming: Weighted -vs- Preemptive Goals

In the advertising example, the goals could readily be weighted by relative importance using the cost penalties ($200,000 for HIM, $100,000 for LIP and $50,000 for HIW).

In many cases, the relative "weighting" of a goal is not easily determined, however the goals can be ranked from most important to least important. In this case, the most important goal pre-empts all the other goals. Once the most important goal is met, the second goal is addressed, and so on.

Goal Programming: Preemptive Goals

Suppose the HIM constraint must be met first, followed by LIP and then HIW.

First rewrite the LP as the following:

Min $Z = d_1^-$

s.t.

HIM) $7x_1 + 3x_2 + d_1^- - d_1^+ = 40$

LIP) $10x_1 + 5x_2 + d_2^- - d_2^+ = 60$

HIW) $5x_1 + 4x_2 + d_3^- - d_3^+ = 35$

Ad Bud.) $100x_1 + 60x_2 \leq 600$

$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$

This LP solves to $Z = 0$, $d_1^- = 0$. So goal HIM is met.

Goal Programming: Preemptive Goals

Since goal HIM is met, now make goal HIM a fixed constraint while trying to satisfy goal LIP.

Min $Z = d_2^-$

s.t.

$$
\begin{array}{lll}
\text{HIM)} & 7x_1 + 3x_2 + d_1^- - d_1^+ & = 40 \\
\text{LIP)} & 10x_1 + 5x_2 + d_2^- - d_2^+ & = 60 \\
\text{HIW)} & 5x_1 + 4x_2 + d_3^- - d_3^+ & = 35 \\
\text{Ad Bud.)} & 100x_1 + 60x_2 & \le 600 \\
 & d_1^- & = 0
\end{array}
$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \ge 0$$

This LP solves to $Z = 0$, $d_2^- = 0$. So goal LIP is met.

Goal Programming: Preemptive Goals

Since both goal HIM and LIP are met, make goal HIM and LIP fixed constraints while trying to satisfy goal HIW.

Min $Z = d_3^-$

s.t.

$$
\begin{array}{lll}
\text{HIM)} & 7x_1 + 3x_2 + d_1^- - d_1^+ & = 40 \\
\text{LIP)} & 10x_1 + 5x_2 + d_2^- - d_2^+ & = 60 \\
\text{HIW)} & 5x_1 + 4x_2 + d_3^- - d_3^+ & = 35 \\
\text{Ad Bud.)} & 100x_1 + 60x_2 & \le 600 \\
 & d_1^- & = 0 \\
 & d_2^- & = 0
\end{array}
$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \ge 0$$

Goal Programming: Additional Example

A company has two machines for manufacturing a product.
Machine 1 make two units per hour, while machine 2 makes three units per hour. The company has an order of 80 units.
Energy restrictions dictate that only one machine can operate at one time.
The company has 40 hours of regular machining time, but overtime is available. It costs $4.00 to run machine 1 for one hour, while machine 2 costs $5.00 per hour.
The company has the following goals:

1) Meet the demand of 80 units exactly.
2) Limit machine overtime to 10 hours.
3) Use the 40 hours of normal machining time.
4) Minimize costs.

Goal Programming: Preemptive Goals

Letting P_i represent the relative weighting of each goal, the example can be formulated as the following LP:

Min $Z = P_1(d_1^- + d_1^+) + P_2 d_3^+ + P_3(d_2^- + d_2^+) + P_{14} d_4^+$

s.t.

$$
\begin{array}{llll}
2x_1 + 3x_2 + d_1^- - d_1^+ & & & = 80 \\
x_1 + x_2 + & d_2^- - d_2^+ & & = 40 \\
 & d_2^+ + d_3^- - d_3^+ & & = 10 \\
4x_1 + 5x_2 + & & d_4^- - d_4^+ & = 0
\end{array}
$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+ \geq 0$$